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On the Fundamental Group of Complex Hyperplane Arrangements: Lower Central Series and Chen Groups

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Abstract. This is a survey of some recent developments in the study of the fundamental group $G(\mathcal{A})$ of the complement of the complex hyperplane arrangement $\mathcal{A}$.

The direct sum of the lower central series quotients of $G(\mathcal{A})$ is a graded Lie algebra. The ranks of the lower central series quotients are numerical invariants of $G(\mathcal{A})$. For the class of hypersolvable arrangements which includes the class of fiber-type, generic and braid arrangements, it is possible to make an explicit calculation.

The Chen groups of $G(\mathcal{A})$ are the lower central series quotients of its maximal metabelian quotient. The Chen groups distinguish non isomorphic $G(\mathcal{A})$ which are not distinguished by the lower central series.

Keywords: Fundamental group; Complex hyperplane arrangement; Lie algebra; Lower Central Series; Chen groups.

1. Introduction

The *fundamental group* $\pi_1(X)$ of a topological space X is an important topological invariant of X; i.e., if two spaces are homeomorphic, their fundamental groups are isomorphic. This gives the possibility of proving that two spaces are not homeomorphic by proving that their fundamental group are not isomorphic. In this paper, we will only consider the particular case of the complements of some algebraic complex hypersurfaces and more precisely of hyperplane arrangements. The fundamental group of the complement of algebraic curves were studied by Zariski almost 70 years ago and Zariski and Van Kampen described a general procedure for calculating these groups. A presentation of the fundamental group of the complement of a complex hyperplane arrangement $\mathcal{A}$ was given by Randell [14], Salvetti [15] and Arvola [1] or using information encoded in the *braid monodromy* of the arrangement. Although topological invariants of the complement $M(\mathcal{A})$ are closely connected to the combinatorics of the arrangement, it is not *a priori* enough to determine the fundamental group $G(\mathcal{A})$ and, as such, it is not easy to handle. According to a classical construction of W. Magnus, the associated graded Lie algebra $\mathrm{gr}\big(G(\mathcal{A})\big)$ defined by the *lower central series* of $G(\mathcal{A})$ reflects many properties of $G(\mathcal{A})$. The ranks of the abelian groups $\mathrm{gr}_k\big(G(\mathcal{A})\big)$, called LCS ranks, are important numerical invariants of $G(\mathcal{A})$. As shown by Kohno [9] (based on foundational work by Sullivan and Morgan), the associated graded Lie algebra $\mathrm{gr}\big(G(\mathcal{A})\big)$ and the *holonomy Lie algebra* $\mathcal{H}(\mathcal{A})$ which is determined by the intersection lattice $\mathcal{L}(\mathcal{A})$ are rationally isomorphic. However, for the class of *hypersolvable* arrangements defined by Jambu and Papadima [6], we have an isomorphism $\mathrm{gr}\big(G(\mathcal{A})\big) \cong \mathcal{H}(\mathcal{A})$. Therefore, for this class of hypersolvable arrangements, an explicit formula for the LCS ranks is known. However, both the direct product and the semi-direct product of free groups, may be realized as the fundamental groups of the complements of (different) arrangements and their LCS ranks are equal. These groups cannot be distinguished by means of their associated Lie algebra.

K.T. Chen [2] introduced a more manageable approximation to the LCS ranks. The *Chen groups* of a group are the lower central series quotients of its maximal metabelian quotient. The direct sum of the Chen groups is a graded Lie algebra. Papadima and Suciu [13] proved that the rational Chen Lie Algebra is combinatorially determined.

In this paper, after a short introduction to arrangements of hyperplanes, we recall the Magnus theory relating groups theory and Lie algebras theory,. Then we proceed by showing different examples. First, we consider the most simple ones, the free groups and direct product of free groups and we get the Witt formula, and its topological meaning which gives a relation between the fundamental group, the cohomology algebra and the holonomy Lie algebra of the complement in $\mathbb{C}$ of a finite set. The second example is the famous braid groups. In the following sections, we introduce some generalizations in terms of hyperplane arrangements, fiber-type and hypersolvable ones. Finally, we introduce the Chen groups and following Cohen and Suciu [3] we give some examples showing that Chen groups allow to distinguish non isomorphic groups which cannot be distinguish by means of lower central series.

2. Hyperplane Arrangements

A *(complex) hyperplane arrangement* is a finite set, $\mathcal{A}$, of codimension 1 affine subspaces in a finite-dimensional complex space, $V = \mathbb{C}^l$. We refer the reader to [12] as a general reference on arrangements.

An important example is the braid arrangement of diagonal hyperplanes in $\mathbb{C}^l$. Loops in the complement can be viewed as (pure) braids on l strings, and the fundamental group can be identified with the pure braid group P_l.

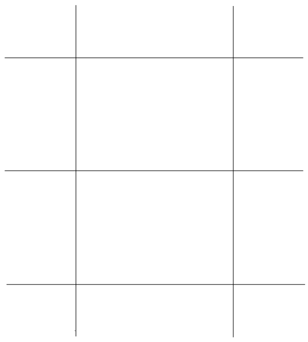
The arrangement $\mathcal{A}$ is called *central* if $\bigcap_{H\in\mathcal{A}} H \neq \emptyset$.

