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Estimates of Sample Paths of Dynamical Systems Described by Stochastic Differential Equations*

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Abstract. In this article, we are concerned with sample trajectory estimates of solutions of an Ito stochastic differential equation

$$\begin{cases} dX_t & = a(t, X_t)\, dt + \sigma(t, X_t)\, dW_t \\ X_0 & \in \mathbb{R}^d, \end{cases}$$

where (W_t) is a Brownian motion. By virtues of the law of iterated logarithm and non-linear inequalities, some estimates of Gronwall-Bellman type for sample trajectories are given.

Keywords: Ito equation; Law of iterated logarithm; Gronwall - Bellman inequality.

1. Introduction

There is current interest in dynamic systems especially after 60-year old conjecture due to Faton who solved by Misiurewicz in 1981[13]. In 1997, W.S. Li and

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F.Y. Ren [8] generalized the results of Mismirewicz, Devaney and Krych [2] by proving that for some λ, μ the Julia set of $f(z) = P(\lambda e^z + \mu d^{-z})$ is the whole plane.

In this paper, we consider the dynamic systems described by stochastic differential equation.

Suppose that (W_t) is a $d-$dimension Brownian motion defined on a stochastic basis $(\Omega, \mathcal{F}_t, P)$ (see [4]). Let us consider a dynamical system described by the Ito stochastic differential equation

$$\begin{cases} dX_t & = a(t, X_t)\, dt + \sigma(t, X_t)\, dW_t \\ X_0 & \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where, for any $x \in \mathbb{R}^d$, $(a(t,x))_{t\geq 0}$ and $(\sigma(t,x))_{t\geq 0}$ are two stochastic progressively measurable processes with value in $\mathbb{R}^d$ and $d \times d-$matrices respectively. We suppose that for every initial value $x \in \mathbb{R}^d$, the equation (1.1) has (not necessarily unique) a solution X_t satisfying $X_0 = x$. As is known, there are some works dealt with an estimate of $p^{\text{th}}-$ moments $E|X_t|^p$ or of the probability $P\{X_t \in A\}$ where A is a subset of $\mathbb{R}^n$ (see [4, 5] for example). The general way to establish these estimates is to use the hypothesis of the linear growth rate of coefficients and Gronwall - Bellman inequality to obtain a formula of the form

$$E|X_t|^p \leq c.\exp\{m.t\} \quad \text{for any} \quad t \in <\alpha, \beta>, \tag{1.2}$$

where α, β are fixed and c, m are suitable constants.

However, it is difficult to use this idea to estimate a sample trajectory of System (1.1). The main reason is that all trajectories of the Brownian motion (W_t) have unbounded variations. Moreover, without the hypothesis of linear growth rate of coefficients, this method is no longer valid. As far as we know, there is no work concerned with such a problem.

We now deal with a method that allows us to give such a type of estimates for the sample trajectory without a uniqueness of solutions. The main idea here is to use the law of iterated logarithm to treat the diffusion component and after that we use Lukshmikantham type of inequalities to obtain

$$\sup_{0\leq t\leq T} |X_t| \leq K(T),$$

where $K(t)$ is a certain function that will be given later.

The article is organized as follows: Section 2 concerns with some properties of stochastic integral with respect to Brownian motions. In Section 3, we give estimates of Gronwall - Bellman type for sample paths of solutions of System (1.1)

2. Some Properties of Stochastic Integral

First, we study some properties of Ito stochastic integral with respect to Brownian motions.

Let $(\mathcal{F}_t)_{t\geq 0}$ be a continuous filtration (see [4]). Suppose that (W_t) is a $\mathcal{F}_t$ standard Brownian motion and $f(t,\omega)$ is a stochastic process, $\mathcal{F}_t$ - progressively measurable such that

$$P\left\{\omega : \int_0^T |f(t,\omega)|^2\,dt < \infty\right\} = 1 \quad \text{for any} \quad T > 0.$$

It is known that the stochastic integral $\int_0^T f(t,\omega)\,dW_t$ is well - defined for any $T > 0$. Set

$$m(t) = \int_0^t |f(s,\omega)|^2\,ds, \quad M(t) = \int_0^t f(s,\omega)\,dW_s. \tag{2.1}$$

