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Small-Essential Submodules and Morita Duality

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Abstract. We introduce and investigate e-small and s-essential submodules, and prove that $Rad_e(M)= \cap\{\ N \trianglelefteq M \mid N$ is maximal in M $\}=\sum\ \{\ N \subseteq M \mid N \ll_e M\}$ and $Soc_s(M)=\sum\{\ N \ll M \mid N$ is minimal in M $\}=\cap\{\ L \leq M \mid L \trianglelefteq_s M\}$. As an application, we introduce the e-gH and s-wcH pairs of modules, and show that they are dual to each other under Morita duality.

Keywords: E-small submodule; S-essential submodule; Morita duality.

1. Introduction

The notion of small submodules (also called superfluous submodules) plays an important role in the theory of modules and rings (see [1, 6, 13]). Recently, Zhou[13] generalizes the concept of small to that of δ-small by considering the class δ of all singular right R-modules in place of right R-modules, and gives various properties of δ-small submodules, which are similar to those of small submodules, and then uses this concept to generalize the notions of perfect, semiperfect and semiregular rings to those of δ-perfect, δ-semiperfect and δ-semiregular rings. Recently, in [12] we define the dual concept of δ-small submodules, which is a generalization of essential submodules, and obtain some interesting properties. It was stated and proved by Xu and Shum [9] some results about the equivalence between the module categories over two rings K and R, where R does not necessary have a unity.

In this paper, we introduce the class of all essential submodules to generalize small submodules and the class of all small submodules to generalize essential submodules respectively. It is of interest to know how far the old theories extend to the new situation. As will be shown later, they do to a very great extent. In Section 2, we give various properties of e-small and s-essential submodules, and characterize the generalized socle and radical of modules, that is, $Rad_e(M) = \cap\{ N \trianglelefteq M \mid N \text{ is maximal in M} \} = \sum \{ N \subseteq M \mid N \ll_e M\}$ and $Soc_s(M) = \sum\{ N \ll M \mid N \text{ is minimal in M} \} = \cap\{ L \leq M \mid L \trianglelefteq_s M\}$. In Section 3, we use the concepts of e-small and s-essential submodules to characterize some properties of homomorphisms of modules. It is proved that e-small homomorphisms and s-essential homomorphisms are dual to each other under Morita duality. In Section 4, as applications, we introduce e-gH pairs and s-wcH pairs of modules, which generalize the generalized Hopfian modules and weakly co-Hopfian modules([3, 4, 8, 10]). We give some conditions under which if a pair of modules satisfies one of e-gH or s-wcH then its dual satisfies the other. In particular, if ${}_SU_R$ defines a Morita duality, M_R and N_R are U-reflexive then (M, N) is s-wcH (resp., e-gH) if and only if (N^*, M^*) is e-gH (resp., s-wcH).

Throughout this paper, R denotes an associative ring with identity and modules are unitary right R-modules. Let $K \trianglelefteq N$ denote that K is an essential submodule of N, and $K \ll N$ denote that K is a small submodule of N. For other definitions and notations in this paper we refer to [2].

2. E-small and s-essential submodules

In this section, as generalizations of small submodules and essential submodules, s-essential submodules and e-small submodules are introduced, and their various properties are given.

Definition 2.1. *Let N be a submodule of a module M.*

1. *N is said to be* e-small *in M (denoted by $N \ll_e M$), if $N + L = M$ with $L \trianglelefteq M$ implies $L = M$;*
2. *N is said to be* s-essential *in M (denoted by $N \trianglelefteq_s M$), if $N \cap L = 0$ with $L \ll M$ implies $L = 0$.*

Obviously, every small (δ-small) submodule of M is e-small in M, and every essential submodule of M is s-essential in M. The converses are false.

Example 2.2. Assume that $R = \mathbb{Z}$, $M = \mathbb{Z}_6$, $N = \{ \overline{0}, \overline{3} \}$ and $K = \{ \overline{0}, \overline{2}, \overline{4} \}$. Then

1. N is e-small in M. But M/K is singular and $N + K = M$. So N is not δ-small in M.
2. K is s-essential in M. However, $K \cap N = 0$, hence K is not essential in M.