The main combinatorial object associated to $\mathcal{A}$ is its *intersection lattice*,

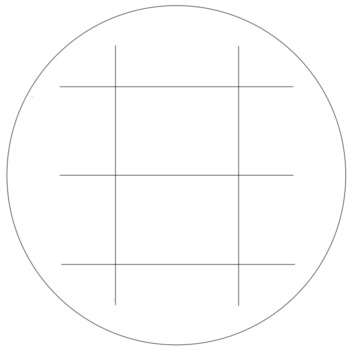
$\mathcal{L}(\mathcal{A}) = \{\emptyset \neq \bigcap_{H \in \mathcal{A}} H \mid \mathcal{B} \subset \mathcal{A}\}$. This is a ranked poset, consisting of all non-empty intersections of $\mathcal{A}$, ordered by reverse inclusion, and with rank function given by codimension. We denote $\mathcal{L}_i(\mathcal{A})$ the set of codimension i elements of $\mathcal{L}(\mathcal{A})$. Then $\mathcal{L}_0(\mathcal{A}) = \{0\} = \{V\}$. When $\mathcal{A}$ is central, then the poset $\mathcal{L}(\mathcal{A})$ is a *geometric lattice*.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central arrangement. A defining polynomial for $\mathcal{A}$ may be written as $Q_{\mathcal{A}} = f_1 \cdots f_n$ where f_i are distinct linear forms, with $H_i = \ker f_i$. Choose coordinates $(z_1, \ldots, z_l)$ in $\mathbb{C}^l$ so that $H_n = \{z_l = 0\}$. The corresponding *decone* of $\mathcal{A}$ is the affine arrangement $\mathbf{d}\mathcal{A}$ in $\mathbb{C}^{l-1}$, with defining polynomial $Q_{\mathbf{d}\mathcal{A}} = Q_{\mathcal{A}}(z_1, \ldots, z_{l-1}, 1)$. Reversing the procedure yields the *cone* $\mathbf{c}\mathcal{A}$ of $\mathcal{A}$.

Example 2.1. Let us consider $Q(\mathcal{A}) = x_1 x_2 (x_1 - 1)(x_2 - 1)(x_2 - 2)$ and $Q(c\mathcal{A}) = x_0 x_1 x_2 (x_1 - x_0)(x_2 - x_0)(x_1 - 2x_0)$.



$\mathcal{A}$



$c(\mathcal{A})$

$(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ be two hyperplane arrangements and let $V = V_1 \oplus V_2$. We define the *product* arrangement $\mathcal{A}_1 \times \mathcal{A}_2$ by

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{H \oplus V_2 \mid H \in \mathcal{A}_1\} \cup \{V_1 \oplus H \mid H \in \mathcal{A}_2\}$$

As an example, let us take the previous one. Let $Q(\mathcal{A}_1) = x(x-1)$, $Q(\mathcal{A}_2) = x(x-1)(x-2)$, then $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, $Q(\mathcal{A}) = xy(x-1)(y-1)(y-2)$ and $Q(c\mathcal{A}) = xyz(x-z)(y-z)(y-2z)$.

Let $M(\mathcal{A}) = V - \bigcup_{H \in \mathcal{A}} H$ be the *complement*. This is an open, l-dimensional complex manifold, whose topological invariants are intimately connected to the combinatorics of the arrangement. However the information encoded in the intersection lattice is not *a priori* enough for finding a finite presentation of the fundamental group of the complement of an arrangement.

Let us point out that $M(c\mathcal{A}) \approx M(\mathcal{A}) \times \mathbb{C}^*$ where $\mathbb{C}^*$ denotes the nonzero complex numbers and $\approx$ denotes homeomorphism.

Let

$$M_1 = V_1 - \bigcup_{H \in \mathcal{A}_1} H, \; M_2 = V_2 - \bigcup_{H \in \mathcal{A}_2} H \text{ and } M = V - \bigcup_{H \in \mathcal{A}_1 \times \mathcal{A}_2} H.$$

Then $\pi_1(M) \simeq \pi_1(M_1) \times \pi_1(M_2)$.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of hyperplanes in V. Let R be an arbitrary commutative ring. Let $E = \bigoplus_{k=1}^{n} E_p$ denote the graded exterior algebra over R generated by 1 and symbols $e_1, \ldots, e_n$. The R-module E_p is free and has a distinguished basis consisting of monomials $e_S = e_{i_1} \cdots e_{i_p}$ where $S = \{i_1, \ldots, i_p\}$ is running through all the subsets of $[n] = \{1, 2, \ldots, n\}$ of cardinality p and $i_1 < i_2 < \cdots < i_p$. Define $\partial : E^p \longrightarrow E^{p-1}$ by

$$\partial(e_1, \ldots, e_n) = \sum_{k=1}^{p} (-1)^{k-1} e_{i_1} \cdots \hat{e}_{i_k} \cdots e_{i_p}$$

For every $S \subset [n]$, put $\cap S = \bigcap_{i \in S} H_i$, and call S *dependent* if $\cap S \neq \emptyset$ and the set of linear polynomials $\{f_i \mid i \in S\}$ is linearly dependent. Let $I = I(\mathcal{A})$ be the ideal of E generated by

$$\{\partial e_S \mid S \subset [n] \text{ is dependent}\}$$

Then I is a homogeneous ideal.

Definition 2.2. *The Orlik-Solomon algebra $A(\mathcal{A})$ is the graded algebra E/I.*

The image of e_i in $A(\mathcal{A})$ is denoted a_i. The generators a_i correspond to logarithmic 1-forms $\mathrm{d}f_i/f_i$ where $f_i : \mathbb{C}^l \longrightarrow \mathbb{C}$ is a linear form with kernel H_i. Let $\mu : \mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathcal{A}) \to \mathbb{Z}$ denote the Möbius function on $\mathcal{L}(\mathcal{A})$. The Poincaré polynomial of $A(\mathcal{A})$ is $P_A(t) = \sum_{k \geq 0} \mathrm{rank}(A_k) t^k$. Then

$$P_A(t) = \sum_{X \in \mathcal{L}(\mathcal{A})} \mu(X)(-t)^{r(X)}$$

where r is the rank function of $\mathcal{L}(\mathcal{A})$ and $\mu(X)$ denotes $\mu(X, 0)$.

