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A Note on Hilbert Algebras

Aldo Figallo Jr.[1] and Alicia Ziliani[2]

[1]Instituto de Ciencias Básicas. Universidad de San Juan, 5400 San Juan, Argentina
E-mail: aldofigallo@gmail.com

[2]Departamento de Matemática, Universidad Nacional del Sur, 8000 Bahía Blanca, Argentina
E-mail: aziliani@criba.edu.ar

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Abstract. In this note, a new proof of the fact that the variety of Hilbert algebras is a congruence-distributive variety using a result from I. Chajda and E. Horváth [Acta Sci. Math. (Szeged) 68 (2002), 29–35] is determined. In addition, the relationship between implicative semilattices, (H)-Hilbert algebras and Hilbert algebras with infimum is described. Finally, a representation theorem for (H)-Hilbert algebras is established which is extended to a duality for finite (H)-Hilbert algebras.

Keywords: Hilbert algebras; (H)-Hilbert algebras; Hilbert algebras with infimum; Congruence distributivity; Duality.

1. Preliminaries

A. Monteiro in [17] (see also [18]) called Hilbert algebra a triple $\langle A, \to, 1\rangle$ where A is a nonempty set, $\to$ is a binary operation on A, 1 is an element of A such that the following conditions are satisfied for every $p, q, r \in A$:

(M1) $p \to (q \to p) = 1$,

(M2) $(p \to (q \to r)) \to ((p \to q) \to (q \to r)) = 1$,

(M3) $p \to 1 = 1$,

(M4) $p \to q = 1$ and $q \to p = 1$ imply $p = q$.

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In 1960, L. Iturrioz proved that (M3) follows from (M1) and (M4) and that (M1), (M2) and (M4) are independent. Besides, that same year A. Diego, answering a problem posed by A. Monteiro, obtained an equational definition of these algebras.

It is worth mentioning that A. Diego was one of the authors who made a fundamental contribution to the development of the theory of these algebras and his conclusions may be consulted in [10] (see also [9]). Later on, many articles have been published about this class of algebras, some of them can be consulted in [2], [3], [6], [11], [18] and [20].

In what follows, we shall denote by **H** the variety of Hilbert algebras.

The result we shall indicate in (H1) is widely used in the theory of Hilbert algebras and according to what is noted in [17], it was L. Henkin who established it in [13].

(H1) For every $A \in \mathbf{H}$ the relation $\leq$ defined by the prescription $p \leq q$ if and only if $p \to q = 1$ is a partial order on A; with respect to this ordering 1 is the last element of A.

Most of the results established in [17] were in fact never published, but some of them were obtained independently by other authors many years later. To the best of our knowledge, A. Monteiro was the first one to prove, among others, the following properties which are necessary for our paper:

(H2) Let $A \in \mathbf{H}$ and $Con_{\mathbf{H}}(A)$ be the set of $\mathbf{H}-$congruences of A. Then $Con_{\mathbf{H}}(A) = \{R(D) \colon D \in \mathcal{D}(A)\}$, where $R(D) = \{(x, y) \in A \times A : x \to y, y \to x \in D\}$ and $\mathcal{D}(A)$ is the family of all deductive systems of A, (i.e. the subsets D of A such that $1 \in D$ and if $x, x \to y \in D$ then $y \in D$).

(H3) Let $A \in \mathbf{H}$ and $X \subseteq A$. Then the deductive system generated by X, denoted by $[X]$, is

(i) $\{1\}$, if $X = \emptyset$,

(ii) $\{b \in A \colon$ there are $x_1, \ldots, x_n \in X$ such that $x_1 \to (x_2 \to (\ldots \to (x_n \to b) \ldots) = 1\}$, if $X \neq \emptyset$.

Furthermore, the deductive system generated by one element $a \in A$ called a principal deductive system is denoted by $[a)$, and it is easy to verify that $[a) = \{x \in A : a \leq x\}$.

Besides, A. Monteiro introduced, among many others, the following notion:

(H4) A proper deductive system D of a Hilbert algebra A is irreducible if for any D_1, $D_2 \in \mathcal{D}(A)$ such that $D = D_1 \cap D_2$, then $D = D_1$ or $D = D_2$.