It is proved in [13, Lemma 1.2] that $N \ll_\delta M$ if and only if $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$ whenever $X + N = M$. We generalize this as follows.

Proposition 2.3. *Let N be a submodule of a module M. The following are equivalent.*

(1) $N \ll_e M$;
(2) *if $X + N = M$, then X is a direct summand of M with M/X a semisimple module.*

Proof. (1) $\Rightarrow$ (2). Let Y be a complement of X in M, then $X \oplus Y \trianglelefteq M$. Since $X + Y + N = M$ and $N \ll_e M$, it follows that $X \oplus Y = M$. To see that $M/X \cong Y$ is semisimple, let A be a submodule of Y. Then $X + A + N = M$. Arguing as above with $X + A$ replacing X, we have that $X + A = X \oplus A$ is a direct summand of M, implying that A is a direct summand of Y, so M/X is semisimple.

(2) $\Rightarrow$ (1). Let $K \trianglelefteq M$ and $K + N = M$, then K is a direct summand of M, so $K = M$. We have $N \ll_e M$. ■

In particular, if M is a projective module, then every e-small submodule N of M is just a $\delta-$small submodule of M by Proposition 2.3 and [13, Lemma 1.2].

The next proposition, which will be used frequently, explains how close the notion of s-essential submodules is to that of essential submodules.

Proposition 2.4. *Let $0 \neq K \leq M$ be a module. Then $K \trianglelefteq_s M$ if and only if for each $0 \neq x \in M$, if $Rx \ll M$, then there is an element $r \in R$ such that $0 \neq rx \in K$.*

Proof. ($\Rightarrow$) Let K be a submodule of M and $K \trianglelefteq_s M$. For each $0 \neq x \in M$, if $Rx \ll M$, then $Rx \neq 0$ and $K \cap Rx \neq 0$. Thus there is an element $r \in R$ such that $0 \neq rx \in K$.

($\Leftarrow$) Suppose L is a small submodule of M and $0 \neq x \in L$. We have $Rx \ll M$, hence there exists an element $r \in R$ such that $0 \neq rx \in K \cap L$. That is, $K \trianglelefteq_s M$. ■

Proposition 2.5. *Let M be a module.*

(1) *Assume that N, K, L are submodules of M with $K \subseteq N$.*
    (*a*) *If $N \ll_e M$, then $K \ll_e M$ and $N/K \ll_e M/K$.*
    (*b*) *$N + L \ll_e M$ if and only if $N \ll_e M$ and $L \ll_e M$.*
(2) *If $K \ll_e M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_e N$. In particular, if $K \ll_e M \subseteq N$, then $K \ll_e N$.*
(3) *Assume that $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then $K_1 \oplus K_2 \ll_e M_1 \oplus M_2$ if and only if $K_1 \ll_e M_1$ and $K_2 \ll_e M_2$.*

Proof.

(1) (a) Suppose that $L \trianglelefteq M$ and $L+K=M$, then $N+L=M$, thus $L=M$ for $N \ll_e M$, so $K \ll_e M$. If $L \leq M$ with $L/K \trianglelefteq M/K$ and $L/K+N/K=M/K$, then $N+L=M$ and $L \trianglelefteq M$. Hence $L=M$ and $L/K=M/K$. Therefore $N/K \ll_e M/K$. (b) The necessity follows immediately from (a). Conversely, suppose $K \trianglelefteq M$ with $N+L+K=M$, then $L+K=M$ since $L+K \trianglelefteq M$ and $N \ll_e M$. Whence $K=M$ for $K \trianglelefteq M$ and $L \ll_e M$.

(2) Suppose that $A \trianglelefteq N$ and $A+f(K)=N$. Then $f^{\leftarrow}(A) \trianglelefteq M$, and $f^{\leftarrow}(A)+K=M$. Since $K \ll_e M$, we have $f^{\leftarrow}(A)=M$. Thus $f(K) \subseteq A$ and $A=N$. So $f(K) \ll_e N$.

