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Operators on Normed Hypervector Spaces

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Abstract. In this paper we prove some results about normed hypervector spaces and linear, antilinear or strong linear operators on these spaces. We also prove that the space of all bounded strong linear functionals is a Banach classical space.

Keywords: Hypervector space; Normed hypervector space; Hyperstructure.

1. Introduction

The concept of hyperstructure was first introduced by Marty [7] in 1934 and has attracted attention of many authors in last decades and has constructed some other structures such as hyperrings, hypergroups, hypermodules, hyperfields, and hypervector spaces. These constructions have been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, and etc. A wealth of applications of this concepts is given in $[1-6]$ and $[8-13]$.

In 1988 the concept of hypervector space was first introduced by Scafati-Tallini. She studied more properties of this new structure in [9] and [10].

This paper is arranged as follows. In section 2, we recall the definition of hypervector space, norm and different types of operators in such space and give some examples. In section 3, we prove some results about normed hypervector spaces and linear, antilinear or strong linear operators on these spaces. We consider the set of all bounded strong linear functionals as a Banach classical space, too. We denote the set of all complex numbers by C and real numbers by R.

2. Preliminaries

Definition 2.1. [10] *Let $(X,+)$ be an abelian group and F be a field. Then a hypervector space is a quadruple $(X,+,o,F)$ where o is a mapping, $o : F \times X \longrightarrow P\star(X)$ such that the following conditions are satisfied:*
(i) $\forall a \in F, \forall x, y \in X,\ ao(x+y) \subseteq aox + aoy$ *(right distributivity),*
(ii) $\forall a, b \in F, \forall x \in X,\ (a+b)ox \subseteq aox + box$ *(left distributivity),*
(iii) $\forall a, b \in F, \forall x \in X,\ ao(box) = (ab)ox$ *(associativity),*
(iv) $(4)\forall a \in F, \forall x \in X,\ ao(-x) = (-a)ox,$
(v) $\forall x \in X,\ x \in 1ox.$

Remark 2.2. We say that $(X,+,o,F)$ is right antidistributive if

$$\forall a \in F, \forall x, y \in X,\ ao(x+y) \supseteq aox + aoy,$$

strongly right distributive if

$$\forall a \in F, \forall x, y \in X,\ ao(x+y) = aox + aoy,$$

In a similar way we define the left antidistributive and strongly left distributive laws, respectively.

Example 2.3. Suppose $0 \neq a \in R$ and $0 \neq z \in C$. The set C with the usual sum and the following product is a hypervector space on R:

$$aoz = \{re^{i\theta}; 0 < r \leq |a||z|, \theta = arg(z)\},$$

if $a = 0$ or $z = 0$, then we define $aoz = 0$.

Example 2.4. Suppose $a \in R$ and $z \in C$. The set C with the usual sum and the following product is a hypervector space on R:

$$a.z = \{re^{i\theta}; 0 \leq r \leq |a||z|, 0 \leq \theta \leq 2\pi\}.$$

Example 2.5. Let X be a vector space over the field F, $\lambda \in F$ and $x \in X$. X with its sum and the following product is a hypervector space on F:

$$\lambda * x = \{0, \lambda x\}.$$

Definition 2.6. [10] *Let $(X,+,o,F)$ be a hypervector space over a field F, that is the field of real or complex numbers. We define a pseudonorm in X as a mapping $\|\|: X \longrightarrow R$, of X into the reals such that:*

(i) $\| 0 \| = 0$,
(ii) $\forall x, y \in X, \quad \| x + y \| \leq \| x \| + \| y \|$,
(iii) $\forall a \in F, \ \forall x \in X, \ \sup \| aox \| = | a | \| x \|$.

A pseudonorm in X is called norm (see [7]) if:
(iv) $\| x \| = 0 \ \Leftrightarrow \ x = 0$.