Later on, in 1999 I. Chajda and H. Halaš [6] introduced the notion of ideal on Hilbert algebras as follows:

(H5) A nonempty subset J of a Hilbert algebra A is an ideal if

(I1) $1 \in J$,

(I2) $a \in A$ and $b \in J$ imply $a \to b \in J$,

(I3) $a \in A$ and $u, v \in J$ imply $(u \to (v \to a)) \to a \in J$.

and these authors also proved

(H6) $Con_{\boldsymbol{H}}(A) = \{T(J) \colon J \in \mathcal{J}(A)\}$, where $T(J) = \{(x, y) \in A \times A : x \to y, y \to x \in J\}$ and $\mathcal{J}(A)$ is the family of all ideals of A.

That same year, W. Dudek proved that in Hilbert algebras the notions of ideals and deductive systems coincide ([11, Theorem 1]). Besides, the characterization of $\mathbf{H}-$congruences obtained in [11, Theorem 2] is another proof of the results indicated in (H2).

On the other hand, D. Busneag [2] proved that if $A \in \mathbf{H}$ and $D_1, D_2 \in \mathcal{D}(A)$ then,

(H7) $D_1 \vee D_2 = \{x \in A \colon$ there are $x_1, \ldots, x_n \in D_1$ such that $(x_1, \ldots, x_n; x) \in D_2\}$,

where $(x_1, \ldots, x_{n-1}; x_n) = \begin{cases} x_n & \text{if } n = 1 \\ x_1 \to (x_2, \ldots, x_{n-1}; x_n) & \text{if } n > 1 \end{cases}$.

(H8) the relative pseudocomplement of D_1 with respect to D_2 is

$D_1 \Rightarrow D_2 = \{x \in A : [x) \cap D_1 \subseteq D_2\}$.

2. Distributivity of Congruences in Hilbert Algebras

The fact that the variety of Hilbert algebras is a congruence distributive variety has been shown, in some way since 1961 because by (H2) the congruences are determined by the deductive systems and A. Diego [9] proved

(H9) Let $A \in \mathbf{H}$. Then the ordered set $(\mathcal{D}(A), \subseteq)$ is a complete distributive lattice where for any D, $D' \in \mathcal{D}(A)$ the infimum and the supremum of $\{D, D'\}$ is $D \cap D'$ and $[D \cup D']$ respectively. More precisely, in $\langle \mathcal{D}(A), \cap, \vee \rangle$, the infinite distributive law $D \cap \bigvee_{i \in I} D_i = \bigvee_{i \in I} (D \cap D_i)$ holds true.

On the other hand, W. Blok and D. Pigozzi in [1] determined the ternary term $p(x,y,z) = (x \to y) \to ((y \to x) \to z)$ for Hilbert algebras, from which it follows that **H** has equationally definable principal congruences and therefore, it is a congruence-distributive variety.

Besides, I. Chajda in [5] proved by a different procedure as the one indicated in (H9) that the lattice of deductive systems of a Hilbert algebra is distributive.

Next, we shall indicate a different proof that the lattice $Con_{\mathbf{H}}(A)$ is distributive, in which we shall use the notion of triangular scheme introduced by I. Chajda and E. Horváth in [7].

In what follows, n is a positive integer and for any congruence δ we shall denote $\delta^1 = \delta$ and $\delta^{n+1} = \delta^n \circ \delta$, where $\circ$ means the composition of relations.

In [7] the following statements were established:

(ChH1) *Weak Triangular Principle.* An algebra $\mathcal{A} = \langle A, F\rangle$ satisfies the Weak Triangular Principle for n if for any $x,y,z \in \mathcal{A}$ and every $\alpha,\beta,\gamma \in Con\,\mathcal{A}$ with $\alpha \cap \beta \subseteq \gamma$ and $\Lambda_n = (\gamma \circ \alpha \circ \gamma)^n$, the following implication holds:

(P_n) $(x,z) \in \alpha$, $(z,y) \in \beta$, $(x,y) \in \Lambda_n$ imply $(z,y) \in \gamma$.

In addition, $\mathcal{A}$ satisfies the Weak Triangular Principle if (P_n) holds true for all n.

(ChH2) An algebra $\mathcal{A}$ satisfies the Weak Triangular Principle if and only if $Con\,\mathcal{A}$ is distributive.

