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On Intuitionistic (S, T)-Fuzzy H_v-Submodules of H_v-Modules*

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Abstract. In this paper, we apply the concept of intuitionistic fuzzy set to H_v-modules. The notion of intuitionistic (S, T)-fuzzy H_v-submodules of an H_v-module is introduced, and some related properties are investigated. Intuitionistic (S, T)-fuzzy relations on an H_v-module are discussed. In particular, we investigate connections H_v-modules with modules.

Keywords: H_v-ring; H_v-module; (Imaginable) intuitionistic (S, T)-fuzzy H_v-submodule; Intuitinistic (S, T)-fuzzy relation; Fundamental module.

1. Introduction

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The theory of algebraic hyperstructures (or hypersystems) is a well established branch of classical algebraic theory. In the literature, the theory of hyperstructure was first initiated by Marty in 1934 [11] when he defined the hypergroups and began to investigate their properties with applications to groups, rational functions and algebraic functions. Later on, many people have observed that the theory of hyper structures also have many applications in both pure and applied sciences, for example, semi-hypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. In a recent monograph of Corsini and Leoreanu [3], the authors have collected numerous applications of algebraic hyperstructures, especially those from the last fifteen years to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Further, Vougiouklis [16] introduced a new class of hyperstructures so-called H_v-structure, and Davvaz [7] surveyed the theory of H_v-structures.

After the introduction of fuzzy sets by Zadeh [19], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. Recently, Dudek et al. [10] considered the intuitionistic fuzzification of the concept of sub-hyperquasigroups in a hyperquasigroup and investigated some properties of such hyperquasigroups. Further, in [9] Davvaz et al. applied the concept of intuitionistic fuzzy sets to H_v-modules and investigated some related properties.

The fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now extensively applied to many disciplines. The relationships between the fuzzy sets and algebraic hyperstructures (structures) have been considered by Corsini, Davvaz, Leoreanu, Vougiouklis, Zhan and others. The reader is refereed to [2-10, 14-17, 20-25].

The work of this paper is organized as follows. In section 2, we first recall some basic definitions and results of H_v-modules. In section 3, by using these new fuzzy concept and new operations defined on intuitionistic fuzzy sets, we are able to extend some results of fuzzy H_v-submodules to intuitionistic (S,T)-fuzzy H_v-submodules of H_v-modules. Consequently, intuitionistic (S,T)-fuzzy relations on an H_v-module M are discussed in section 4. In particular, in section 5, we investigate connections H_v-submodules with submodules.

2. Preliminaries

A hyperstructure is a non-empty set H together with a map $\cdot : H \times H \to P^*(H)$ be a *hyperoperation*, where $P^*(H)$ is the set of all the non-empty subsets of H. The image of pair (x, y) is denoted by $x \cdot y$. If $x \in H$ and $A, B \subseteq H$, then by $A \cdot B$, $A \cdot x$ and $x \cdot B$, we mean $A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b$, $A \cdot x = A \cdot \{x\}$ and $x \cdot B = \{x\} \cdot B$, respectively. A hyperstructure $(H, \cdot)$ is called an H_v*-semigroup* if $(x \cdot y) \cdot z \bigcap x \cdot (y \cdot z) \neq \emptyset$ for all $x, y, z \in H$.

Definition 2.1. [15] *An H_v-ring is a system $(R,+,\cdot)$ with two hyperoperations satisfying the following axioms:*

(i) $(R,+)$ *is an H_v-group, i.e., $(x+y)+z\bigcap x+(y+z)\neq\emptyset$ for all $x,y,z\in R, a+R=R+a=R$ for all $a\in R$;*
(ii) $(R,\cdot)$ *is an H_v-semigroup;*
(iii) $(\cdot)$ *is weak distributive with respect to (+), i.e., $x\cdot(y+z)\bigcap(x\cdot y+x\cdot z)\neq\emptyset$ and $(x+y)\cdot z\bigcap(x\cdot z+y\cdot z)\neq\emptyset$.*

Definition 2.2. [15] *A non-empty set M is an H_v-module over an H_v-ring R if $(M,+)$ is a weak commutative H_v-group and there exists the map $\cdot : R\times M\to P^*(M)$ by $(r,x)\mapsto r\cdot x$ such that for all $a\in R$ and $x,y\in M$, we have*

(i) $a\cdot(x+y)\bigcap(a\cdot x+a\cdot y)\neq\emptyset$;
(ii) $(a+b)\cdot x\bigcap(a\cdot x+b\cdot x)\neq\emptyset$;
(iii) $(ab)\cdot x\bigcap a\cdot(b\cdot x)\neq\emptyset$.