We define a sequence of stopping times $\tau(t);\ t > 0$, given by

$$\tau(t) = \begin{cases} \inf\{s : m(s) > t\}, \\ \infty, \ \text{if} \ \ t \geq m(\infty) = \lim_{s\uparrow\infty} m(s). \end{cases}$$

Then, from Theorem 7.2', §7.2, in [5, pp. 92], it follows that on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}_t, \widetilde{P})$ of $(\Omega, \mathcal{F}, P)$, there exists an $\widetilde{\mathcal{F}}_t-$ Brownian motion $\mu(t)$ such that $\mu(t) = M(\tau(t)),\quad t \in [0, m(\infty))$. Consequently, we can represent $M(t)$ by an $\widetilde{\mathcal{F}}_t-$ Brownian motion $\mu(t)$ and the stopping times $m(t)$, i.e.,

$$\int_0^t f(s,\omega)\,dW_s = \mu(m(t)).$$

On the other hand, since $(\mu(t))$ is a Brownian motion, then by virtue of the law of iterated logarithm, it follows that

$$\limsup_{t\to\infty} \frac{|\mu(t)|}{\sqrt{2t\ln\ln t}} = 1, \quad \limsup_{t\to 0} \frac{|\mu(t)|}{\sqrt{2t|\ln|\ln t||}} = 1 \quad \text{a.s..}$$

Therefore,

$$\limsup_{t\to\infty} \frac{|\mu(t)|}{\sqrt{2t\ln t}} = 0, \quad \limsup_{t\to 0} \frac{|\mu(t)|}{\sqrt{2t|\ln t|}} = 0 \quad \text{a.s..}$$

Let $c = (e - e^{-1})^{-1}$. It is easy to see that the parabola $y = c(t - e^{-1})^2 + e^{-1}$ is tangent to the curve $y = t\ln t$ at the point e. We define

$$\mathbf{h}(t) = \begin{cases} -t\ln t & \text{if} \quad 0 < t < e^{-1}, \\ y = c(t - e^{-1})^2 + e^{-1} & \text{if} \quad e^{-1} \leq t \leq e, \\ t\ln t & \text{if} \quad t > e,. \end{cases} \tag{2.2}$$

Let

$$\theta := \sup_{t\geq 0} \frac{|\mu(t)|}{\mathbf{h}(t)},$$

then it is easy to see that $P\{\theta < \infty\} = 1$. Hence,

$$|\mu(t)| \leq \theta\, \mathbf{h}(t) \quad \text{for any} \quad t \geq 0 \quad \text{a.s.}$$

or, equivalently,

$$\Big| \int_0^t f(s,\omega)\, dW_s \Big| \leq \theta \mathbf{h}\Big(\int_0^t |f(s,\omega)|^2\, ds \Big). \tag{2.3}$$

We study some further properties of θ.

Lemma 2.1. *1. The distribution of θ is independent of f.*
2. For any $s \in R$ we have $E\, \exp\{s\,\theta\} < \infty$.

Proof. 1. The fact that the distribution of θ is independent of f follows from the definition of θ.
2. Denote by $F(t)$ the distribution function of θ and $G(x) = 1 - F(x)$. We have

$$E\, \exp\{s\theta\} < \infty \Longleftrightarrow \int_1^\infty e^{sx} G(x)\, dx < \infty$$

(see [8], pp. 256). On the other hand

$$\begin{aligned} G(x) = P\{\theta \geq x\} &= P\Big\{ \frac{|\mu(t)|}{\mathbf{h}(t)} \geq x \quad \text{for some} \quad 0 < t < \infty \Big\} \\ &= P\Big\{ |\mu(t)| \geq \mathbf{h}(t) x \quad \text{for some} \quad 0 < t < \infty \Big\}. \end{aligned}$$

Since $(\mu(t))$ is a Brownian motion then

$$\begin{aligned} P\Big\{&|\mu(t)| \geq \mathbf{h}(t)x \text{ for some } 0 < t < \infty \Big\} \\ &= 2P\Big\{ \mu(t) \geq \mathbf{h}(t)x \text{ for some } \ 0 < t < \infty \Big\} \\ &\leq 2 \int_0^\infty \frac{x\mathbf{h}(t)}{\sqrt{2\pi t^3}} e^{-\frac{x^2 \mathbf{h}^2(t)}{2t}}\, dt \end{aligned}$$

for any $x > 1$ (see [9], pp. 34).