Define the rational 1-forms $\eta_i = \frac{1}{2\pi i} \frac{df_i}{f_i}$ on V. Then the integral cohomology ring $H^*(M(\mathcal{A}), \mathbb{Z})$ is generated by 1 and the classes of η_i for $1 \leq i \leq n$.

Theorem 2.3. *Let $R = \mathbb{Z}$. Then $A(\mathcal{A}) \cong H^*(M(\mathcal{A}); \mathbb{Z})$ and the Poincaré polynomial of $H^*\big(M(\mathcal{A}); \mathbb{Z}\big)$ is equal to $P_A(t)$.*

Remark 2.4. Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, then $P_{\mathcal{A}}(t) = P_{\mathcal{A}_1}(t).P_{\mathcal{A}_2}(t)$.

Definition 2.5. *The holonomy Lie algebra of $M(\mathcal{A})$ (over $\mathbb{Z}$) is defined as*

$$\mathcal{H}_M = \mathbf{L}(X_k; 1 \leq k \leq n)/\mathcal{I}$$

where $\mathbf{L}(X_k; 1 \leq k \leq n)$ is the free Lie algebra on the set $\{X_1, \ldots, X_n\}$ and $\mathcal{I}$ is the ideal generated by the elements $[X_i, \sum_{j \in J} X_j]$ for all $i \in J$ such that codim$\bigcap_{j \in J} H_j = 2$ and codim$(\bigcap_{j \in J}) \cap H_k = 3$ for all $k \notin J$.

This theory is intimately connected with the theory of knots and links in 3-spaces with its varied applications to biology, chemistry, and physics. A more direct link to physics is provided by the deep connections between arrangement theory and hypergeometric functions. There are implications in the study of Knizhnik-Zamolodchikov equations in conformal field theory.

Hyperplane arrangements are used in numerous areas, including robotics, graphics, molecular biology, computer vision,....

3. The Magnus Theory

There is a very strong analogy between the theory of *groups* and the theory of *Lie algebras*. The most well-known is the one between Lie groups and Lie algebras; every finite dimensional (complex) Lie algebra is the Lie algebra of some (complex) Lie group.

Another connection is due to Magnus and developed by Lazard and we will consider it in the following.

Let G be an arbitrary group. How *"far"* is it from an abelian group?

Let G^{ab} be the *abelianization* of G, that is $G^{\mathrm{ab}} = G/[G,G]$ where $[G,G]$ is the subgroup of commutators.

If G is abelian, then $[G,G] = 0$ and $G^{\mathrm{ab}} = G$.

If G is *perfect*, then $[G,G] = G$ and $G^{\mathrm{ab}} = 0$.

Therefore, replacing G by its abelianization is too strict.

Let us consider the *Lower Central Series* of G which is denoted by $(\Gamma_n G)_{n\geq 1}$ where:

(i) $\Gamma_1 G = G$
(ii) $\Gamma_{n+1} G = [G, \Gamma_n G]$

Properties:

(i) $\Gamma_{n+1} G$ is a subgroup of $\Gamma_n G$.
(ii) $\Gamma_n G/\Gamma_{n+1} G$ is an abelian group which is finitely generated if G is finitely generated.
(iii) $[\Gamma_m G, \Gamma_n G] \subset \Gamma_{m+n} G$.

Define

$$\mathrm{gr}_n G = \Gamma_n G/\Gamma_{n+1} G$$

which is an abelian group for any $n \geq 1$ and

$$\mathrm{gr} G = \bigoplus_{n\geq 1} \mathrm{gr}_n G$$

There is a natural structure of Lie algebra on $\mathrm{gr}G$ over $\mathbb{Z}$ where the Lie bracket $[x, y]$ is induced from the group commutator $(x, y) = xyx^{-1}y^{-1}$.

Let denote $\phi_n(G) = \mathrm{rank}(\mathrm{gr}_n G)$. They are important numerical invariants of G. Although they may be very difficult to determine, many properties of the group G are reflected into properties of its associated Lie algebra gr$G$.

Then a natural question is, given the group G, to determine the Lie algebra gr$G$ and to compute $\phi_n(G)$ for every n.

4. Some Examples

4.1. Free Groups

Let $G = \mathbf{F}_l$ be a free group of rank l. Magnus showed that

$$\mathrm{gr}(\mathbf{F}_l) = \mathbf{L}_l$$

where $\mathbf{L}_l$ is the free Lie algebra on l generators, whose ranks were computed by Witt.

Let R be a commutative ring with unit and let $R<A>$ be the free associative algebra over the set A of l elements. The product $[x, y] = xy - yx$ turns $R<A>$ into a Lie algebra. The Lie subalgebra $\mathbf{L}_A(R)$ generated by A is called the *free Lie algebra* over A. Notice that $R<A>$ is the universal enveloping algebra of $\mathbf{L}_A(R)$.

When $R = \mathbb{Z}$, we denote $\mathbf{L}_A(R)$ by $\mathbf{L}_A$. So, $\mathbf{L}_l$ is the free Lie algebra on l generators over $\mathbb{Z}$.

Theorem 4.1. [1] *Let $\mathbf{F}_l$ be the free group of rank l. Then*

$$\phi_k(\mathbf{F}_l) = \mathrm{rank}(\mathbf{L}_l)_k = \frac{1}{k}\sum_{d|k}\mu(d)l^{k/d}$$

where $(\mathbf{L}_l)_k$ is the homogeneous component of rank k of the free Lie algebra $\mathbf{L}_l$ and μ is the classical Möbius function.