We note that an H_v-module is a generalization of a module. For more definitions, results and applications on H_v-modules, we refer the reader to [4, 6-9, 15-16, 21, 25]. Note that by using P-hyperoperations, we can consider the structure of H_v-modules on modules.

The concept of fuzzy sets was introduced by Zadeh in 1965. A mapping $\mu : X\to[0,1]$, where X is an arbitrary non-empty set, is called a *fuzzy set* in X. The complement of μ, denoted by the $\overline{\mu}$, is the fuzzy set in X defined by $\overline{\mu}(x)=1-\mu(x)$.

For any fuzzy set μ in X and $\alpha\in[0,1]$, the set $U(\mu;\alpha)=\{x\in X\mid\mu(x)\geq\alpha\}$ is called an *upper α-level cut* of μ and the set $L(\mu;\alpha)=\{x\in X\mid\mu(x)\leq\alpha\}$ is called a *lower α-level cut* of μ.

Definition 2.3. [13] *Let f be a mapping from a set X to a set Y. Let μ be a fuzzy set in X and λ be a fuzzy set in Y. Then the inverse image $f^{-1}(\lambda)$ of λ is a fuzzy set in X defined by $f^{-1}(\lambda)=\lambda(f(x))$ for al $x\in X$.*

The image $f(\mu)$ of μ is the fuzzy set in Y defined by

$$f(\mu)(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)}\mu(x) & \text{if } f^{-1}(y)\neq\emptyset,\\ 0 & \text{otherwise,}\end{cases}$$

for all $y\in Y$. We have always $f(f^{-1}(\lambda))\leq\lambda$ and $\mu\leq f^{-1}(f(\mu))$.

In [5], Davvaz applied the concept of fuzzy sets to the algebraic hyperstructures. In particular, he defined the concept of a fuzzy H_v-submodule of an H_v-module which is a generalization of the concept of a fuzzy submodule (see [4]), and he studied further properties in [6-9].

Definition 2.4. [4] *Let M be an H_v-module over an H_v-ring R and μ be a fuzzy set of M. Then μ is said to be a fuzzy H_v-submodule of M if the following axioms hold:*

(FH1) $\min\{\mu(x), \mu(y)\} \leq \inf_{\alpha \in x+y} \mu(\alpha)$ *for all* $x, y \in M$*;*
(FH2) *for all* $x, a \in M$*, there exists* $y \in M$ *such that* $x \in a + y$ *and* $\min\{\mu(a), \mu(x)\} \leq \mu(y)$*;*
(FH3) *for all* $x, a \in M$*, there exists* $z \in M$ *such that* $x \in z + a$ *and* $\min\{\mu(a), \mu(x)\} \leq \mu(z)$*;*
(FH4) $\mu(x) \leq \inf_{\alpha \in r \cdot x} \mu(\alpha)$ *for all* $r \in R$ *and* $x \in M$*.*

Lemma 2.5. [9] *Let M_1 and M_2 be two H_v-modules over an H_v-ring R and $f : M_1 \to M_2$ a strong epimorphism. If N is an H_v-submodule of M_2, then $f^{-1}(N)$ is an H_v-submodule of M_1.*

As it is well-known, any function $\delta : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm, resp., (s-norm)* if it satisfies (i) $\delta(1,1) = 1$ (resp.,$\delta(x,0) = x$), (ii) $\delta(x,y) = \delta(y,x)$, (iii) $\delta(\delta(x,y),z) = \delta(x,\delta(y,z))$ and (iv) $\delta(x,u) \leq \delta(x,w)$ for all $x, y, z, u, w \in [0,1]$, where $u \leq w$. In particular, a t-norm (resp., s-norm) is called *imaginable* if $\delta(x,x) = x$, for all $x \in [0,1]$, see [18].

Now, we give the following definitions:

Definition 2.6. [9] *Let M be a module over a ring R. An intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ in M is called an intuitionistic fuzzy submodule of M if the following axioms hold:*

(IF1) $\alpha_A(0) = 1$ *and* $\beta_A(0) = 0$*;*
(IF2) $\min\{\alpha_A(x), \alpha_A(y)\} \leq \alpha_A(x-y)$ *and* $\max\{\beta_A(x), \beta_A(y)\} \geq \beta_A(x-y)$ *for all* $x, y \in M$*;*
(IF3) $\alpha_A(x) \leq \alpha_A(r \cdot x)$ *and* $\beta_A(x) \geq \beta_A(r \cdot x)$ *for all* $x \in M$ *and* $r \in R$*.*

Definition 2.7. [9] *An intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ in an H_v-module M over an H_v-ring R is said to be an intuitionistic fuzzy H_v-submodule of M if the following axioms hold:*