(3) Immediate from (1) and (2). ■

It is proved in [13, Lemma 1.3] that if $K \ll_\delta M$ and $N/K \ll_\delta M/K$, then $N \ll_\delta M$. The following example shows that the converse of Proposition 2.5 (a) is false.

Example 2.6. Assume that $R=\mathbb{Z}$, $M=\mathbb{Z}_{24}$ $K=6\mathbb{Z}_{24}$ and $N=3\mathbb{Z}_{24}$. Then $K \ll M$ and $N/K \ll_e M/K$. But N is not e-small in M.

Dually, we have the following conclusions on s-essential submodules.

Proposition 2.7. *Let M be a module.*

(1) *Assume that N, K, L are submodules of M with $K \subseteq N$.*
  (a) *If $K \trianglelefteq_s M$, then $K \trianglelefteq_s N$ and $N \trianglelefteq_s M$.*
  (b) *$N \cap L \trianglelefteq_s M$ if and only if $N \trianglelefteq_s M$ and $L \trianglelefteq_s M$.*
(2) *If $K \trianglelefteq_s N$ and $f: M \to N$ is a homomorphism, then $f^{\leftarrow}(K) \trianglelefteq_s M$.*
(3) *Assume that $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M=M_1 \oplus M_2$, then $K_1 \oplus K_2 \trianglelefteq_s M_1 \oplus M_2$ if and only if $K_1 \trianglelefteq_s M_1$ and $K_2 \trianglelefteq_s M_2$.*

The converse of Proposition 2.7 (1)(a) is not true.

Example 2.8. Let $R=\mathbb{Z}$, $M=\mathbb{Z}_{36}$, $N=6\mathbb{Z}_{36}$ and $K=18\mathbb{Z}_{36}$. Then $K \trianglelefteq_s N$, $N \trianglelefteq_s M$. But K is not s-essential in M.

The socle and radical of a module are important in the study of modules and rings. In [13], the radical of a module M is generalized as follows

$$\delta(M)=\cap\{\ K \leq M \ \mid \ M/K \text{ is singular and simple}\}.$$

Furthermore, we have

Definition 2.9. *Let M be a module. Define*

$$Rad_e(M)=\cap\{\ N \trianglelefteq M \ \mid \ N \ \textit{ is maximal in } M\ \},$$

and

$$Soc_s(M)=\sum\{\ N \ll M \ \mid \ N \ \textit{ is minimal in } M\ \}.$$

Obviously,

$$Soc_s(M) \subseteq Rad(M) \subseteq \delta(M) \subseteq Rad_e(M)$$

and

$$Soc_s(M) \subseteq Soc(M) \subseteq Rad_e(M).$$

In the following we use e-small submodules and s-essential submodules to characterize $Rad_e(M)$ and $Soc_s(M)$.

Theorem 2.10. *Let M be a module. Then*

(1) $Rad_e(M) = \sum \{ N \subseteq M \mid N \ll_e M\}$.
(2) $Soc_s(M) = \cap\{ L \leq M \mid L \trianglelefteq_s M\}$.

Proof. (1). Let $U = \sum \{ N \subseteq M \mid N \ll_e M\}$. Suppose that $L \ll_e M$ and $K \trianglelefteq M$ is maximal in M, hence $L \leq K$. Otherwise, we have $K + L = M$. But $L \ll_e M$, hence $K = M$, a contradiction. It follows that $U \subseteq Rad_e(M)$.

On the other hand, for $x \in Rad_e(M)$ suppose that Rx is not e-small in M. Set

$$\Gamma = \{ B \mid B \neq M, \ \ B \trianglelefteq M \text{ and } Rx + B = M\}.$$