In fact, we will consider the following equalities coming from the proof of the Witt theorem, which is called *Witt formula* or *LCS formula*:

$$\prod_{k\geq 1}(1-t^k)^{-\phi_k(\mathbf{F}_l)} = \sum_{n\geq 1} l^n t^n = (1-lt)^{-1}.$$

Let us now consider a direct product of free groups $G = \mathbf{F}_{i_1} \times \mathbf{F}_{i_2} \times \ldots \times \mathbf{F}_{i_n}$. Then, we get the following LCS formula for the direct product of free groups:

$$\prod_{k\geq 1}(1-t^k)^{-\phi_k(G)} = \prod_{k\geq 1}(1-t^k)^{-\phi_k(\mathbf{F}_{i_l})} \ldots \prod_{k\geq 1}(1-t^k)^{-\phi_k(\mathbf{F}_{i_n})} = \prod_{j=1}^{n}(1-i_j t)^{-1}$$

4.2. Topological and Geometric Meaning of the Witt Formula

Let $M = \mathbb{C} - \{a_1, \dots, a_l\}$.

The *fundamental group* $\pi_1(M)$ is the free group $\mathbf{F}_l$ of rank l.

The *Differential Equation* $dY = \omega Y$ where $\omega = \sum_{k=1}^{l} A_k \omega^k$, $\omega^k = \frac{dt}{t-a_k}$, and $A_k \in End(\mathbb{C}^m)$, is completely integrable, ($d\omega = 0$ and $\omega \wedge \omega = 0$). Then let be the monodromy representation:

$$\rho : \pi_1(M) \longrightarrow Gl(m; \mathbb{C})$$

which is defined by Chen iterated integrals:

$$\rho(\gamma) = I + \int_\gamma \omega + \int_\gamma \omega\omega + \cdots$$

In order to get a *universal* expression, let $\mathbb{C}\langle\langle X_1, \cdots, X_l \rangle\rangle$ be the I-adic completion of $\mathbb{C}\langle X_1, \cdots, X_l \rangle$ where I is the augmentation ideal.

Define the homomorphism:

$$\begin{aligned} \theta \, : \, \pi_1(M) &\longrightarrow \mathbb{C}\langle\langle X_1, \cdots, X_l \,\rangle\rangle \\ \gamma &\longmapsto 1 + \sum_{\substack{k \geq 1 \\ 1 \leq i_1, \cdots, i_k \leq l}} \int_\gamma \omega^{i_1} \cdots \omega^{i_k} X_{i_1} \cdots X_{i_k} \end{aligned}$$

Then ρ is obtained by substituting A_k to X_k.

Finally, $\mathbb{C}\langle X_1, \cdots, X_l \rangle = \mathcal{U}(\mathbf{L}_l)(\mathbb{C})$ is the universal enveloping algebra of the free Lie algebra $\mathbf{L}_l(\mathbb{C})$.

$\mathbf{L}_l(\mathbb{C})$ is called the *Holonomy Lie algebra* of M and is denoted $\mathcal{H}_M(\mathbb{C})$.

$\mathbb{C}\langle X_1, \cdots, X_l \rangle$ is the enveloping algebra of $\mathcal{H}_M(\mathbb{C})$ and $\sum_{n \geq 0} l^n t^n$ is its *Poincaré series*.

The *integral cohomology ring*

$$H^*(M(\mathcal{A}), \mathbb{Z}) = H^0(M(\mathcal{A}); \mathbb{Z}) \bigoplus H^1(M(\mathcal{A}); \mathbb{Z}) = \mathbb{Z} \bigoplus \mathbb{Z}^l$$

and its *Poincaré polynomial* is $P_M(t) = 1 + lt$. Notice that $M(\mathcal{A}) \cong \bigvee_l S^1$ is a wedge of l circles.

The 3th term of the LCS formula is $\Big(P_M(-t)\Big)^{-1}$.

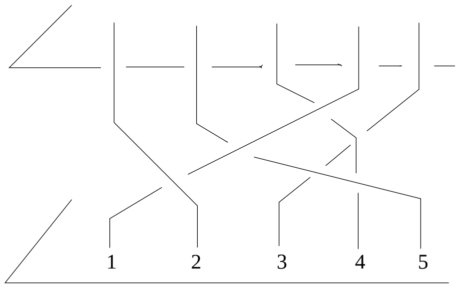
Remark 4.2. Let $\mathcal{A}_j = \mathbb{C} - \{a_{i_1}, \dots, a_{i_j}\}$ and $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. Let $M(\mathcal{A}) = \mathbb{C}^n - \bigcup_{H \in \mathcal{A}} H$. Then $G(\mathcal{A}) \cong \prod_{j=1}^{n} \mathbf{F}_{i_j}$ and using the LCS formula, we obtain the LCS ranks $\phi_k(G(\mathcal{A}))$.

4.3. Braid Groups

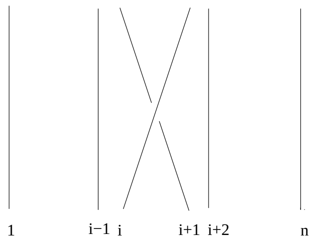
The *Braid group with l strands* denoted B_l admits the following presentation:

(i) generators: $\sigma_1, \cdots, \sigma_{l-1}$
(ii) relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|\, i-j \,|> 1$ and
$\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$ for $1 \leq i \leq l-2$

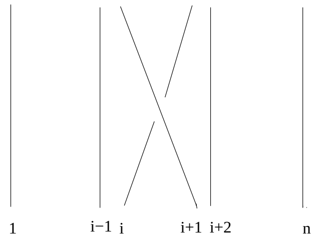
Braids can be viewed as *isotopy* classes of collection of n connected curves in 3-dimensional space.



