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Some Combinatorial Properties of the Alternating Group

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Abstract. In this note we obtain and discuss formulae for the number of even permutations (of an n-element set) having exactly k fixed points. Moreover, we obtain generating functions for these numbers. We also obtain similar results for the number of odd permutations.

Keywords: Permutation; Derangement; Even (odd) permutation; Partial one-one transformation; Exponential generating function.

1. Introduction and Preliminaries

Let $X_n = \{1, 2, \ldots n\}$ be a finite n-element set, and let S_n and A_n be the symmetric and alternating groups of X_n, respectively. Another closely related algebraic structure to S_n and A_n is I_n, the semigroup of partial one-one transformations of X_n. This semigroup is also known as the finite symmetric inverse semigroup of X_n. This paper investigates certain combinatorial properties of A_n.

Combinatorial properties of S_n have been studied over a long period and many interesting and delightful results have emerged (see, for example [1, 3, 4, 5, 12]). In particular, the number of permutations (of X_n) having exactly k fixed points and their generating functions are known [12]. Recently, inspired by the works of Gomes and Howie[8], Laradji and Umar [9] obtained some corresponding results in the semigroup I_n. However, the number of even (odd) permutations

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(of X_n) having exactly k fixed points and their generating functions do not seem to have been studied. The only exception is the number of even derangements (permutations without fixed points) which we found in [12] recorded as sequence number $A003221$, see also [11]. The number of even derangements could also be easily deduced from [2, Corollary 2.7]. At the end of this introductory section we gather some known combinatorial results that we shall need in later sections. In Section 2 we establish certain combinatorial results for A_n, the main result being proposition 2.2 which gives recurrence formulae for e_n, the number of even derangements α (of X_n), having observed that the number of permutations with exactly k fixed points can be deduced from the number of derangements. In Section 3 we obtain exponential generating functions for the number of even permutations with exactly k fixed points, and deduce that for the number of odd permutations with exactly k fixed points.

Recall from [6] that an *even* permutation is a permutation which can be expressed as a product of an even number of cycles of even length and/or a product of any number of cycles of odd length. A permutation that is not even is called *odd*. The set of even permutations of X_n, called the *alternating group* is usually denoted by A_n.

Recall also that, a *derangement* σ is a permutation such that $\sigma(x) \neq x$, that is, a permutation without fixed points. The number of derangements of X_n is usually denoted by d_n, while the number of permutations having exactly k fixed points will be denoted by $d(n,k)$. We list some known combinatorial results which may be found in [1, 3, 12], that we shall need later.

Result 1.1 Let d_n be as defined above. Then

$$d_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!} = (n-1)(d_{n-1}+d_{n-2}) = nd_{n-1}+(-1)^n,$$

where $d_0 = 1$.

Result 1.2 [10, p. 24]. Suppose that X is some set of objects and P is a set of properties. For $R \subseteq P$, let $N_=(R)$ be the number of objects in X that have exactly the properties in R and none of the properties in $P \setminus R$. We let $N_\geq(R)$ denote the number of objects in X that have all the properties in R and possibly some of those in $P \setminus R$. The principle of inclusion-exclusion says that

$$N_=(R) = \sum_{R\subseteq Q\subseteq P}(-1)^{|Q\setminus R|}N_\geq(Q).$$

Result 1.3 Let A_n be the alternating group on X_n. Then $|A_n| = n!/2\,(n \geq 2)$, where $|A_0| = 1 = |A_1|$.

Result 1.4 Let $d(x,k) = \sum_{n\geq 0}\frac{d(n,k)}{n!}x^n$. Then $d(x,k)$ converges for $|x| < 1$ to the function $\frac{x^k e^{-x}}{k!(1-x)}$.

Corollary 1.5 *Let* $d(x) = \sum_{n\geq 0} \frac{d_n}{n!} x^n$. *Then* $d(x)$ *converges for* $|x| < 1$ *to the function* $\frac{e^{-x}}{(1-x)}$.