It is easy to see that there is a constant k such that:

$$\int_0^\infty \frac{\mathbf{h}(t)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{x^2\mathbf{h}^2(t)}{2t}\right\} dt = \frac{1}{\sqrt{2\pi}}\left[\int_0^{e^{-1}} \frac{\sqrt{-\ln t}}{t} \exp\left\{\frac{x^2 \ln t}{2}\right\} dt \right.$$
$$+ \int_{e^{-1}}^{e} \frac{c(t-e^{-1})^2+e^{-1}}{\sqrt{t^3}} \exp\left\{-\frac{x^2(c(t-e^{-1})^2+e^{-1})}{2t}\right\} dt$$
$$\left. + \frac{1}{\sqrt{2\pi}} \int_e^\infty \frac{\sqrt{\ln t}}{t} \exp\left\{-\frac{x^2 \ln t}{2}\right\} dt\right] \leq k \exp\left\{-\frac{x^2}{4}\right\}$$

for any $x > 1$. Therefore, for any $s \in R$ we have

$$\int_1^\infty e^{sx} G(x)\, dx \leq \int_1^\infty e^{sx}\left(\int_0^\infty \frac{x\mathbf{h}(t)}{\sqrt{2\pi t^3}} e^{-\frac{x^2\mathbf{h}^2(t)}{2t}}\, dt\right) dx$$
$$\leq k \int_1^\infty x\, e^{sx} \exp\left\{-\frac{x^2}{4}\right\} dx < m \exp\{s^2\}, \tag{2.4}$$

where

$$m = k \int_1^\infty x \exp\left\{-\left(\frac{x}{2}-s\right)^2\right\}.$$

This implies that $E[e^{s\theta}] < \infty$ for any $s \in R$. The proof is completed. ∎

Remark 2.2. Inequality (2.3) is still valid when (W_t) is a Brownian motion on $\mathbb{R}^d$

3. Estimates for Sample Paths

3.1 Estimation for Sample Paths with the Coefficients Having a Linear Growth Rate

We now turn to the estimation problem of solutions of Equation (1.1). Suppose that the coefficients $a(t,x)$ and $\sigma(t,x)$ are two stochastic processes, progressively measurable with respect to $(\mathcal{F}_t)$ such that (1.1) has a solution. Furthermore, there exists a continuous function $k(t)$ defined on $\mathbb{R}^+$ such that

$$|a(t,x)| \leq k(t).(1+|x|) \qquad t \geq 0, \quad x \in \mathbb{R}^d, \tag{3.1}$$
$$|\sigma(t,x)| \leq k(t).(1+|x|) \qquad t \geq 0, \quad x \in \mathbb{R}^d. \tag{3.2}$$

Let (X_t) be a solution of Equation (1.1). Then, by using the Ito formula we have

$$d|X_t|^2 = 2X_t'.a(t,X_t)\, dt + |\sigma(t,X_t)|^2\, dt + 2X_t'\sigma(t,X_t)\, dW_t,$$

where X' denotes the transpose vector of X. Thus,

$$d\ln(1+|X_t|^2) = \frac{1}{1+|X_t|^2}\Big(2X_t'a(t,X_t)dt + |\sigma(t,X_t)|^2\Big)dt - 2\bigg(\frac{X_t'\sigma(t,X_t)}{1+X_t^2}\bigg)^2 dt + 2\frac{X_t'\sigma(t,X_t)}{1+X_t^2}\,dW_t.$$

Therefore,

$$\ln(1+|X_t|^2) - \ln(1+|X_0|^2) \leq \int_0^t \frac{2X_s'a(s,X_s)+|\sigma(s,X_s)|^2}{1+|X_s|^2}\,ds + 2\int_0^t \frac{X_s'\sigma(s,X_s)}{1+X_s^2}\,dW_s.$$

