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On Pseudo-P-Injectivity*

Samruam Baupradist
Department of Mathematics, Mahidol University, Bangkok, Thailand
E-mail: samruam_b@hotmail.com

Hoang Dinh Hai
International Education Center, Hong Duc University, Thanh Hoa, Vietnam
Email: haiedu93@yahoo.com

Nguyen Van Sanh
Department of Mathematics, Mahidol University, Center of Excellence in Mathematics, Bangkok, Thailand
Email: frnvs@mahidol.ac.th

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Abstract. For a given right R-module M, a right R-module N is called pseudo-M-p-injective if every monomorphism from an M-cyclic submodule X of M to N can be extended to a homomorphism from M to N. A right R-module M is said to be a quasi-pseudo-p-injective module if it is pseudo-M-p-injective. We study the structure of the endomorphism ring of a quasi-pseudo-p-injective module M which is a quasi-projective Kasch module. In this case, we show that there is a bijection between the class of all maximal submodules of M and the class of all left ideals of its endomorphism rings. Especially, for a right self-pseudo-p-injective right Kasch ring, we get a bijection between the class of all maximal right ideals and the class of all minimal left ideals.

Keywords: Pseudo-M-p-injective module; Quasi-pseudo-p-injective module; Kasch module; Self-generators.

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1. Introduction and Preliminaries

Throughout this paper, R is an associative ring with identity and Mod-R is the category of unitary right R-modules. Let M be a right R-module, and $S = End_R(M)$, its endomorphism ring. A right R-module N is called *M-generated* if there exists an epimorphism $M^{(I)} \longrightarrow N$ for some index set I. If I is finite, then N is called *finitely M-generated.* In particular, N is called *M-cyclic* if it is isomorphic to M/L for some submodule L of M. Hence, any M-cyclic submodule X of M can be considered as the image of an endomorphism of M. Following Wisbauer [19], $\sigma[M]$ denotes the full subcategory of Mod-R, whose objects are the submodules of M-generated modules. A module M is called a *self-generator* if it generates all of its submodules. M is called a *subgenerator* if it is a generator of $\sigma[M]$.

Let M be a right R-module. A right R-module N is called *M-p-injective* if every homomorphism from an M-cyclic submodule of M to N can be extended to a homomorphism from M to N. For more details of M-p-injective modules, we can refer to [11].

Pseudo-injective modules have been studied in [2, 6, 13–18]. Recently, Hai Quang Dinh [4] introduced the notion of pseudo-M-injective modules (the original terminology is M-pseudo-injective), following which a right R-module N is called *pseudo-M-injective* if for every submodule A of M, any monomorphism $\alpha : A \longrightarrow N$ can be extended to a homomorphism $\beta : M \longrightarrow N$. A right R-module N is called *quasi-pseudo-injective* if N is pseudo-N-injective. In 1999, Sanh et. al., introduced the notion of M-p-injective modules and studied the endomorphism rings of quasi-p-injective modules (see [11]).

In this paper, we will investigate pseudo-M-p-injectivity, and endomorphism rings of quasi-pseudo-p-injective modules. As an application, we can get some results to all right self-pseudo-p-injective rings as corollaries when $M = R_R$.

Definition 1.1. *Let M be a right R-module and $S = End_R(M)$. A right R-module N is said to be pseudo-M-p-injective (resp. M-p-injective) if for any $s \in S$, and every monomorphism (resp. homomorphism) from $s(M)$ to N can be extended to a homomorphism from M to N.*

A right R-module M is called a quasi-pseudo-p-injective if M is pseudo-M-p-injective. A right R-module N is pseudo-p-injective if it is pseudo-R_R-p-injective. A ring R is right self-pseudo-p-injective if R_R is quasi-pseudo-p-injective as a right R-module.

Example 1.2.

(1) Clearly, if N is pseudo-M-injective, then N is pseudo-M-p-injective. Moreover, if N is M-*p*-injective, then it is pseudo-M-p-injective too (see [3, 7]).

(2) The following example (see [8, Exercise(2), p. 361]) shows that pseudo-M-p-injective modules need not to be pseudo-M-injective.

Let K be a field and let $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ be the ring of all matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a, b, c \in K$, $M = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$. Then we have the following statements (see [3]):

(i) N is pseudo-M-p-injective,

(ii) N is not pseudo-M-injective.