Clearly, Γ is a non-empty subposet of the lattice of submodules of M. By the Maximal Principle, Γ has a maximal element, say B_0. Now we claim that B_0 is maximal in M. Otherwise, there is a submodule C of M such that $B_0 \subsetneqq C \subsetneqq M$, thus

$$Rx + C \supseteq Rx + B_0 = M$$

and $C \trianglelefteq M$, hence $C \in \Gamma$, which contradicts the maximality of B_0. So B_0 is maximal in M and $B_0 \trianglelefteq M$. Thus $x \in Rad_e(M) \subseteq B_0$ and $Rx \subseteq B_0$. Since $Rx + B_0 = M$, it follows that $B_0 = M$, a contradiction. So $Rx \ll_e M$, hence $Rad_e(M) \subseteq U$. Therefore

$$Rad_e(M) = \sum\{ N \subseteq M \mid N \ll_e M\}.$$

(2). Let $S = \cap \{ L \leq M \mid L \trianglelefteq_s M\}$. Suppose that $L \trianglelefteq_s M$ and $K \ll M$ is minimal in M, then $K \leq L$. Otherwise, $K \cap L = 0$, hence $K = 0$, a contradiction. So $Soc_s(M) \subseteq S$. Note that $S \subseteq Soc(M)$, thus $Soc_s(M)$ and S are semisimple modules. If $S \nsubseteq Soc_s(M)$, there is a simple module T such that $T \leq S$ and T is not small in M. Let K be a proper submodule such that $K + T = M$.

(a) If $K \cap T \neq 0$, then $T \subseteq K$, hence $K = M$, a contradiction.
(b) If $K \cap T = 0$, then $M = K \oplus T$. For each $H \leq M$, if $H \ll M$ and $K \cap H = 0$, then $H + K$ is a proper submodule of M and $H \cong (H + K)/K$ is a submodule of M/K, where $M/K \cong T$ is a simple module. Thus $H = 0$. Then $K \trianglelefteq_s M$, that is, $T \subseteq S \subseteq K$, a contradiction.

Thus $T \ll M$, a contradiction. Therefore $S = Soc_s(M)$. ■

Corollary 2.11. *Let M and N be modules.*

1. *If $f : M \to N$ is an $R-$homomorphism, then $f(Rad_e(M)) \subseteq Rad_e(N)$. In particular, $Rad_e(M)$ is a fully invariant submodule of M.*
2. *If every proper essential submodule of M is contained in a maximal submodule of M, then $Rad_e(M)$ is the unique largest e-small submodule of M.*

Proof.

(1) By Proposition 2.5 and Proposition 2.10.

(2) For each essential submodule K of M, if $K \neq M$, there is a maximal submodule L of M such that $K \subseteq L$, then $L \trianglelefteq M$. By the definition of $Rad_e(M)$, $Rad_e(M) \subseteq L$. So $Rad_e(M) + K \subseteq L \subsetneq M$. Thus $Rad_e(M) \ll_e M$. ■

Dually, we have

Corollary 2.12. *Let M and N be modules. Then*

1. *If $f : M \to N$ is an $R-$homomorphism, then $f(Soc_s(M)) \subseteq Soc_s(N)$. Therefore, $Soc_s(M)$ is a fully invariant submodules of M.*
2. *If $M = \oplus_{i=1}^{n} M_i$, then $Soc_s(M) = \oplus_{i=1}^{n} Soc_s(M_i)$.*
3. *If every non-zero small submodule of M contains a minimal submodule of M, then $Soc_s(M)$ is the unique least s-essential submodule of M.*

Example 2.13. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_{24}$ and $N \leq M$. All submodules of M have the following properties.

$N \leq M$	small	e-small	essential	s-essential
$\mathbb{Z}_{24}$	✗	✗	✓	✓
$2\mathbb{Z}_{24}$	✗	✓	✓	✓
$3\mathbb{Z}_{24}$	✗	✗	✗	✓
$4\mathbb{Z}_{24}$	✗	✓	✓	✓
$6\mathbb{Z}_{24}$	✓	✓	✗	✓
$8\mathbb{Z}_{24}$	✗	✓	✗	✗
$12\mathbb{Z}_{24}$	✓	✓	✗	✓
0	✓	✓	✗	✗

According to the above chart, we have

(1) $Rad(M) = 6\mathbb{Z}_{24}$, $Rad_e(M) = 2\mathbb{Z}_{24}$, $Soc(M) = 4\mathbb{Z}_{24}$ and $Soc_s(M) = 12\mathbb{Z}_{24}$.

(2) $Soc_s(M) \subsetneqq Rad(M) \subsetneqq Rad_e(M)$ and $Soc_s(M) \subsetneqq Soc(M) \subsetneqq Rad_e(M)$.