From (2.3), it yields

$$\ln(1+|X_t|^2) - \ln(1+|X_0|^2) \leq \int_0^t \frac{2X_s'a(s,X_s)+|\sigma(s,X_s)|^2}{1+|X_s|^2}\,ds + 2\theta\mathbf{h}\bigg(\int_0^t \bigg(\frac{X_s'\sigma(s,X_s)}{1+X_s^2}\bigg)^2 ds\bigg).$$

Taking into account Conditions (3.1) and (3.2), we get

$$\left|\frac{2X_s'a(s,X_s)+|\sigma(s,X_s)|^2}{1+|X_s|^2}\right| \leq 3(k(s)+k^2(s)); \quad \frac{X_s'\sigma(s,X_s)}{1+X_s^2} \leq \sqrt{2}k(s).$$

Hence,

$$\ln(1+|X_t|^2) - \ln(1+|X_0|^2) \leq 3\int_0^t \big(k(s)+k^2(s)\big)\,ds + 4\theta\mathbf{h}\bigg(\int_0^t k^2(s)\,ds\bigg).$$

Thus, we have proved:

Theorem 3.1. *Suppose that the coefficients of Equation* (1.1) *satisfy Conditions* (3.1) *and* (3.2), *then any solution* X_t *of* (1.1) *can be estimated by the inequality*

$$|X_t|^2 \leq (1+X_0^2)\exp\bigg\{3\int_0^t \big(k(s)+k^2(s)\big)\,ds + 4\theta\mathbf{h}\bigg(\int_0^t k^2(s)\,ds\bigg)\bigg\}.$$

Corollary 3.2. *Suppose that* $E[|X_0|^2] < \infty$ *then*

$$E|X_t| \leq K.\exp\bigg\{\frac{3}{2}\int_0^t \big(k(s)+k^2(s)\big)\,ds + 8\mathbf{h}^2\bigg(\int_0^t k^2(s)\,ds\bigg)\bigg\}.$$

Proof. The proof follows immediately from Theorem 3.1 and Inequality (2.4) ■

3.2 Estimates of Sample Paths with the Coefficients Having Non-linear Growth Rate

We study the case where solutions of (1.1) may not be defined on the whole interval $[0, \infty)$. For Equation (1.1), instead of (3.1) and (3.2) we assume that

$$|a(t,x)| \le k_t(1+|x|)^{\alpha} \qquad t \ge 0, \quad x \in \mathbb{R}^d, \tag{3.3}$$

$$|\sigma(t,x)| \le k_t(1+|x|)^{\alpha} \qquad t \ge 0, \quad x \in \mathbb{R}^d, \tag{3.4}$$

where $k > 0$; $\alpha > 0$. If $0 < \alpha \le 1$ then (3.3) and (3.4) imply (3.1) and (3.2). So, we consider only the case $\alpha > 1$.

Let (X_t) be a solution of Equation (1.1). Then, we have

$$|X_t| \le |X_0| + \int_0^t |a(s,X_s)|\,ds + \left|\int_0^t \sigma(s,X_s)\,dW_s\right|.$$

From (2.3) it follows

$$|X_t| \le |X_0| + \int_0^t |a(s,X_s)|\,ds + \theta\mathbf{h}\left(\int_0^t |\sigma(s,X_s)|^2\,ds\right).$$

By virtue of (3.3) and (3.4) it yields

$$|X_t| \le |X_0| + \int_0^t k_s(1+|X_s|)^{\alpha}\,ds + \theta\mathbf{h}\left(\int_0^t k_s^2(1+|X_s|)^{2\alpha}\,ds\right).$$

Putting $u_t = 1 + |X_t|$ we get

$$\begin{aligned} u_t &\le u_0 + \int_0^t k_s u_s^{\alpha}\,ds + \theta\mathbf{h}\left(\int_0^t k_s^2 u_s^{2\alpha}\,ds\right) \\ &\le u_0 + \sqrt{t}\sqrt{\int_0^t k_s^2 u_s^{2\alpha}\,ds} + \theta\mathbf{h}\left(\int_0^t k_s^2 u_s^{2\alpha}\,ds\right). \end{aligned}$$