Definition 2.7. [1] *Let X and Y be hypervector spaces over F. A map $T : X \longrightarrow Y$ is called*

(i) *linear if and only if $T(x + y) = T(x) + T(y), \ T(aox) \subseteq aoT(x), \ \forall x, y \in X, a \in F$,*
(ii) *antilinear if and only if $T(x+y) = T(x)+T(y), \ T(aox) \supseteq aoT(x), \ \forall x, y \in X, a \in F$,*
(iii) *strong linear if and only if $T(x + y) = T(x) + T(y), \ T(aox) = aoT(x), \ \forall x, y \in X, a \in F$.*

Definition 2.8. *Let X be a hypervector space over F and $f : X \to F$ be a map. If f satisfies in the condition (ii) of Definition 2.7, then f is called antilinear functional. If f be an additive map such that $f(aox) - \{0\} = af(x)$ for all $a \in F$ and $x \in X$, then it is called strong linear functional. We denote the set of all antilinear functionals by X^*_a and the set of all strong linear functionals by X^*_s.*

Example 2.9. Let T be a map on the hypervector space C into C over R (that was defined in Example 2.3) and defined by $x \mapsto x + x$. Easily, we prove that T is a strong linear operator, because in this space for any $a \in R$ and $x \in C$ we have

$$\{y + y; y \in aox\} = ao(x + x).$$

Example 2.10. Let T be a map on the hypervector space C(that was defined in Example 2.3) into C (that was defined in Example 2.4) over R and defined by $x \mapsto x$. We see that T is a linear operator, because in this space for any $a \in R$ and $x \in C$ we have

$$T(aox) = \{Ty; y \in aox\} = \{y; y \in aox\} = \{re^{i\theta}; 0 < r \leq | a || z |, \theta = arg(z)\},$$
$$a.Tx = a.x = \{re^{i\theta}; 0 \leq r \leq | a || z |, 0 \leq \theta \leq 2\pi\}.$$

So $T(aox) \subseteq a.Tx$ and hence T is linear.

Example 2.11. Let in Example 2.5 X be C and F be R. Also let f be a map on the hypervector space C into R by letting $x \mapsto x + \overline{x}$. Easily, we prove that f is a strong linear functional, because in this space for any $\lambda \in R$ and $x \in C$ we have

$$f(\lambda * x) - \{0\} = \lambda(x + \overline{x}) = \lambda f(x).$$

3. Main Results

Proposition 3.1. *Let X and Y be hypervector spaces over F and $T : X \longrightarrow Y$ be an injective antilinear operator. Then the inverse of T is linear.*

Proof. Additivity of T^{-1} is obvious. Let $y \in T(X)$, then $y = Tx$ for some x in X. We know $T(aox) \supseteq aoT(x)$ for all $a \in F$. So $aox \supseteq T^{-1}(ao(Tx))$, and hence $aoT^{-1}(y) \supseteq T^{-1}(aoy)$. Thus by Definition 2.7(i), T^{-1} is a linear operator. ∎

Definition 3.2. *Let X and Y be normed hypervector spaces over F and $T : X \longrightarrow Y$ be a linear, antilinear or strong linear operator. T is said to be bounded if there exists a positive real number K such that*

$$\| Tx \| \leq K \| x \| \qquad \forall x \in X.$$

Theorem 3.3. *Let X and Y be normed hypervector spaces over F and $T : X \longrightarrow Y$ is an antilinear operator, then T is continuous if and only if T be bounded.*

Proof. Let T be a continuous antilinear operator, $x_0 \in X$ and $\epsilon > 0$ be arbitrary. So there exists $\delta > 0$ such that

$$\| x - x_0 \| < \delta \quad \Rightarrow \quad \| Tx - Tx_0 \| < \epsilon. \quad (*) \tag{1}$$

Let $y \in X$ be arbitrary, not zero. Set $Z = x_0 + (\delta \| y \|^{-1})oy$. Since $sup \| Z - x_0 \| = sup \| (\delta \| y \|^{-1})oy \| = (\delta \| y \|^{-1}) \| y \| = \delta$, so the norm of all elements of $Z - x_0$ or $(\delta \| y \|^{-1})oy$ is less than δ. Hence by $(*)$, the norm of the elements of $T(Z - x_0)$ and thus $T((\delta \| y \|^{-1})oy)$ is less than ϵ. This implies

$$sup \| T((\delta \| y \|^{-1})oy) \| = sup \| T(Z - x_0) \| < \epsilon. \quad (**) \tag{2}$$

Since T is antilinear, we have

$$T((\delta \| y \|^{-1})oy) \supseteq (\delta \| y \|^{-1})oTy.$$

This condition and $(**)$ imply

$$sup \| (\delta \| y \|^{-1})oTy \| \leq sup \| T((\delta \| y \|^{-1})oy) \| < \epsilon,$$

and this implies

$$\delta \| y \|^{-1} \| Ty \| < \epsilon \Rightarrow \| Ty \| < \epsilon\delta^{-1} \| y \| .$$

This relation holds for $y = 0$, too. So by Definition 3.2, T is bounded. The proof of inverse is easy. ∎

Corollary 3.4. *If T is a strong linear operator, then the continuity of T is equivalent to the boundedness of T.*

Remark 3.5. If T is a bounded linear operator, then T is continuous.