2. Even and Odd Permutations

As in [9] we define an equivalence on A_n by the equality of number of fixed points, that is,

$$e(n,k) = |\{\alpha \in A_n : f(\alpha) = k\}|, \tag{2.1}$$

where $f(\alpha) = |\{x \in X_n : x\alpha = x\}|$. Then it is not difficult to see that

$$e(n,k) = \binom{n}{k} e(n-k,0) = \binom{n}{k} e_{n-k}. \tag{2.2}$$

Thus to compute $e(n,k)$ it is sufficient to compute $e(n,0) = e_n$. However, note that e_n is the number of even permutations without fixed points, that is, the number of even derangements. Now we have

Theorem 2.1 *Let* e_n *be as defined in* (2.2). *Then* $e_0 = 1, e_1 = 0$, *and for all* $n \geq 2$, *we have*

$$e_n = \frac{n!}{2} \sum_{i=0}^{n-2} \frac{(-1)^i}{i!} + (-1)^{n-1}(n-1).$$

Proof. By the Inclusion-Exclusion Principle we see that

$$\begin{aligned} e_n &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} |A_{n-i}| = \sum_{i=0}^{n-2} (-1)^i \binom{n}{i} |A_{n-i}| + (-1)^{n-1} n + (-1)^n \\ &= \sum_{i=0}^{n-2} (-1)^i \frac{n!}{(n-i)!i!} . \frac{(n-i)!}{2} + (-1)^{n-1}(n-1) \\ &= \frac{n!}{2} \sum_{i=0}^{n-2} \frac{(-1)^i}{i!} + (-1)^{n-1}(n-1). \end{aligned}$$

■

The number e_n satisfies some recurrences similar to those for d_n in Result 1.1.

Proposition 2.2 *Let* e_n *be as defined in* (2.2). *Then*

(*a*) $e_n = (n-1)(e_{n-1} + e_{n-2}) + (-1)^{n-1}(n-1), e_0 = 1, e_1 = 0$;

(*b*) $e_n = ne_{n-1} + (-1)^n (n-2)(n+1)/2$, $e_0 = 1$.

Proof. (a) Using Theorem 2.1 and algebraic manipulations successively we have

$$\begin{aligned}
e_n &= \frac{n!}{2}\sum_{i=0}^{n-2}\frac{(-1)^i}{i!} + (-1)^{n-1}(n-1) \\
&= (n-1)\left[\frac{\{(n-1)+1\}(n-2)!}{2}\sum_{i=0}^{n-2}\frac{(-1)^i}{i!}\right] + (-1)^{n-1}(n-1) \\
&= (n-1)\left[\frac{(n-1)(n-2)!}{2}\sum_{i=0}^{n-2}\frac{(-1)^i}{i!} + \frac{(n-2)!}{2}\sum_{i=0}^{n-2}\frac{(-1)^i}{i!}\right] \\
&\quad +(-1)^{n-1}(n-1) \\
&= (n-1)\left[\frac{(n-1)!}{2}\sum_{i=0}^{n-3}\frac{(-1)^i}{i!} + \frac{(n-1)!}{2}\cdot\frac{(-1)^{n-2}}{(n-2)!} + \frac{(n-2)!}{2}\sum_{i=0}^{n-4}\frac{(-1)^i}{i!}\right. \\
&\quad \left. +\frac{(n-2)!}{2}\cdot\frac{(-1)^{n-3}}{(n-3)!} + \frac{(n-2)!}{2}.\frac{(-1)^{n-2}}{(n-2)!}\right] + (-1)^{n-1}(n-1) \\
&= (n-1)\left[\frac{(n-1)!}{2}\sum_{i=0}^{n-3}\frac{(-1)^i}{i!} + \frac{(-1)^{n-2}}{2}.(n-1) + \frac{(n-2)!}{2}\sum_{i=0}^{n-4}\frac{(-1)^i}{i!}\right. \\
&\quad \left. +\frac{(-1)^{n-3}}{2}.(n-2) + \frac{(-1)^{n-2}}{2}\right] + (-1)^{n-1}(n-1) \\
&= (n-1)\left[\frac{(n-1)!}{2}\sum_{i=0}^{n-3}\frac{(-1)^i}{i!} + \frac{(n-2)!}{2}\sum_{i=0}^{n-4}\frac{(-1)^i}{i!} + (-1)^{n-2}\right] \\
&\quad +(-1)^{n-1}(n-1) \\
&= (n-1)\left[\frac{(n-1)!}{2}\sum_{i=0}^{n-3}\frac{(-1)^i}{i!} + \frac{(-1)^{n-2}(n-2)}{1} + \frac{(n-2)!}{2}\sum_{i=0}^{n-4}\frac{(-1)^i}{i!}\right. \\
&\quad \left. +\frac{(-1)^{n-3}(n-3)}{1}\right] + (-1)^{n-1}(n-1) \\
&= (n-1)(e_{n-1}+e_{n-2}) + (-1)^{n-1}(n-1),
\end{aligned}$$

as required.