Since $\sqrt{v} \le \mathbf{h}(v)$ for any $v > 0$ then

$$u_t \le u_0 + (\sqrt{t}+\theta)\mathbf{h}\left(\int_0^t k_s^2 u_s^{2\alpha}\,ds\right). \tag{3.5}$$

We are going to use a nonlinear inequality due to V. Lakshmikantham [12, Theorem 6.1, p. 111] which is formulated as follows

Lemma 3.3. *Suppose that*

a) *$H(t,v)$ is a continuous, increasing in v function defined on $\mathbb{R}^+ \times \mathbb{R}^+$.*

b) *$F(t,u)$ is a continuous, positive function defined on $\mathbb{R}^+ \times \mathbb{R}^+$.*

c) The functions $u(t)$ and $b(t)$ are continuous on $\mathbb{R}^+$.

d) (u_t) satisfies the inequality

$$u_t \le b(t) + H\big(t, \int_0^t F(s, u_s)\, ds\big) \quad t \ge 0.$$

e) There exists a unique solution of the equation

$$\dot{\phi} = F(t, b(t) + H(t, \phi)), \quad \phi(0) = 0.$$

Then the following inequality holds

$$u_t \le b(t) + H\big(t, \phi(t)\big) \quad t \ge 0.$$

We apply this lemma to Inequality (3.5) by putting

$$H(t, v) = (\sqrt{t} + \theta)\mathbf{h}(v); \;\; F(t, u) = k_t^2 . u^{2\alpha} \quad u > 0; v > 0,$$

then (3.5) leads to the inequality

$$u_t \le u_0 + H\left(t, \int_0^t F(s, u_s)\, ds\right)$$

Suppose that ϕ_t be the solution of the equation

$$\dot{\phi}_t = k_t^2 \big[u_0 + (\sqrt{t} + \theta)\mathbf{h}\big(\phi_t\big)\big]^{2\alpha}, \quad \phi_0 = 0. \tag{3.6}$$

Then, from Lemma 3.3 it follows that

$$u_t \le u_0 + (\sqrt{t} + \theta).\mathbf{h}(\phi_t).$$

We estimate the solution ϕ_t of (3.6). It is obvious that

$$\dot{\phi} \le 2^{2\alpha-1} k_t^2 \big[u_0^{2\alpha} + (\sqrt{t} + \theta)^{2\alpha} \cdot \mathbf{h}^{2\alpha}(\phi_t)\big], \quad \phi_0 = 0,$$

or

$$\phi_t \le 2^{2\alpha-1} \left[u_0^{2\alpha} K(t) + \int_0^t k_s^2 (\sqrt{s} + \theta)^{2\alpha} \mathbf{h}^{2\alpha}(\phi_s)\; ds\right],$$

where $K(t) = \int_0^t k_s^2\, ds$.

Set

$$\delta_t = 2^{2\alpha-1} K(t); \;\; \Delta_t = 2^{2\alpha-1} k_t^2 (\sqrt{t} + \theta)^{2\alpha} \tag{3.7}$$

then

$$\phi_t \le \delta_t u_0^{2\alpha} + \int_0^t \Delta_s \mathbf{h}^{2\alpha}(\phi_s)\; ds$$

Set

$$G(u) = \int_1^u \frac{dx}{\mathbf{h}^{2\alpha}(x)}. \tag{3.8}$$

It is easy to see that the limit $\lim_{u\to\infty} G(u)$ exists as a finite number $\gamma > 0$ and $\lim_{u\to 0} G(u) = -\infty$. Moreover, G is strictly monotone. Therefore, the inverse function G^{-1} is defined on $(-\infty, \gamma)$.