3. E-small and s-essential homomorphisms

In this section, we use the concepts of e-small and s-essential submodules to characterize some properties of homomorphisms.

Definition 3.1. *Let M and N be modules.*

(1) *An epimorphism $g : M \to N$ is e-small in case $Ker g \ll_e M$.*
(2) *A monomorphism $f : M \to N$ is s-essential in case $Im f \trianglelefteq_s M$.*

In the following, we give a useful characterization of e-small homomorphisms and s-essential homomorphisms.

Proposition 3.2. *Let M and N be modules.*

(1) *An epimorphism $g : M \to N$ is e-small if and only if for each essential monomorphism h, if gh is epic, then h is epic.*
(2) *A monomorphism $f : M \to N$ is s-essential if and only if for each small epimorphism h, if hf is monic, then h is monic.*

Proof.
(1) Let $g : M \to N$ be an epimorphism and $K = Ker g$. Then there is a unique isomorphism $v : M/K \to N$, such that $v\pi = g$ where $\pi : M \to M/K$. Thus it follows that for each homomorphism h, $v\pi h = gh$ is epic if and only if πh is epic.

($\Rightarrow$) If g is e-small, then $K \ll_e M$. Since πh is epic, we have $Imh + K = M$. Note that h is an essential monomorphism, hence $Imh \trianglelefteq M$, thus $Imh = M$. So h is epic.

($\Leftarrow$) Let L be an essential submodule of M. Let $i_L : L \to M$ be the inclusion. Then i_L is essential. If $K + L = M$, then πi_L is epic. By hypothesis, i_L is epic, that is, $L = M$. So $K \ll_e M$, hence g is e-small.

(2) Dual to (1). ■

Proposition 3.3. *Suppose that the following diagram of modules and homomorphisms*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

is commutative and has exact rows.

(1) *If α is epic and g is e-small, then g' is e-small.*
(2) *If γ is monic and f' is s-essential, then f is s-essential.*

Proof.

(1) Assume that g is e-small, then $Kerg \ll_e B$ and $\beta(Kerg) \ll_e B'$. It suffices to show $Kerg' \leq \beta(Kerg)$. Let $b' \in Kerg'$. Since the bottom row is exact, there is an element $a \in A$ with $\alpha(a) = b'$. Since the diagram commutes and the top row is exact, $b' = f'\alpha(a) = \beta f(a)$ and $gf(a) = 0$. Thus there is a $f(a) \in Kerg$ such that $\beta(f(a)) = b'$. So $b' \in \beta(Kerg)$, hence $Kerg' \ll_e B'$.

(2) Dual to (1). ∎

Corollary 3.4. *Consider the following diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & B \arrow[d, "\beta_2"] \\ C \arrow[r, "\beta_1"'] & D \end{tikzcd}$$

(1) *Assume that the diagram is a pullback of β_1 and β_2. If β_1 is a s-essential monomorphism, so is α_1.*
(2) *Assume that the diagram is a pushout of α_1 and α_2. If α_1 is an e-small epimorphism, so is β_1.*

Proof.

(1) Assume that the diagram is a pullback of β_1 and β_2 with β_1 a s-essential monomorphism. Then we have a full commutative diagram with exact rows by [7, Proposition 5.1].

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & B \arrow[r, "\pi_1"] \arrow[d, "\beta_2"] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C \arrow[r, "\beta_1"'] & D \arrow[r, "\pi_2"] & L \arrow[r] & 0 \end{tikzcd}$$

By Proposition 3.3, α_1 is a s-essential monomorphism.

(2) Dual to (1). ∎

Let R and S be two rings, if $F : \text{Mod}-R \to \text{Mod}-S$ define a Morita equivalence, by Proposition 3.2 we note that $f : M \to N$ is e-small (resp., s-essential) if and only if $F(f) : F(M) \to F(N)$ is e-small (resp., s-essential).