Theorem 2.1 will allow us to conclude that **H** is congruence-distributive.

Theorem 2.1. *Let $A \in \mathbf{H}$. Then A satisfies the Weak Triangular Principle.*

Proof. Let $\alpha,\beta,\gamma \in Con(A)$, and

(1) $\alpha \cap \beta \subseteq \gamma$, (2) $\Lambda_n = (\gamma \circ \alpha \circ \gamma)^n$.

Suppose

(3) $(x,z) \in \alpha$, (4) $(z,y) \in \beta$, (5) $(x,y) \in \Lambda_n$.

Then, taking into account the definition of supremum between equivalence relations, we have

(6) $\alpha \circ \Lambda_n \subseteq \alpha \vee \gamma$,

and therefore,

(7) $(z, y) \in \alpha \vee \gamma$. [(3), (5), (6)]

On the other hand, from (H2) there are $D_\alpha, D_\beta, D_\gamma \in \mathcal{D}(A)$ such that

(8) $\alpha = R(D_\alpha)$, (9) $\beta = R(D_\beta)$, (10) $\gamma = R(D_\gamma)$.

Then,

(11) $(z, y) \in R(D_\alpha \vee D_\gamma)$, [(7), (8), (10)]

(12) $z \to y,\ y \to z \in D_\alpha \vee D_\gamma$, [(11), (H2)]

(13) there are $a_1, \ldots, a_l \in D_\gamma$ such that $(a_1, \ldots, a_l; z \to y) \in D_\alpha$, [(12), (H7)]

(14) there are $b_1, \ldots, b_t \in D_\gamma$ such that $(b_1, \ldots, b_t; y \to z) \in D_\alpha$. [(12), (H7)]

Besides,

(15) $z \to y,\ y \to z \in D_\beta$, [(4), (9), (H2)]

(16) $(a_1, \ldots, a_l; z \to y) \in D_\alpha \cap D_\beta$, [(13), (15)]

(17) $(b_1, \ldots, b_t; y \to z) \in D_\alpha \cap D_\beta$, [(14), (15)]

(18) $(a_1, \ldots, a_l; z \to y) \in (D_\alpha \cap D_\beta) \vee D_\gamma$, [(13), (16)]

(19) $(b_1, \ldots, b_t; y \to z) \in (D_\alpha \cap D_\beta) \vee D_\gamma$. [(14), (17)]

Since

(20) $a_1, \ldots, a_l,\ b_1, \ldots, b_t \in (D_\alpha \cap D_\beta) \vee D_\gamma$, [(13), (14)]

we deduce that

(21) $z \to y,\ y \to z \in (D_\alpha \cap D_\beta) \vee D_\gamma$, [(18), (19), (20), (H2)]

(22) $(z, y) \in (\alpha \cap \beta) \vee \gamma$, [(21), (H2)]

(23) $(z, y) \in \gamma$. [(22), (1)]

Then, A verifies (P_n) for all n, ending the proof. ■

3. On (H)-Hilbert Algebras

In this section, we shall firstly indicate the relationship between implicative semilattices, Hilbert algebras with infimum and (H)-Hilbert algebras. Next, we shall determine a representation theorem for (H)-Hilbert algebras which can be extended to a duality for finite algebras.

Recall that an implicative semilattice is an algebra $\langle A, \to, \wedge, 1\rangle$ of type $(2, 2, 0)$ such that the reduct $\langle A, \wedge, 1\rangle$ is a meet–semilattice with last element 1, which satisfies the following property:

(B) $x \wedge y \leq z \Leftrightarrow x \leq y \to z$. (see [19])

This notion was introduced by H.B. Curry in [8, page 66] as implicative logical group whereas A. Monteiro in [16] called them implicative systems. For simplicity, in what follows, we call them $IS-$algebras.

It was determined that the class of $IS-$algebras form a variety in [16]. More precisely, it was proved that an $IS-$algebra is an algebra $\langle A, \to, \wedge, 1\rangle$ of type $(2, 2, 0)$ which satisfies the following identities:

(He1) $x \to x = 1$,

(He2) $(x \to y) \wedge y = y$,

(He3) $x \wedge (x \to y) = x \wedge y$,

(He4) $x \to (y \wedge z) = (x \to z) \wedge (x \to y)$.