By applying estimate (6.3.1) in [12, p. 116] we obtain

$$\phi_t \le G^{-1}\left\{G(\delta_t u_0^{2\alpha}) + \int_0^t \Delta_s\, ds\right\}$$

for any t satisfying the following relation:

$$G(\delta_t u_0^{2\alpha}) + \int_0^t \Delta_s\, ds < \gamma.$$

Summing up, we get

Theorem 3.4. *Under Hypotheses* (3.3) *and* (3.4)*, every solution of Equation* (1.1) *has an estimate*

$$|X_t| \le |X_0| + (\sqrt{t} + \theta).\mathbf{h}\left(G^{-1}\left\{G(\delta_t(1+|X_0|)^{2\alpha}) + \int_0^t \Delta_s\, ds\right\}\right) \tag{3.9}$$

on the interval $\left\{t : G(\delta_t(1+|X_0|)^{2\alpha}) + \int_0^t \Delta_s\, ds < \gamma\right\}$ *where* G *is given by* (3.8) *and* δ_t *are given by* (3.7)*.*

Next theorem is concerned with the stability of a stochastic differential equation by first approximations

Theorem 3.5. *Consider the equation*

$$\begin{cases} dX_t &= (q_t X_t + a(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \\ X_0 &\in \mathbb{R}^d, \end{cases} \tag{3.10}$$

where $\limsup_{t\to\infty} \frac{1}{t}\int_0^t q_s ds < 0$*. Suppose that there exist constants* $k,\ \alpha > 1,\ \beta > 1$ *such that*

$$|a(t,x)| \le k|x|^\alpha \qquad |\sigma(t,x)| \le k|x|^\beta \qquad t \ge 0, \quad x \in \mathbb{R}^d,$$

If there is a $\delta > 0$ *such that every solution starting from* x *with* $|x| < \delta$ *is defined on* $[0,\infty)$ *then the solution* $X = 0$ *is stable.*

Proof. Putting $Z_t = \exp\{\int_0^t q_s\, ds\}$ and $Y_t = Z_t^{-1} X_t$ we have

$$dY_t = Z_t^{-1} a(t, X_t)\, dt + Z_t^{-1}\sigma(t, X_t)\, dW_t.$$

From (2.3) it follows that

$$\begin{aligned}|Y_t| &\le |Y_0| + \int_0^t Z_s^{-1}|a(s,X_s)|\,ds + \theta\mathbf{h}\Big(\int_0^t |Z_s^{-2}\sigma^2(s,X_s)|\,ds\Big)\\ &\le |Y_0| + k\int_0^t Z_s^{-1}|X_s|^{\alpha}\,ds + \theta\mathbf{h}\Big(\int_0^t k^2 Z_s^{-2}|X_s)|^{2\beta}\,ds\Big)\\ &\le |Y_0| + k\int_0^t Z_s^{\alpha-1}|Y_s|^{\alpha}\,ds + \theta\mathbf{h}\Big(\int_0^t k^2 Z_s^{2\beta-2}|Y_s)|^{2\beta}\,ds\Big).\end{aligned}$$

We suppose that $\alpha \ge \beta$ (the case $\alpha < \beta$ can be done similarly). For any $\zeta > 0$, we put

$$\tau = \tau(X_0,\omega) = \inf\{t>0 : |X_t| > \zeta\} = \inf\{t>0 : |Y_t| > Z_t^{-1}\zeta\}$$

then for any $0 < t < \tau$

$$|Y_t| \le |Y_0| + k\zeta^{\alpha-\beta}\int_0^t Z_s^{\beta-1}|Y_s|^{\beta}\,ds + \theta\mathbf{h}\Big(\int_0^t k^2 Z_s^{2\beta-2}|Y_s|^{2\beta}\,ds\Big).$$

By a similar way as in the proof of Theorem 3.4 we get

$$|Y_t| \le |Y_0| + (\zeta^{\alpha-\beta}\sqrt{t}+\theta)\mathbf{h}\Big(\int_0^t k^2 Z_s^{2\beta-2}|Y_s|^{2\beta}\,ds\Big).$$