Proposition 1.3. *Let M, N be right R-modules, $S = End_R(M)$.*

1. *If N is pseudo-M-p-injective, then N is pseudo-$s(M)$-p-injective for all $s \in S$. Especially, if $B \subset^{\oplus}_{>} M$, then N is pseudo-B-p-injective.*
2. *If N is pseudo-M-p-injective, then every direct summand of N is pseudo-M-p-injective.*
3. *For any $s \in S$, if $s(M)$ is pseudo-M-p-injective, then $s(M)$ is a direct summand of M.*

Proof. (1) Let $s \in S$. Take any $h \in End_R(s(M))$ and any monomorphism $\varphi : h(s(M)) \longrightarrow N$. We can see that $h(s(M)) = k(M)$ for some endomorphism k of M. Let $\iota_1 : hs(M) = h(s(M)) \to s(M)$ and $\iota_2 : s(M) \to M$ be inclusions. Since N is pseudo-M-p-injective, there is a homomorphism $\alpha : M \to N$ such that $\alpha|_{k(M)} = \varphi$. We can see that $\overline{\varphi} = \alpha\iota_2$ is an extension of φ from $s(M)$ to N. It follows that N is $s(M)$-pseudo-p-injective.

The last statement follows from the fact that any direct summand of M can be considered as $e(M)$ for some idempotent e of S.

(2) The proof is routine.

(3) The result follows from the fact that the inclusion map $i : s(M) \hookrightarrow M$ splits. ∎

The following Corollary is straightforward.

Corollary 1.4. *Let M be a quasi-pseudo-p-injective module which is quasi-projective, and $s \in S = End_R(M)$. The following statements are equivalent:*

1. *$Im(s)$ is a direct summand of M;*
2. *$Im(s)$ is a pseudo-M-p-injective;*
3. *$Im(s)$ is M-projective.*

2. Quasi-pseudo-p-injective Modules

Lemma 2.1. *Let M be a right R-module. If M is a quasi-pseudo-p-injective and $Im(s) \subset^{*}_{>} M$ where $s \in S = End_R(M)$, then every monomorphism $\varphi : s(M) \to M$ can be extended to a monomorphism in S.*

Proof. Since M is quasi-pseudo-p-injective, there exists $\overline{\varphi} : M \to M$ such that $\overline{\varphi}s = \varphi s$. It follows that $Im(s) \cap Ker(\overline{\varphi}) = 0$. From $Im(s) \subset^{*}_{>} M$, we get $Ker(\overline{\varphi}) = 0$. ∎

Recall that a right R-module M satisfies the condition C_2 if every submodule of M isomorphic to a direct summand is again a direct summand.

Lemma 2.2. *Every quasi-pseudo-p-injective module satisfies C_2.*

Proof. The proof is routine by applying Proposition 1.3. ∎

Following [9], a right R-module M is said to be direct projective if every epimorphism $f : M \to X$ splits for any direct summand X of M. By Lemma 2.2 above, referring to 37.7 in [19] and considering the Theorem 2.7 in [11], we have the following theorem:

Theorem 2.3. *Let M be a direct projective module and $S = End_R(M)$. Then the following conditions are equivalent:*

(1) *S is von Neumann regular;*
(2) *Every M-cyclic submodule of M is M-p-injective;*
(3) *Every M-cyclic submodule of M is pseudo-M-p-injective.*

Proof. By [11, Theorem 2.7], we get (1)⇔(2).

(2)⇒(3) is obvious from the definitions.

(3)⇒(1) Let $s \in S$. By assumption, $s(M)$ is pseudo-M-p-injective and by Proposition 1.3(3) we have $s(M) \subset^{\oplus}_{>} M$. Since M is direct projective, the epimorphism $s : M \to s(M)$ splits. It follows that $Ker(s) \subset^{\oplus}_{>} M$. By Proposition 37.7 in [19], S is von Neumann regular. ∎

Proposition 2.4. *Let M be a quasi-pseudo-p-injective module. Then $J(S) \subset \{s \in S | Ker(s) \neq 0\}$ where $J(S)$ is the Jacobson radical of S. Moreover if M is uniform, then S is a local ring and in this case $J(S) = \{s \in S | Ker(s) \neq 0\}$.*