It is a well-known result that the reduct $\langle A, \to, 1\rangle$ of an $IS-$algebra is a Hilbert algebra and since the reduct $\langle A, \wedge, 1\rangle$ is a meet–semilattice with last element 1, we conclude that $IS-$algebras are Hilbert algebras with infimum.

In [12], the authors realized that Hilbert algebras with infimum are not the same as $IS-$algebras. Then they introduced the notion of $iH-$algebras as algebras $\langle A, \to, \wedge, 1\rangle$ of type $(2, 2, 0)$ such that the reduct $\langle A, \to, 1\rangle$ is a Hilbert algebra and the following identities are fulfilled:

(iH1) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,

(iH2) $x \wedge x = x$,

(iH3) $x \wedge (x \to y) = x \wedge y$,

(iH4) $(x \to (y \wedge z)) \to ((x \to z) \wedge (x \to y)) = 1$,

and they proved that the notions of $iH-$algebras and Hilbert algebras with infimum are equivalent.

On the other hand, (H)-Hilbert algebras are introduced in [14] as Hilbert algebras A which satisfy the following condition:

(H) the set $A(a,b) = \{x \in A : a \to (b \to x) = 1\}$ has least element for all $a, b \in A$, and this element is denoted by $a + b$.

M. Kondo in [15] proved that $IS-$algebras are the same as (H)-Hilbert algebras where $a+b$ is the infimum of $\{a,b\}$. It is noteworthy to remark that the result is correct although the proof is incomplete. In fact, this author proved that (H)-Hilbert algebras are Hilbert algebras with infimum. Then, taking into account [16] to complete the proof it only remains to prove (B). Indeed, from $x \wedge y \leq z$, it follows $x \to (y \to (x \wedge y)) \leq x \to (y \to z)$. Since in every (H)-Hilbert algebra $x \to (y \to (x \wedge y)) = 1$ we have that $x \to (y \to z) = 1$. The converse is immediate by (iH3).

On the other hand, in [12, Lemma 2.2], it was proved that $iH-$algebras which satisfy the additional identity

(iH5) $x \to (y \to (x \wedge y)) = 1$,

are $IS-$algebras. The proof is based on the above mentioned Kondo's result, although the same proof can be obtained without it.

It is worth mentioning that the converse of this statement also holds. That is, $IS-$algebras coincide with $iH-$algebras which verify (iH5).

Next, we shall describe a representation theorem for (H)-Hilbert algebras and Hilbert algebras with infimum.

In [10] it was proved that Hilbert algebras can be represented by a subalgebra of the Hilbert algebra of all open subsets of a topological space following an analogous reasoning to the one used by M. Stone [21] for Heyting algebras. Later on, S. Celani in [4] obtained a new representation theorem for these algebras by means of posets.

Let $(X, \leq)$ be a poset. We shall denote by $\mathcal{S}_c(X)$ the set of all increasing subsets of X, where $Y \subseteq X$ is increasing if $Y = \{x \in X : y \leq x \text{ for some } y \in Y\}$.

Now, we shall prove that Hilbert algebras described in [4] are (H)-Hilbert algebras. More precisely, we have the following proposition.

Proposition 3.1. *Let $(X, \leq)$ be a poset. Then $\langle \mathcal{S}_c(X), \Rightarrow, X \rangle$ is an (H)-Hilbert algebra where the implication operation is defined as in (H8).*