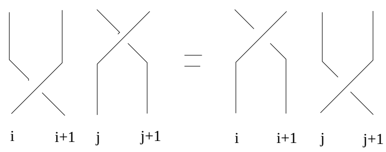
$$\gamma = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ 4\ 1\ 5\ 3\ 2 \end{pmatrix} \in B_5$$



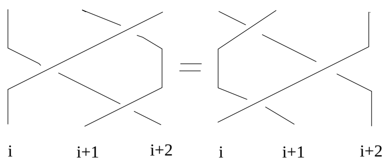
σ_i



σ_i^{-1}



$\sigma_i\sigma_j = \sigma_j\sigma_i$



$\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$

Let $\pi : B_l \longrightarrow \mathrm{S}_l$ be the natural homomorphism where S_l is the symmetric group on l and define $P_l = \ker\pi$ as the *pure braid group.* The symmetric group S_l has the following presentation:

(i) generators: $s_1, \cdots, s_{l-1}$
(ii) relations: $s_is_j = s_js_i$ for $|\, i-j \,|> 1$ and
$s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$ for $1 \leq i \leq l-2$
$s_i^2 = 1$ for any i

and $\pi(\sigma_i) = s_i$ is the transposition $(i, i+1)$.

The group P_l can be realized as the fundamental group of the complement M_l of the diagonal hyperplanes H_{ij} of $\mathbb{C}^l$ defined by $z_i = z_j$ for $1 \leq i < j \leq l$,

$$M_l = \mathbb{C}^l - \bigcup_{1\leq i<j\leq l} H_{ij}, \quad P_l \cong \pi_1(M_l).$$

Moreover $\mathbb{C} - \{p-1 \text{ points}\} \hookrightarrow M_p \longrightarrow M_{p-1}$ is a linear fibration where $M_p \longrightarrow M_{p-1}$ is the restriction of the map $\mathbb{C}^p \longrightarrow \mathbb{C}^{p-1}$ which forgets the last coordinate.

Consequences:

(i) M_l is a $K(\pi, 1)$-space

(ii) $\pi_1(M_l) \cong \mathbf{F}_{l-1} \rtimes \mathbf{F}_{l-2} \rtimes \cdots \rtimes \mathbf{F}_2 \rtimes \mathbf{F}_1$ (iterated semi-direct product of free groups)

(iii) $H^*(M_l) \cong \bigotimes_{k=1}^{l-1} H^1(\bigvee_k S^1)$

(iv) $P_{M_l}(t) = \prod_{k=1}^{l-1}(1+kt)$

(v) Let $dY = \omega Y$ be the differential equation where $\omega = \sum_{1\leq i<j\leq l} A_{ij}\omega^{ij}$, $A_{ij} \in gl(m, \mathbb{C})$ and $\omega^{ij} = d\log(z_i - z_j)$.

Lemma 4.3. *$dY = \omega Y$ is completely integrable if and only if $[A_{ij}, A_{ik} + A_{jk}] = 0$, for i, j, k distinct and $[A_{i_1j_1}, A_{i_2j_2}] = 0$, for i_1, j_1, i_2, j_2 distinct.*

Then as in the case of the free groups, we define the monodromy representation and we get the *holonomy Lie algebra* of M_l as $\mathcal{H}_{M_l} = \mathbf{L}(X_{ij}; 1 \leq i < j \leq l)/\mathcal{J}$ where $\mathbf{L}(X_{ij}; 1 \leq i < j \leq l)$ is the free Lie algebra on the generators X_{ij} and $\mathcal{J}$ is the ideal generated by the *infinitesimal braid relations*:

$$[X_{ij}, X_{ik} + X_{jk}] = 0, \quad i, j, k \quad \text{distinct}$$

$$[X_{i_1j_1}, X_{i_2j_2}] = 0, \ i_1, j_1, i_2, j_2 \quad \text{distinct}$$

Theorem 4.4. [Kohno] $gr(P_l) \bigotimes \mathbb{Q} \ \cong H_{M_l}(\mathbb{Q})$

$$\prod_{k\geq 1}(1-t^k)^{-\phi_k(M_l)} = \sum_{p\geq 0}\chi(p)t^p = \prod_{1\leq k\leq l-1}(1-kt)^{-1}$$

where $\phi_k(M_l) = \text{rank gr}_k(M_l)$ and $\sum \chi(p)t^p$ is the Poincaré series of the universal enveloping algebra of the rational holonomy Lie algebra $H_{M_l}(\mathbb{Q})$.

Remarks:

(i) Both the semi-direct product P_l and the direct product $\Pi_l = \mathbf{F}_{l-1} \times \cdots \times \mathbf{F}_2 \times \mathbf{F}_1$, may be realized as the fundamental groups of the complements of (different) arrangements of hyperplanes. Neither homology nor the lower central series can distinguish between Π_l and P_l.

(ii) T. Kohno [10] showed that the first equality in the theorem 3 remains true for any arrangement of hyperplanes, therefore in the following, the LCS formula will be the equality between the first and the third term.

4.4. Fiber-type Arrangements

This is a natural generalization of the braid arrangements.

Definition 4.5. [5] *$\mathcal{A}$ is a fiber-type arrangement if there exists a sequence of sub-arrangements:*

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_l = \mathcal{A}$$

such that $|\mathcal{A}_1| = 1$, rank$(\mathcal{A}_i) = i$ and $M(\mathcal{A}_{i+1}) \longrightarrow M(\mathcal{A}_i)$ is a locally trivial fibration with fiber $\mathbb{C} - \{|\mathcal{A}_{i+1} - \mathcal{A}_i| \text{ points}\}$.