Proof. Let $s \in J(S)$, the Jacobson radical of S. Suppose on the contrary that s is a monomorphism. Then $s(M) \cong M$ and hence $s(M)$ is pseudo-M-p-injective. It follows that $s(M)$ is a direct summand of M and therefore s has a left inverse, φ says. Then $\varphi s = 1_M$ or $Ss = S$, a contradiction. This shows that $J(S) \subset \{s \in S | Ker(s) \neq 0\}$. We now suppose that M is uniform. Then it is clear that every monomorphism from M to M is an automorphism. Hence any non-invertible element φ of S has $Ker(\varphi) \neq 0$. Let φ_1 and φ_2 be non-invertible elements of S. Then $Ker(\varphi_1) \neq 0$ and $Ker(\varphi_2) \neq 0$. Since M is uniform, $Ker(\varphi_1) \cap Ker(\varphi_2) \neq 0$ and hence $Ker(\varphi_1 + \varphi_2) \neq 0$, proving that $\varphi_1 + \varphi_2$ is not invertible. Thus S is a local ring, as desired. ∎

Theorem 2.5. *Let M_R be a quasi-pseudo-p-injective module which is a Kasch module. Consider the map*

$$\theta : T \mapsto l_S(T)$$

from the set of all maximal submodules T of M to the set of all minimal left ideals of $S = End_R(M)$. Then we have:

(1) *θ is an injective map;*

(2) *If M is finitely generated, then θ is a bijection if and only if $l_S r_M(K) = K$ for all minimal left ideals K of S. In this case θ^{-1} is given by $K \mapsto r_M(K)$.*

Proof. (1) Let T be a maximal submodule of M. Since M is a Kasch module, M/T can be considered as a submodule of M, and hence $M/T = s(M)$ for some $s \in S = End_R(M)$. Take any $0 \neq t \in l_S(T)$. Then $T \subset Ker(t) \neq M$. It follows that $Ker(s) = T = Ker(t)$. By the homomorphism Theorem, there is a monomorphism φ from $s(M)$ to $t(M)$ such that $\varphi s = t$. Since $t(M)$ and $s(M)$ are simple, the monomorphism φ must be isomorphic. Consider φ as a monomorphism from $s(M)$ to M. Then φ can be extended to a homomorphism $\psi : M \to M$, that is $\psi s = \varphi s$ and hence $t = \varphi s$ proving that $t \in Ss$. This shows that $l_S(T) \subset Ss$. Moreover, we always have $Ss \subset l_S(Ker(s)) = l_S(T)$. This means that $l_S(T) = Ss$. We now show that Ss is a minimal left ideal of S. Take any $0 \neq v \in Ss$. We have $v = gs$ for some $g \in S$. It follows that $Ker(s) \subset Ker(v) \neq M$ and hence $Ker(v) = Ker(s)$ by the maximality of $Ker(s)$. Then there is a monomorphism $f : v(M) \to s(M)$ which is also an isomorphism satisfying $fv = s$. Consider f as a monomorphism from $v(M)$ to M. By the quasi-pseudo-p-injectivity of M, we can find $h \in S$ such that $hv = fv = s$. This shows that $Ss \subset Sv$, proving that $Ss = Sv$ or Ss is a minimal left ideal of S.

(2) If θ is surjective and K is a minimal left ideal of S, then we can write $K = l_S(T)$ where T is maximal in M. Then $l_S r_M(K) = K$ follows. Now, let $K \subset S$ be a minimal left ideal of S. Then $K = Ss$ for some $s \in S$. We now show that $r_M(K)$ is maximal in M. Note that $r_M(K) = Ker(s)$. Since M is finitely generated, $r_M(K)$ is contained in a maximal submodule T of M. Then $K = l_S r_M(K) \supset l_S(T) \neq 0$ since M is a Kasch module. Therefore $K = l_S(T)$ because K is simple. This leads to $r_M(K) = r_M l_S(T) \supset T$. Therefore by the maximality of T in M, we have $r_M(K) = T$, proving that θ is surjective. ■

As an application, putting $M = R_R$. We get the following result.