(b) As in (a) above, using Theorem 2.1 and algebraic manipulations successively we have

$$\begin{aligned}
e_n &= \frac{n!}{2}\sum_{i=0}^{n-2}\frac{(-1)^i}{i!} + (-1)^{n-1}(n-1) \\
&= n\left[\frac{(n-1)!}{2}\sum_{i=0}^{n-3}\frac{(-1)^i}{i!} + \frac{(n-1)!}{2}.\frac{(-1)^{n-2}}{(n-2)!}\right] + (-1)^{n-1}(n-1) \\
&= n\left[\frac{(n-1)!}{2}\sum_{i=0}^{n-3}\frac{(-1)^i}{i!} + \frac{(n-1)}{2}.(-1)^{n-2}\right] + (-1)^{n-1}(n-1)
\end{aligned}$$

$$= n\left[\frac{(n-1)!}{2}\sum_{i=0}^{n-3}\frac{(-1)^i}{i!} + (-1)^{n-2}(n-2) - \frac{(-1)^{n-2}}{2}.(n-3)\right]$$
$$+(-1)^{n-1}(n-1)$$
$$= ne_{n-1} + (-1)^{n-1}\frac{1}{2}n(n-3) + (-1)^{n-1}(n-1)$$
$$= ne_{n-1} + (-1)^{n-1}\frac{1}{2}(n-2)(n+1),$$

as required. ∎

We now turn our attention to finding the number of odd permutations with k fixed points. Let

$$e'(n,k) = |\{\alpha \in A_n^{'} : f(\alpha) = k\}|. \tag{2.3}$$

Then it is not difficult to see that

$$e'(n,k) = \binom{n}{k} e'(n-k,0) = \binom{n}{k} e_{n-k}^{'}. \tag{2.4}$$

As in the even case above, to compute $e'(n,k)$ it is sufficient to compute $e'(n,0) = e_n^{'}$. Also, note that $e_n^{'}$ is the number of odd permutations without fixed points, that is, the number of odd derangements. We can certainly deduce results for $e_n^{'}$ in exactly the same manner as above, however, we shall take advantage of Theorem 2.1 and Result 1.1, since it is clear that

$$e_n^{'} = d_n - e_n$$
$$= n!\sum_{i=0}^{n}\frac{(-1)^i}{i!} - \left[\frac{n!}{2}\sum_{i=0}^{n-2}\frac{(-1)^i}{i!} + (-1)^{n-1}(n-1)\right]$$
$$= \frac{n!}{2}\sum_{i=0}^{n-2}\frac{(-1)^i}{i!}.$$

Thus we have proved the following result

Theorem 2.3 *Let* $e_n^{'}$ *be as defined in* (2.4). *Then* $e_n^{'} = \frac{n!}{2}\sum_{i=0}^{n-2}\frac{(-1)^i}{i!}$.

Proposition 2.4 *Let* $e_n^{'}$ *be as defined in* (2.4). *Then*
(*a*) $e_n^{'} = (n-1)(e_{n-1}^{'} + e_{n-2}^{'}) + (-1)^n(n-1), e_0^{'} = e_1^{'} = 0$;
(*b*) $e_n^{'} = ne_{n-1}^{'} + (-1)^n(n-1)(n-2)/2, e_0^{'} = 0$.

Proof. It follows directly from Result 1.1 and Proposition 2.2. ∎

Alternative recurrences the first of which is in [12] are

Proposition 2.5 *Let e_n and $e_n^{'}$ be as defined in* (2.2) *and* (2.4), *respectively. Then*

(*a*) $e_n = \frac{1}{2}\left[d_n - (-1)^n(n-1)\right], d_0 = 1$;

(*b*) $e_n^{'} = \frac{1}{2}\left[d_n + (-1)^n(n-1)\right], d_0 = 1$.