Let $\{d_{i+1} = |\mathcal{A}_{i+1} - \mathcal{A}_i|,\ 0 \leq i < l-1\}$ where $\mathcal{A}_0 = \emptyset$, be denoted the set of the *exponents* of $\mathcal{A}$ and $P_M(t) = \prod_i (1 + d_i t)$ for all exponents d_i.

Notice that the ideal defining the Orlik-Solomon algebra is generated by quadratic relations.

Theorem 4.6. [5]

(i) *The fiber-type arrangements satisfy the LCS formula:*

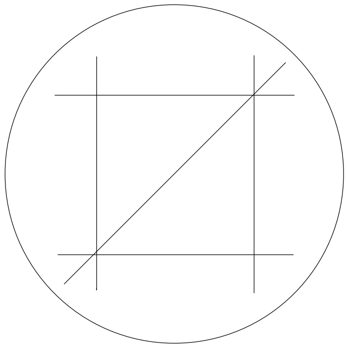
$$\prod_{k \geq 1} (1 - t^k)^{\phi_k(M)} = \prod_{i=1}^{l} (1 - d_i t)$$

(ii) *The fiber-type arrangements are $K(\pi, 1)$ and $\pi_1(M_{d_l}) \cong \mathbf{F}_{d_l} \rtimes \mathbf{F}_{d_{l-1}} \rtimes \cdots \rtimes \mathbf{F}_{d_2} \rtimes \mathbf{F}_{d_1}$.*

Remark 4.7. A fiber-type arrangement is also defined as *supersolvable* [8] if its intersection lattice is supersolvable, in the sense of Stanley. As a consequence, the exponents are combinatorially determined.

Notice that Π_l and P_l may be realized as the fundamental groups of different fiber-type arrangements with same exponents $\{1, 2, \ldots, l-1\}$.

Example 4.8. The braid arrangement associated to the braid group B_4 is fiber-type with exponents $(1, 2, 3)$.



Notice that $c(\mathcal{A}_1 \times \mathcal{A}_2)$ given as example of product arrangement is also fiber-type with exponents $(1,2,3)$.

4.5. Hypersolvable Arrangements

The class of hypersolvable arrangements [6] [7] contains both fiber-type and *generic* arrangements and many others.

An arrangement is called *generic* if and only if it is a cone over a general position arrangement.

Notice that all the fiber-type arrangements are $K(\pi,1)$ which means that the complement is $K(\pi,1)$ and the generic arrangements are never $K(\pi,1)$.
Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central arrangement in the complex vector space V. We also denote $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{P}(V^*)$ its set of defining equations, viewed as points in the dual projective space. Let $\mathcal{B} \subset \mathcal{A}$ be a proper, non-empty sub-arrangement, and set $\overline{\mathcal{B}} = \mathcal{A} - \mathcal{B}$. We say that $(\mathcal{A}, \mathcal{B})$ is a *solvable extension* if the following conditions are satisfied.

(i) No points $a \in \overline{\mathcal{B}}$ sits on a projective line determined by $\alpha, \beta \in \mathcal{B}$.
(ii) For every $a, b \in \overline{\mathcal{B}}$, there exists a point $\alpha \in \mathcal{B}$ on the line passing through a and b. (In the presence of condition (I), this point is uniquely determined, and will be denoted by $f(a,b)$.
(iii) For every distinct points $a, b, c \in \overline{\mathcal{B}}$, the three points $f(a,b), f(a,c)$ and $f(b,c)$ are either equal or collinear.

Note that only two possibilities may occur: either $\text{rank}(\mathcal{A}) = \text{rank}(\mathcal{B}) + 1$ (*fibered case*), or $\text{rank}(\mathcal{A}) = \text{rank}(\mathcal{B})$ (*singular case*).

Definition 4.9. [6, 7] *The arrangement $\mathcal{A}$ is called hypersolvable if it has a hypersolvable composition series, i.e., an ascending chain of sub-arrangements, $\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_i \subset \mathcal{A}_{i+1} \subset \cdots \subset \mathcal{A}_l = \mathcal{A}$, where rank $\mathcal{A}_1 = 1$, and each extension $(\mathcal{A}_{i+1}, \mathcal{A}_i)$ is solvable.*

The *quadratic* Orlik-Solomon algebra is defined by

$$\overline{A}^*(\mathcal{A}) = \bigwedge{}^*(e_1, \ldots, e_n)/J$$

where J is the homogeneous ideal generated by

$$(\mathcal{R}_{\mathcal{A}}) \qquad e_{i_1} \wedge e_{i_2} - e_{i_1} \wedge e_{i_3} + e_{i_2} \wedge e_{i_3}$$

with $\operatorname{codim}(H_{i_1} \cap H_{i_2} \cap H_{i_3}) = 2$.

The *quadratic* Poincaré polynomial $\overline{P}_M(t)$ is the Poincaré polynomial of $\overline{A}^*(\mathcal{A})$.

Theorem 4.10. [6, 7] *Let $\mathcal{A}$ be a hypersolvable arrangement, with composition series $\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_l = \mathcal{A}$. Then*

(i) $\overline{P}_M(t) = \prod_{i=1}^{l}(1 + d_i t)$ *where* $d_i = |\mathcal{A}_i - \mathcal{A}_{i+1}|$.

(ii) $\mathcal{A}$ *is fiber-type if and only if* $l = \operatorname{rank}(\mathcal{A})$ *if and only if* $\overline{P}_M(t) = P_M(t)$.