(IFH1) $\min\{\alpha_A(x), \alpha_A(y)\} \leq \inf_{z \in x+y} \alpha_A(\alpha)$ *and* $\max\{\beta_A(x), \beta_A(y)\} \geq \sup_{z \in x+y} \beta_A(\alpha)$ *for all* $x, y \in M$*;*
(IFH2) *for all* $x, a \in M$ *there exists* $y \in M$ *such that* $x \in a + y$ *and* $\min\{\alpha_A(a), \alpha_A(x)\} \leq \alpha_A(y)$ *and* $\max\{\beta_A(a), \beta_A(x)\} \geq \beta_A(y)$;
(IFH3) *For all* $x, a \in M$ *there exists* $z \in M$ *such that* $x \in z + a$ *and* $\min\{\alpha_A(a), \alpha_A(x)\} \leq \alpha_A(z)$ *and* $\max\{\beta_A(a), \beta_A(x)\} \geq \beta_A(z)$;
(IFH4) $\alpha_A(x) \leq \inf_{z \in r \cdot x} \alpha_A(z)$ *and* $\beta_A(x) \geq \sup_{z \in r \cdot x} \beta_A(z)$ *for all* $x \in M$ *and* $r \in R$*.*

3. Intuitionistic (S,T)-Fuzzy H_v-Submodules

In what follows, let M denote an H_v-module over an H_v-ring R unless otherwise.

Based on [9], we can extend the concept of the intuitionistic fuzzy H_v-submodules to the concept of intuitionistic (S,T)-fuzzy H_v-submodules in the following way:

Definition 3.1. *An intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ in M is called an intuitionistic fuzzy H_v-submodule of M with respect to t-norm T and s-norm S (briefly, intuitionistic (S,T)-fuzzy H_v-submodule of M) if it satisfies the following conditions:*

(ISTFH1) $T(\alpha_A(x), \alpha_A(y)) \le \inf_{z\in x+y} \alpha_A(\alpha)$ *and* $S(\beta_A(x), \beta_A(y)) \ge \sup_{z\in x+y} \beta_A(\alpha)$ *for all* $x, y \in M$;

(ISTFH2) *for all* $x, a \in M$ *there exists* $y \in M$ *such that* $x \in a + y$ *and* $T(\alpha_A(a), \alpha_A(x)) \le \alpha_A(y)$ *and* $S(\beta_A(a), \beta_A(x)) \ge \beta_A(y)$;

(ISTFH3) *for all* $x, a \in M$ *there exists* $z \in M$ *such that* $x \in z + a$ *and* $T(\alpha_A(a), \alpha_A(x)) \le \alpha_A(z)$ *and* $S(\beta_A(a), \beta_A(x)) \ge \beta_A(z)$.

(ISTFH4) $\alpha_A(x) \le \inf_{z\in r\cdot x} \alpha_A(z)$ *and* $\beta_A(x) \ge \sup_{z\in r\cdot x} \beta_A(z)$, *for all* $x \in M$ *and* $r \in R$.

Definition 3.2. *The norms T and S are called dual if for all $a, b \in [0,1]$, $\overline{T(a,b)} = S(\overline{a}, \overline{b})$.*

Lemma 3.3. *Let T and S be dual norms. If $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy H_v-submodule of M, then so is $\square A = (\alpha_A, \overline{\alpha_A})$.*

Proof. It is sufficient to show that $\overline{\alpha_A}$ satisfies the conditions of Definition 3.1. For all $x, y \in M$, we have $T(\alpha_A(x), \alpha_A(y)) \le \inf_{z\in x+y} \alpha_A(z)$ and so $T(1 - \overline{\alpha_A}(x), 1 - \overline{\alpha_A}(y)) \le \inf_{z\in x+y}(1 - \overline{\alpha_A}(z))$. Hence $T(1 - \overline{\alpha_A}(x), 1 - \overline{\alpha_A}(y)) \le 1 - \sup_{z\in x+y} \overline{\alpha_A}(z)$ which implies $\sup_{z\in x+y} \overline{\alpha_A}(z) \le 1 - T(1 - \overline{\alpha_A}(x), 1 - \overline{\alpha_A}(y)) = S(\overline{\alpha_A}(x), \overline{\alpha_A}(y))$ since T and S are dual.