Theorem 2.6. *If a ring R is right Kasch, right self-pseudo-p-injective ring, then there is a bijection map between the class of all minimal left ideals of R and the class of all maximal right ideals of R.*

We consider a right R-module as an S-R-bimodule, where $S = End_R(M)$ is the endomorphism ring of M. The following proposition give a relation between the socle of M_R and ${}_SM$ in a special case.

Proposition 2.7. *Let M be a right R-module. If M is quasi-pseudo-p-injective which is self-generator, then $Soc(M_R) \subset Soc({}_SM)$.*

Proof. Suppose that xR is a simple submodule of M where $x \in M$. Since M is a self-generator, there exists an element $s \in S$ such that $xR = s(M)$, and hence $x = s(m)$ for some $m \in M$. Take any $0 \neq u \in Sx$. Then $u = \varphi x$ for some $\varphi \in S$. So $u = \varphi s(m)$. Since M is quasi-pseudo-p-injective, every monomorphism from $\varphi s(M)$ to M can be extended to an endomorphism of M.

Consider the following map:

$$\begin{array}{cccc} \xi : xR = s(M) & \longrightarrow & \varphi s(M) = \varphi xR \\ xr & \longmapsto & \varphi xr \end{array}$$

Clearly, ξ is a non-zero homomorphism. Since φxR is simple and $\xi \neq 0$, we see that ξ is an isomorphism. Let $\psi = \iota \xi^{-1}$ where $\iota : xR \longrightarrow M$ is the embedding. Then $\psi(\varphi xr) = xr$ for all $r \in R$. Thus $x = \psi(\varphi x) = \overline{\psi} u$, where $\overline{\psi}$ is an extending of ψ on M. It shows that $x \in Su$, that is $Sx \subset Su$. It is clear that $Su \subset Sx$, and hence $Su = Sx$, proving that Sx is a simple submodule of ${}_SM$.

We now let $x \in Soc(M_R) = \sum_{i \in I} X_i$, where each X_i is a simple submodule of M. Then $x = x_1 + x_2 + ... + x_n, 0 \neq x_i \in X_i$ and $Sx \subset Sx_1 + Sx_2 + ... + Sx_n$. Since $X_i = x_iR$ is simple, Sx_i is a simple submodule of ${}_SM$. This shows that $x \in Soc({}_SM)$, proving that $Soc(M_R) \subset Soc({}_SM)$. ■

As an application let $M = R_R$ we get the following result:

Corollary 2.8. *If R is a right self -pseudo-p-injective ring, then $Soc(R_R) \subset Soc({}_SR)$.*

Proposition 2.9. *Let M be a quasi-pseudo-p-injective module and $s, t \in S = End_R(M)$. If $s(M) \cong t(M)$, then $Ss \cong St$.*

Proof. Let $f : s(M) \to t(M)$ be an isomorphism. Embedding $s(M)$ and $t(M)$ to M, f can be considered as a monomorphism from $s(M)$ to M. By the property of M, there exists a homomorphism $\varphi \in S$ such that $\varphi|_{s(M)} = f$. We now define $\sigma : St \to Ss$ by $\sigma(ut) = u\varphi s$. Since $Im(fs) \subset Im(t)$, the map σ is well-defined and we can check that σ is an S-homomorphism.

We first show that σ is an epimorphism. Let $g : t(M) \to s(M)$ be the inverse of f. Embedding $s(M)$ to M, we can consider g as a monomorphism from $t(M)$ to M. By using the property of M, we can find $\psi \in S$ such that $\psi|_{t(M)} = g$. For any $vs \in Ss$, we take $u = v\psi$, and by a routine calculation, we get $\sigma(ut) = \sigma(v\psi t) = vs$, proving that σ is an epimorphism. We now suppose $u_1t, u_2t \in St$ such that $\sigma(u_1t) = \sigma(u_2t)$. By the definition, $u_1\varphi s = u_2\varphi s$. It follows that $Im(t) = Im(\varphi s) \subset Ker(u_1 - u_2)$ and then $u_1t = u_2t$, showing that σ is a monomorphism. The proof of our proposition is now complete. ■

Corollary 2.10. *For a right self-pseudo-p-injective ring, if $aR \cong bR$ then $Ra \cong Rb$.*

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