Question 3.6. Does the converse of the above remark still hold?

Proposition 3.7. *Let X and Y be normed hypervector spaces over F and $T : X \longrightarrow Y$ be a bounded antilinear operator. Then for all $x \in X$ we have*

$$\| Tx \| \leq \| T \| \| x \|,$$

where $\| T \| = sup\{\| Tx \| ; \| x \| \leq 1\}$.

Proof. Let $x \in X$ be arbitrary, not zero. Then $sup \| \| x \|^{-1} ox \| = \| x \|^{-1} \| x \| = 1$. Since T is antilinear operator, we have

$$T(\| x \|^{-1} ox) \supseteq \| x \|^{-1} oTx.$$

Hence by the definition of $\| T \|$, we must have

$$\| x \|^{-1} \| Tx \| = sup \| \| x \|^{-1} oTx \| \leq sup \| T(\| x \|^{-1} ox) \| \leq \| T \| .$$

This implies that

$$\| Tx \| \leq \| T \| \| x \| .$$

This relation is also true for $x = 0$. Therefore, our proof is completed. ∎

Theorem 3.8. *Let X be a normed hypervector space over F. Then X_s^* is a Banach classical space by the following defined actions and norm:*

(i) $(f + g)(x) = f(x) + g(x), \quad \forall\ f, g \in X_s^*,\ \forall x \in X,$

(ii) $(aof)(x) = af(x), \quad \forall a \in F,\ \forall f \in X_s^*,\ \forall x \in X,$

(iii) $\| f \| = \sup\{|f(x)|\ ;\ \ \| x \| \leq 1\}.$

Proof. It is easy to check that X_s^* is a classical normed space. To show that X_s^* is complete, let $B = \{x \in X; \| x \| \leq 1\}$. If $f \in X_s^*$, define $\rho(f) : B \to F$ by $\rho(f)(x) = f(x)$. That is $\rho(f)$ is the restriction of f to B. We denote the set of all bounded and continuous functions on B by $C_b(B)$. Note that $\rho : X_s^* \to C_b(B)$ is a linear isometry. Since $C_b(B)$ is complete, it suffices to show that $\rho(X_s^*)$ is closed in $C_b(B)$. Let $\{f_n\} \subseteq X_s^*$ and $g \in C_b(B)$ such that

$$\| \rho(f_n) - g \| \to 0 \ as\ n \to \infty.$$

Since for each $x \in X$ we have $\rho(f_n)(x) \to g(x)$, so for each $\alpha \in F$ we have $\alpha\rho(f_n)(x) \to \alpha g(x)$ and hence $\rho(f_n)(\alpha ox) \to g(\alpha ox)$. Now let $x \in X$. If $\alpha,\ \beta \in F,\ \alpha,\ \beta \neq 0$, such that $\alpha ox, \beta ox \subseteq B$, then

$$\alpha^{-1}g(\alpha ox) = lim\ \alpha^{-1}f_n(\alpha ox) = lim\ \beta^{-1}f_n(\beta ox) = \beta^{-1}g(\beta ox).$$

Define $f : X \to F$ by letting $f(x) = \alpha^{-1}g(\alpha ox)$, for any $\alpha \neq 0$ such that $\alpha ox \subseteq B$. It is not difficult to check that $f \in X_s^*$ and $\rho(f) = g$. ∎

Remark 3.9. Note that nontrivial strong linear functionals in an arbitrary hypervector space do not necessarily exist. As an example, we cite the defined hypervector space in 2.3. Let $z \in C$ be arbitrary, not zero and $x \in 2oz$. So $f(x) = f(2oz) = 2f(z)$. On the other hand, since $x \in 3oz$ we have $f(x) = f(3oz) = 3f(z)$. These imply $f(z) = 0$ and hence $f \equiv 0$.

Moreover, there exist hypervector spaces such that nontrivial strong linear functionals are defined on them. See Example 2.11.

Question 3.10. What is a sufficient condition on a hypervector space in order that nontrivial strong linear functionals on it exist?

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