Proof. From [4] it only remains to prove condition (H). If U, $V \in \mathcal{S}_c(X)$, then $U \cap V \in \mathcal{S}_c(X)$. Besides, for all $W \in \mathcal{S}_c(X)$ (1) $U \Rightarrow (V \cap W) = (U \Rightarrow V) \cap (U \Rightarrow W)$ holds true. Indeed, let $p \in (U \Rightarrow V) \cap (U \Rightarrow W)$, then by (H8) we have $[p) \cap U \subseteq V$ and $[p) \cap U \subseteq W$. Therefore, $[p) \cap U \subseteq V \cap W$ and so we conclude that $p \in U \Rightarrow (V \cap W)$. The other inclusion follows immediately. On the other hand, by (1) and properties of Hilbert algebras, we have $U \Rightarrow (V \Rightarrow (U \cap V)) = X$. Therefore, $U \cap V \in \mathcal{S}_c(X)(U, V)$. Furthermore, if $W \in \mathcal{S}_c(X)(U, V)$ then $X = (U \cap V) \Rightarrow (U \Rightarrow (V \Rightarrow W)) = ((U \cap V) \Rightarrow U) \Rightarrow ((U \cap V) \Rightarrow (V \Rightarrow W)) = ((U \cap V) \Rightarrow V) \Rightarrow ((U \cap V) \Rightarrow W) = (U \cap V) \Rightarrow W$. Hence, $U \cap V$ is the least element of $\mathcal{S}_c(X)(U, V)$. ∎

Proposition 3.1 and [4] allow us to obtained a representation theorem for (H)-Hilbert algebras.

Theorem 3.2. *If A is an (H)-Hilbert algebra, then A is isomorphic to a subalgebra of $\mathcal{S}_c(X(A))$ where $X(A)$ is the set of all irreducible deductive systems of A.*

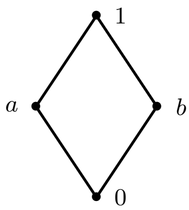
Proof. From [4] the application $\beta : A \longrightarrow \mathcal{S}_c(X(A))$ definided by the prescription $\beta(a) = \{P \in X(A) : a \in P\}$ is the desired immersion. ∎

We remark that Theorem 3.2 is also valid for Hilbert algebras with infimum.

Theorem 3.3. *If A is a finite (H)-Hilbert algebra, then $A \simeq \mathcal{S}_c(X(A))$.*

Proof. The proof is a consequence of the fact that each finite (H)-Hilbert algebra can turn into a Heyting algebra. ∎

Remark 3.4. Theorem 3.3 is not true for finite Hilbert algebras with infimum. For that, it is enough to consider the following algebra A:



$\to$	0	a	b	1
0	1	1	1	1
a	0	1	b	1
b	0	a	1	1
1	0	a	b	1

Then the irreducible deductive systems of A are $D_1 = \{a, 1\}$, $D_2 = \{b, 1\}$ and $D_3 = \{a, b, 1\}$. Therefore, $\mathcal{S}_c(X(A)) = \{\emptyset, \{D_3\}, \{D_1, D_3\}, \{D_2, D_3\}, X(A)\}$ and the aplication β of Theorem 3.2 is defined by

$x \in A$	0	a	b	1
$\beta(x)$	$\emptyset$	$\{D_1, D_3\}$	$\{D_2, D_3\}$	$X(A)$

In the sequel, Theorem 3.2 allows us to obtain a duality for finite (H)-Hilbert algebras as follows:

Let $\mathcal{HH}$ be the category of (H)-Hilbert algebras and their corresponding homomorphism and $\mathcal{P}$ be the category whose objects are posets and whose morphisms are mappings $f : X \longrightarrow Y$ with $X, Y \in \mathcal{P}$ satisfying the following conditions:

(P1) f is increasing,

(P2) for all $(z, t) \in X \times Y$ such that $f(z) \leq t$, there is $d \in X$ such that $z \leq d$ and $f(d) = t$.

For each $X \in \mathcal{P}$ we define $\varphi(X) = \mathcal{S}_c(X)$ and by Proposition 3.1 we have that $\varphi(X) \in \mathcal{HH}$. Let X_1, $X_2 \in \mathcal{P}$. If $f : X_1 \longrightarrow X_2$ is a $\mathcal{P}$-morphism and we define $\varphi(f) : \mathcal{S}_c(X_2) \longrightarrow \mathcal{S}_c(X_1)$ by the prescription $\varphi(f)(U) = f^{-1}(U)$, for all $U \in \mathcal{S}_c(X_2)$. It is simple to verify that φ is a contravariant functor between the categories $\mathcal{P}$ and $\mathcal{HH}$. Besides, the restriction φ_f of φ to the full subcategory $\mathcal{P}_f$ of finite posets is a duality (coequivalence) between $\mathcal{P}_f$ and category $\mathcal{HH}_f$ of finite (H)-Hilbert algebras because φ_f is representative, full and faithful.

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