For two rings R and S, a bimodule ${}_SU_R$ is said to *define a Morita duality*, if ${}_SU_R$ is a faithfully balanced bimodule such that ${}_SU$ and U_R are injective cogenerators. A presentation of Morita duality can be found in [2, §23, §24] and

[11]. If M is a right R-module (left S-module), we let $M^* = {}_S\mathrm{Hom}_R(M, U)$ ($=$ $\mathrm{Hom}_S(M, U)_R$), and M is said to be *U-reflexive* if the evaluation homomorphism $e_M : M \to M^{**}$ is an isomorphism. According to [2], let $\mathbf{R}_R[U]$ and ${}_S\mathbf{R}[U]$ denote the class of all U-reflexive right R-modules and that of all U-reflexive left S-modules, respectively.

Theorem 3.5. *Assume that ${}_SU_R$ defines a Morita duality and $f : M \to N$. If M, N are U-reflexive, then*

(1) *f is an e-small epimorphism if and only if $f^* : N^* \to M^*$ is a s-essential monomorphism.*
(2) *f is a s-essential monomorphism if and only if $f^* : N^* \to M^*$ is an e-small epimorphism.*

Proof. (1). Let $f : M \to N$ be an e-small epimorphism, then $f^* : N^* \to M^*$ is monic by [2, Corollary 24.2]. We claim that f^* is a s-essential monomorphism.

Suppose that $h : M^* \to H$ is such that hf^* is a monomorphism and h is a small epimorphism, then $(hf^*)^* = f^{**}h^*$ is an epimorphism and h^* is an essential monomorphism. Since M_R and N_R are U-reflexive, the evaluation homomorphisms $\sigma_M : M \to M^{**}$ and $\sigma_N : N \to N^{**}$ are isomorphisms, that is, the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\sigma_M"'] & N \arrow[d, "\sigma_N"] \\
M^{**} \arrow[r, "f^{**}"] & N^{**}
\end{tikzcd}$$

Since f is an e-small epimorphism, f^{**} is an e-small epimorphism. By Proposition 3.2, h^* is epic. By [2, Corollary 24.2] h is monic. Therefore f^* is a s-essential monomorphism by Proposition 3.2.

Conversely, let $f^* : N^* \to M^*$ be a s-essential monomorphism. By [2, Corollary 24.2], $f : M \to N$ is an epimorphism. We shall prove that f is e-small.

Suppose that $h : H \to M$ is an essential monomorphism such that fh is epimorphic, then $(fh)^* = h^*f^*$ is monic and h^* is a small epimorphism. By Proposition 3.2, h^* is monomorphism. By [2, Corollary 24.2], h is an epimorphism. So f is an e-small epimorphism by Proposition 3.2.

Dually, (2) can be proved. ■

4. On generalizations of Hopfian modules

According to [3, 4], a module M is generalized Hopfian (gH for short) if every surjective R-endomorphism f of M has a small kernel in M. Dually, a module M is weakly co-Hopfian (abbreviated wcH) if every injective endomorphism of M is essential. These notions have been extensively studied in [3, 4, 8, 10]. We generalize these notions as follows.

Definition 4.1. *Let M and N be modules.*

(1) *a pair (M, N) is called an e-gH pair of modules, if every epimorphism from M to N is e-small.*
(2) *a pair (M, N) is called a s-wcH pair of modules, if every monomorphism from M to N is s-essential.*

Clearly, (M, M) is an e-gH pair for each gH module M, and (N, N) is a s-wcH pair for each wcH module N.

Let R and S be two fixed rings and ${}_SU_R$ a left S right R bimodule. For $M \in$ Mod-R and $N \in S$-Mod, the U-duals $\mathrm{Hom}_R(M, U)$ and $\mathrm{Hom}_S(N, U)$ are denoted by M^* and N^* respectively. In this section we give some conditions under which if a pair of modules satisfies one of e-gH or s-wcH then its dual satisfies the other.