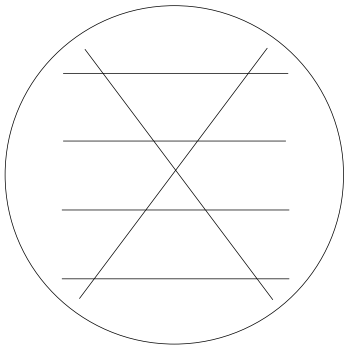
(iii) $gr^*\big(G(\mathcal{A})\big) \cong \mathcal{H}_M$

(iv) $\prod_{k\geq 1}(1 - t^k)^{\phi_k(M)} = \overline{P}_M(-t)$ *(called the generalized LCS formula).*

(v) $\pi_1(M(\mathcal{A}))$ *is an iterated almost-direct product of free groups,* $\pi_1(M_{d_l}) \cong \mathbf{F}_{d_{l-1}} \rtimes \mathbf{F}_{d_{l-2}} \rtimes \cdots \rtimes \mathbf{F}_{d_2} \rtimes \mathbf{F}_{d_1}$

Remark 4.11. The fundamental group of a hypersolvable arrangement is combinatorial although it is known that it is not true for an arbitrary arrangement.

Example 4.12. This arrangement factors, $P_M(t) = (1 + t)(1 + 3t)^2$, is neither fiber-type nor generic but is hypersolvable with $\overline{P}_M(t) = (1 + t)^3(1 + 4t)$.



5. Chen Groups

The *Chen groups* of a group G are the lower central series quotients of G modulo its second commutator subgroup G''. Recall that $G^{(i+1)} = [G^{(i)}, G^{(i)}]$, then $G' = [G, G]$ and $G'' = [\Gamma_1 G, \Gamma_1 G]$.

The group G/G'' is metabelian and finitely-generated if G is finitely-generated. It fits into the exact sequence:

$$0 \longrightarrow G'/G'' \longrightarrow G/G'' \longrightarrow G/G' \longrightarrow 0$$

The k**-th Chen** group of G is, by definition, $\mathrm{gr}_k(G/G'')$. Let $\theta_k(G) = \phi_k(G/G'')$ be its rank. The projection $G \longrightarrow G/G''$ induces surjections $\mathrm{gr}_k G \longrightarrow \mathrm{gr}_k(G/G'')$. Thus $\phi_k \geq \theta_k$ for all k and $\phi_k = \theta_k$ for $k \leq 3$.

Assume $G/G' \cong \mathbb{Z}^n$. The Chen groups of G can be determined from the *Alexander invariant* $B := G'/G''$ (viewed as a module over $\mathbb{Z}[G/G']$)
Then $\mathrm{gr}_k(G/G'') = \mathrm{gr}_{k-2}B$, for $k \geq 2$ and we have

$$\sum_{k\geq 0} \theta_{k+2} t^k = \mathrm{Hilb}(\mathrm{gr}B)$$

where $\mathrm{gr}B = \bigoplus_{k\geq 0} \mathrm{gr}_k B$ (viewed as a module over $\mathrm{gr}\mathbb{Z}[G/G']$)and $\mathrm{Hilb}(\mathrm{gr}B)$ is the Hilbert series of the graded module $\mathrm{gr}B$. A presentation for $\mathrm{gr}B$ can be obtained from a presentation for B via the well-known Gröbner basis algorithm for finding the tangent cone to a variety.

Theorem 5.1. [11, 2] *Let* $G = \mathbf{F}_l$. *Then*

$$\theta_k(\mathbf{F}_l) = (k-1).\binom{l+k-2}{k}, \quad k \geq 2$$

Theorem 5.2. [3]

(i) *Let* $G = \pi_1(M_1 \times M_2)$, $\pi_1(M_1) = G_1$, $\pi_1(M_2) = G_2$, *then*

$$\theta_k(G) = \theta_k(G_1) + \theta_k(G_2)$$

(ii) *Let* $G = \mathbf{F}_{d_1} \times \cdots \times \mathbf{F}_{d_l}$ *be a direct product of free groups, then the Chen groups of* G *are free abelian and*

$$\theta_1(G) = \sum_{i=1}^{l} d_i,$$
$$\theta_k(G) = (k-1)\sum_{i=1}^{l} \binom{k+d_i-2}{k} \quad \textit{for } k \geq 2.$$

In particular, let $\Pi_l = \mathbf{F}_{l-1} \times \cdots \times \mathbf{F}_1$ *then*

$$\theta_1(\Pi_l) = \binom{l}{2},$$
$$\theta_k(\Pi_l) = (k-1)\binom{k+l-2}{k+1} \quad \textit{for } k \geq 2$$

(iii) *The Chen groups of the pure braid group P_l are free abelian. The rank θ_k are given by*

$$\theta_1 = \binom{l}{2}, \qquad \theta_2 = \binom{l}{3}$$

and

$$\theta_k = (k-1).\binom{l+1}{4} \text{ for } k \geq 3$$

Theorem 5.3. *For $l \geq 4$, the groups P_k/P_k'' and Π_k/Π_k'' are not isomorphic. For $l \geq 4$, the groups P_l and Π_l are not isomorphic.*

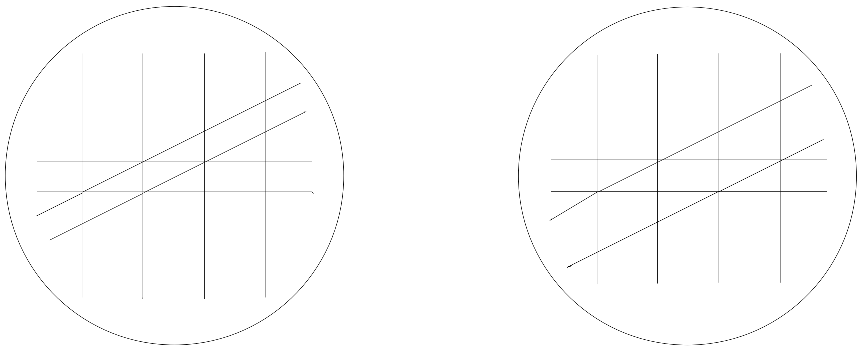
Remark 5.4. $P_2 \cong \mathbf{F}_1$ and $P_3 \cong \mathbf{F}_2 \times \mathbf{F}_1$.