Remark 2.6. The sequences $e(n,k)$ and $e'(n,k)$ with the exception of $e_n = e(n,0)$, as at the time of writing are not yet listed in Sloane's encyclopaedia of integer sequences [12]. For some selected values of $e(n,k)$ and $e'(n,k)$ see Tables 1 and 2, respectively.

$n\backslash k$	0	1	2	3	4	5	6	7	$\Sigma e(n,k)$
0	1								1
1	0	1							1
2	0	0	1						1
3	2	0	0	1					3
4	3	8	0	0	1				12
5	24	15	20	0	0	1			60
6	130	144	45	40	0	0	1		360
7	930	910	504	105	70	0	0	1	2520

Table 1. *e(n, k)*

$n\backslash k$	0	1	2	3	4	5	6	7	$\Sigma e'(n,k)$
0	0								0
1	0	0							0
2	1	0	0						1
3	0	3	0	0					3
4	6	0	6	0	0				12
5	20	30	0	10	0	0			60
6	135	120	90	0	15	0	0		360
7	924	945	420	210	0	21	0	0	2520

Table 2. *e'(n, k)*

3. Generating Functions

Let $f(x)$ be the exponential generating function for e_n. Then using Proposition

2.5, Result 1.4 and algebraic manipulations successively we see that

$$\begin{aligned} f(x) &= \sum_{i\geq 0} e_i \frac{x^i}{i!} = \sum_{i\geq 0} \frac{1}{2}\left[d_i - (-1)^i(i-1)\right]\frac{x^i}{i!} \\ &= \frac{1}{2}\sum_{i\geq 0} d_i \frac{x^i}{i!} - \frac{1}{2}\sum_{i\geq 0}(-1)^i(i-1)\frac{x^i}{i!} \\ &= \frac{1}{2}\frac{e^{-x}}{1-x} + \frac{x}{2}\sum_{i\geq 1}(-1)^{i-1}\frac{x^{i-1}}{(i-1)!} + \frac{1}{2}\sum_{i\geq 0}(-1)^i\frac{x^i}{i!} \\ &= \frac{1}{2}\frac{e^{-x}}{1-x} + \frac{x}{2}e^{-x} + \frac{1}{2}e^{-x} \\ &= \frac{(1-x^2/2)}{1-x}e^{-x}. \end{aligned}$$

Proposition 3.1 *Let $f_k(x)$ be the exponential generating function for $e_{i,k} = \binom{i}{k} e_{i-k}$. Then $f_k(x) = \dfrac{x^k(1-x^2/2)e^{-x}}{k!(1-x)}$.*

Proof.

$$\begin{aligned} \text{lhs} &= f_k(x) = \sum_{i\geq k} \frac{\binom{i}{k} e_{i-k}\cdot x^i}{i!} \\ &= \sum_{i\geq k}\frac{e_{i-k}x^i}{k!(i-k)!} \\ &= \frac{x^k}{k!}\sum_{i\geq k}\frac{e_{i-k}x^{i-k}}{(i-k)!} \\ &= \frac{x^k}{k!}f(x) = \frac{x^k(1-x^2/2)e^{-x}}{k!(1-x)} = \text{rhs}, \end{aligned}$$

as required. ∎

Proposition 3.2 *Let $g_k(x)$ be the exponential generating function for $e'(i,k) = \binom{i}{k} e'_{i-k}$. Then $g_k(x) = \dfrac{x^k(x^2/2)e^{-x}}{k!(1-x)}$.*

Proof. From the obvious fact that $d(i,k) = e(i,k) + e'(i,k)$, Result 1.4 and

Proposition 3.1 it follows that

$$\begin{aligned}\frac{x^k e^{-x}}{k!(1-x)} &= \sum_{i\geq k} d(i,k)\frac{x^i}{i!} = \sum_{i\geq r}[e(i,r)+e'(i,r)]\frac{x^i}{i!} \\ &= \sum_{i\geq k} e(i,k)\frac{x^i}{i!} + \sum_{i\geq k} e'(i,k)\frac{x^i}{i!} \\ &= \frac{x^k(1-x^2/2)e^{-x}}{k!(1-x)} + g_k(x).\end{aligned}$$

Hence the result follows. ∎

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