Proposition 4.2. *Assume that $M, N \in$ Mod-R and for each proper essential submodule L of M there exists $0 \neq g \in M^*$ such that $Sg \ll M^*$ and $g(L) = 0$. If (N^*, M^*) is s-wcH, then (M, N) is e-gH.*

Proof. Let $\varphi : M \to N$ be an epimorphism, then $\varphi^* : N^* \to M^*$ is a monomorphism, where $\varphi^*(f) = f\varphi$ for each $f \in N^*$. Suppose that $\ker \varphi + L = M$ and $L \trianglelefteq M$, then $\varphi(L) = \varphi(M) = N$. Since (N^*, M^*) is s-wcH, $Im\varphi^* \trianglelefteq_s M^*$. For each $0 \neq g \in M^*$ with $Sg \ll M^*$, by Proposition 2.4 there is an element $s \in S$ such that $0 \neq sg \in Im\varphi^*$, that is, there is a homomorphism $f \in N^*$ such that $0 \neq sg = \varphi^*(f) = f\varphi$. Thus

$$sg(L) = f\varphi(L) = f\varphi(M) \neq 0,$$

hence for every $0 \neq g \in M^*$ with $Sg \ll M^*$, we have that $g(L) \neq 0$. By hypothesis, $L = M$. Therefore (M, N) is e-gH. ■

Corollary 4.3. *Assume that ${}_SU_R$ defines a Morita duality, $M \in \boldsymbol{R}_R[U]$, $N \in$ Mod-R. If (N^*, M^*) is s-wcH, then (M, N) is e-gH.*

Proof. Let L be a proper essential submodule of M. Since U_R is a cogenerator, there is a map $f : M/L \to U$ with $0 \neq f\pi \in M^*$, where $\pi : M \to M/L$ is the canonical map. Thus $f\pi(L) = 0$. Since $L \trianglelefteq M$, it follows that $\pi^*((M/L)^*) \ll M^*$ by [2, Ex24.5]. Note that $Sf\pi = S\pi^*(f) \subseteq \pi^*((M/L)^*)$, thus $Sf\pi \ll M^*$. Therefore (M, N) is e-gH by Proposition 4.2. ■

In the following, we give the converse of above propositions.

Proposition 4.4. *Assume that ${}_SU_R$ defines a Morita duality, $M \in$ Mod-R and $N \in \boldsymbol{R}_R[U]$. If (N^*, M^*) is an e-gH pair of left S-modules, then (M, N) is a s-wcH pair of right R-modules.*

Proof. Suppose that $\varphi : M \to N$ is a monomorphism, $h : N \to H$ is a small epimorphism and $h\varphi$ is monic. Since U_R is injective, $\varphi^* : N^* \to M^*$ is an epimorphism, where $\varphi^*(f) = f\varphi$ for each $f \in N^*$. By [2, Ex24.5] $h^* : H^* \to N^*$ is an essential monomorphism and $\varphi^* h^*$ is epic. Since (N^*, M^*) is an e-gH pair, it follows that φ^* is e-small. Thus by Proposition 3.2 h^* is an epimorphism, hence an isomorphism. Since U_R is an injective cogenerator, h is an isomorphism, whence φ is s-essential by Proposition 3.2. So (M, N) is a s-wcH pair. ∎

Theorem 4.5. *Assume that ${}_SU_R$ defines a Morita duality, M_R and N_R are U-reflexive. Then*

(1) *(M, N) is s-wcH if and only if (N^*, M^*) is e-gH.*
(2) *(M, N) is e-gH if and only if (N^*, M^*) is s-wcH.*

Proof. Since ${}_SU_R$ define a Morita duality, it follows that ${}_SU_R$ is a faithfully balanced bimodules, U_R and ${}_SU$ are injective cogenerators.

(1) Let (M, N) be s-wcH. Since $M \cong M^{**}$ and $N \cong N^{**}$, we have that (M^{**}, N^{**}) is s-wcH. By Corollary 4.3, (N^*, M^*) is e-gH. Conversely, let (N^*, M^*) be e-gH. By Proposition 4.4, (M, N) is s-wcH.

(2) Let (M, N) be e-gH. Since $M \cong M^{**}$ and $N \cong N^{**}$, it follows that (M^{**}, N^{**}) is e-gH. By (1), (N^*, M^*) is s-wcH. Conversely, let (N^*, M^*) be s-wcH. By Corollary 4.3, (M, N) is e-gH. ∎

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