Example 5.5. Let be the two arrangements $\mathcal{A}$ and $\mathcal{B}$ defined by:

$$Q(\mathcal{A}) = xyz(x-y)(x-z)(x-2z)(x-3z)(y-z)(x-y-z)$$

and

$$Q(\mathcal{B}) = xyz(x-y)(x-z)(x-2z)(x-3z)(y-z)(x-y-2z)$$



$\mathcal{A}$ $\qquad$ $\mathcal{B}$

They are fiber-type with the same exponents $(1,4,4)$. The sequences are:

$$\{x=0\} \subset \{x=0, z=0, x-z=0, x-2z=0, x-3z=0\} \subset \mathcal{A}$$

$$\{x=0\} \subset \{x=0, z=0, x-z=0, x-2z=0, x-3z=0\} \subset \mathcal{B}$$

Therefore homology groups and lower central series quotients are isomorphic.

However,

$$\theta_1\big(G(\mathcal{A})\big) = \theta_1\big(G(\mathcal{B})\big) = 9$$
$$\theta_2\big(G(\mathcal{A})\big) = \theta_2\big(G(\mathcal{B})\big) = 12$$
$$\theta_3\big(G(\mathcal{A})\big) = \theta_3\big(G(\mathcal{B})\big) = 40$$

and for $k \geq 4$

$$\theta_k\big(G(\mathcal{A})\big) = \frac{1}{2}(k-1)(k^2+3k+24)$$

$$\theta_k\big(G(\mathcal{B})\big) = \frac{1}{2}(k-1)(k^2+3k+22)$$

then

$$G(\mathcal{A}) \ncong G(\mathcal{B})$$

Notice that the groups $G(\mathcal{A})$ and $G(\mathcal{B})$ cannot be distinguished by means of the LCS formula.

Theorem 5.6. [13] *For any complex arrangement $\mathcal{A}$,*

$$\mathrm{gr}\big(G(\mathcal{A})/G''(\mathcal{A})\big) \otimes \mathbb{Q} \cong (\mathcal{H}_M/\mathcal{H}''_M) \otimes \mathbb{Q}$$

In particular, the rational holonomy Lie algebra of the arrangement is combinatorially determined by the level 2 of the intersection lattice, $\mathcal{L}_2(\mathcal{A})$. Hence, the Chen ranks are combinatorially determined.

There is a conjecture (*Suciu*) [16] which makes this combinatorial dependence explicit.

This is related to the cohomology of the Orlik-Solomon algebra which figures in the Aomoto-Gelfand theory of generalized hypergeometric functions, and in solutions of the Knizhnik-Zamolodchikov equations of conformal field theory. Let consider the cohomology $H^*\big(A(\mathcal{A}), d_\omega\big)$, where d_ω is the degree one mapping defined by left multiplication by a fixed element $\omega \in A^1(\mathcal{A})$. The cohomology $H^*\big(A(\mathcal{A}), d_\omega\big)$ is isomorphic to the cohomology of the complement $M(\mathcal{A})$ with coefficients in a local system determined by ω, when ω satisfies some *non-resonance* conditions dependent only on $M(\mathcal{A})$. Then Falk [4] showed that $H^1\big(A(\mathcal{A}), d_\omega\big) \neq 0$ precisely when ω belongs to an affine variety called the *resonance variety* $R_1(\mathcal{A})$ of the arrangement $\mathcal{A}$. He showed that $R_1(\mathcal{A})$ is a linear subspace of $\mathbf{C}^n$ which is a union of subspaces of dimension at least 2, as follows.

A partition $\mathsf{P} = (p_1 \mid \cdots \mid p_q)$ of $\mathcal{A}$ is called *neighborly* if

$$p_j \cap I \geq \mid I \mid -1 \Rightarrow I \subset p_j \text{ for all } I \in \mathcal{L}_2(\mathcal{A})$$

To a neighborly partition corresponds an irreducible subvariety of $R_1(\mathcal{A})$

$$L_{\mathsf{P}} = \Delta_n \cap \bigcap_{\{I \in \mathcal{L}_2(\mathcal{A}) \mid I \not\subset p_j, \text{ any } j\}} \{\lambda \mid \sum_{i \in I} \lambda_i = 0\}$$

where $\Delta_n = \{\lambda \in \mathbb{C}^n \mid \mathcal{A})$ arise from neighborly partition of a sub-arrangement of $\mathcal{A}$.

Then $R_1(\mathcal{A}) = \bigcup L_i$ and for any $r \geq 0$ let $h_r = \mid \{L_i \mid \dim L_i = r\} \mid$ be the number of components of $R_1(\mathcal{A})$ of dimension r.
h_r can be computed directly from the lattice $\mathcal{L}(\mathcal{A})$ by computing neighborly partitions of sub-arrangements of $\mathcal{A}$ and finding $\dim L_{\mathsf{P}}$.

This is the conjecture (*Suciu*) [16]:

$$\theta_k\big(G(\mathcal{A})\big) = \sum_{r \geq 2} h_r \theta_k(\mathbf{F}_r), \text{ for } k \geq 4$$

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