Hence, by definition $G^{-1}(x) = \infty$ if $x \ge \gamma$, we can show that

$$|Y_t| \le |Y_0| + (\zeta^{\alpha-\beta}\sqrt{t}+\theta)\mathbf{h}\Big(G^{-1}\Big\{G(m_t|Y_0|^{2\alpha}) + \int_0^t \Delta_s\,ds\Big\}\Big),$$

where $m_t = 2^{2\beta-1}k^2\int_0^t Z_s^{2(\beta-1)}\,ds$ and $\Delta_t = 2^{2\alpha-1}k^2 Z_s^{2(\beta-1)}(\zeta^{\alpha-\beta}\sqrt{t}+\theta)^{2\beta}$. Indeed, this inequality is obvious if $G(m_t|Y_0|^{2\alpha}) + \int_0^t \Delta_s\,ds \ge \gamma$. In the case $G(m_t|Y_0|^{2\alpha}) + \int_0^t \Delta_s\,ds < \gamma$ this one follows from Lemma 3.3. Thus,

$$|X_t| \le exp\Big\{\int_0^t q_s\,ds\Big\}\Big\{|X_0| + (\zeta^{\alpha-\beta}\sqrt{t}+\theta)\mathbf{h}\Big(G^{-1}\Big(G(m_t|X_0|^{2\beta}) + \int_0^t \Delta_s\,ds\Big)\Big)\Big\}.$$

We see that $\sup_{0<t<\infty} m_t := m < \infty$; $\max\{\sup_{0<t<\infty} Z_t\sqrt{t},\ \sup_{0<t<\infty} Z_t\}$ $:= M < \infty$ and $\sup_{0<t<\infty}\int_0^t \Delta_s\,ds := \Delta(\theta) < \infty$ a.s. Hence,

$$|X_t| \le M|X_0| + M(\zeta^{\alpha-\beta}+\theta)\mathbf{h}\Big(G^{-1}\Big(G(m|X_0|^{2\beta}) + \Delta(\theta)\Big)\Big)$$

for any $0 \le t \le \tau$.

On the other hand, by the definition of θ, its distribution is as same as the distribution of the random variable $\eta := \sup_{0<t<\infty}|W_t|/\mathbf{h}(t)$, then for any X_0

$$\begin{aligned}&P\{\sup_t |X_t| > \zeta\} = P\{\tau < \infty\} \le P\{X_\tau = \zeta\}\\&\le P\Big\{\zeta \le M|X_0| + M(\zeta^{\alpha-\beta} + \eta)\mathbf{h}\Big(G^{-1}\Big(G(m|X_0|^{2\beta}) + \Delta(\eta)\Big)\Big)\Big\}\end{aligned}$$

It is clear that $\lim_{X_0\to 0} \mathbf{h}(G^{-1}(G(m|X_0|^{2\beta}) + \Delta(\eta))) = 0$ a.s.Thus,

$$\begin{aligned}&P\{\sup_t |X_t| > \zeta\}\\&\le P\Big\{\zeta \le M|X_0| + M(\zeta^{\alpha-\beta} + \eta)\mathbf{h}\Big(G^{-1}\Big(G(m|X_0|^{2\beta}) + \Delta(\eta)\Big)\Big)\Big\}\end{aligned}$$

tends to 0 as $X_0 \to 0$. This mean that the trivial solution $X = 0$ is stable. Theorem is proved. ■

Applications. Consider the quasi-linear stochastic differential equation

$$\begin{cases} dX_t &= (a_t X_t + X_t^3)\,dt + \sqrt{2}X_t^2\,dW_t,\\ X_0 &\in R,\end{cases} \tag{3.11}$$

where $\limsup_{t\to\infty} \frac{1}{t}\int_0^t a_s\,ds = \sigma < 0$. Let $X_0 > 0$. By using the Ito's formula we have

$$\begin{aligned}d(\ln X_t) &= \left(\frac{a_t X_t + X_t^3}{X_t} - \frac{X_t^4}{X_t^2}\right)dt + \frac{\sqrt{2}X_t^2}{X_t}\,dW_t\\&= a_t\,dt + \sqrt{2}X_t\,dW_t\end{aligned}$$

Hence, for $X_0 > 0$ we have $\mathbb{E}X_t < \infty$. Similarly, $\mathbb{E}X_t > -\infty$ for $X_0 < 0$. This means that the solution $X(t, X_0)$ exists on $[0, \infty)$ which implies that the solution $X = 0$ of the equation (3.11) is stable by Theorem 3.5.

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