Now, let $a, x \in M$. Then exists $y \in M$ such that $x \in a + y$ and $T(\alpha_A(a), \alpha_A(x)) \le \alpha_A(y)$. It follows that that $T(1 - \overline{\alpha_A}(a), 1 - \overline{\alpha_A}(x)) \le 1 - \overline{\alpha_A}(y)$, so that $\overline{\alpha_A}(y) \le 1 - T(1 - \overline{\alpha_A}(a), 1 - \overline{\alpha_A}(x)) = S(\overline{\alpha_A}(a), \overline{\alpha_A}(x))$.

Similarly, let $a, x \in M$, then there exists $z \in M$ such that $x \in z + a$ and $\overline{\alpha_A}(z) \le S(\overline{\alpha_A}(a), \overline{\alpha_A}(x))$.

Now, let $x \in M$ and $r \in R$, we have $\alpha_A(x) \le \inf_{z\in r\cdot x} \alpha_A(z)$ since α_A is a T-fuzzy H_v-submodule of M. Hence $1 - \overline{\alpha_A}(x) \le \inf_{z\in r\cdot x}(1 - \overline{\alpha_A}(z))$ which implies $\sup_{z\in r\cdot x} \overline{\alpha_A}(z) \le \overline{\alpha_A}(x)$. Therefore $\square A = (\alpha_A, \overline{\alpha_A})$ is an intuitionistic (S,T)-fuzzy H_v-submodule of M. ∎

Lemma 3.4. *Let T and S be dual norms. If $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy H_v-submodule of M, then so is $\lozenge A = (\overline{\beta_A}, \beta_A)$.*

Proof. The proof is similar to the proof of Lemma 3.3. ■

Combining the above two lemmas it is not difficult to verify that the following theorem is valid.

Theorem 3.5. *Let T and S be dual norms. Then $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy H_v-submodule of M if and only if $\Box A$ and $\Diamond A$ are intuitionistic (S,T)-fuzzy H_v-submodules of M.*

Corollary 3.6. *Let T and S be dual norms. Then $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy H_v-submodule of M if and only if α_A and $\overline{\beta_A}$ are T-fuzzy H_v-submodules of M.*

Definition 3.7. *An intuitionistic (S,T)-fuzzy H_v-submodule $A = (\alpha_A, \beta_A)$ of M is said to be imaginable if α_A and β_A satisfy the imaginable property.*

The following are obvious.

Lemma 3.8. *Every imaginable intuitionistic (S,T)-fuzzy H_v-submodule of M is intuitionistic fuzzy H_v-submodule.*

Lemma 3.9. [4] *A fuzzy set μ in M is a fuzzy H_v-submodule of M if and only if the non-empty $U(\mu;\alpha), \alpha \in [0,1]$ is an H_v-submodule of M.*

Lemma 3.10. [4] *A fuzzy set μ in M is a fuzzy H_v-submodule of M if and only if the non-empty $\overline{\mu}$ is an anti-fuzzy H_v-submodule of M.*

By the above Lemmas, we can give the following results.

Theorem 3.11. *If $A = (\alpha_A, \beta_A)$ is an imaginable intuitionistic fuzzy set in M. Then $A = (\alpha_A, \beta_A)$ is an intuitionistic (S,T)-fuzzy H_v-submodule of M if and only if the non-empty sets $U(\alpha_A;\alpha)$ and $L(\beta_A;\alpha)$ are H_v-submodules of M, for every $\alpha \in [0,1]$.*

Theorem 3.12. *Let $A = (\alpha_A, \beta_A)$ be an intuitionistic (S,T)-fuzzy H_v-submodule of M. Then $\alpha_A(x) = \sup\{\alpha \in [0,1] | x \in U(\alpha_A;\alpha)\}$ and $\beta_A(x) = \inf\{\alpha \in [0,1] | x \in L(\beta_A;\alpha)\}$, for all $x \in M$.*

Definition 3.13. *Let $f : M \to M'$ be a strong epimorphism of H_v-modules. If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy set in M', then the inverse image of A under f, denoted by $f^{-1}(A)$, is an intuitionistic fuzzy set in M, defined by $f^{-1}(A) = (f^{-1}(\alpha_A), f^{-1}(\beta_A))$.*

By the above Definition, we can give the following result.

Theorem 3.14. *Let $f : M \to M'$ be a strong epimorphism of H_v-modules. If $A = (\alpha_A, \beta_A)$ is an intuitionistic (S, T)-fuzzy H_v-submodule of M'. Then the inverse image $f^{-1}(A) = (f^{-1}(\alpha_A), f^{-1}(\beta_A))$ of A under f is an intuitionistic (S, T)-fuzzy H_v-submodule of M.*

4. Intuitionistic (S, T)-Fuzzy Relations

We first recall that a fuzzy relation on any set X is a fuzzy set $\mu : X \times X \to [0, 1]$. We now give the following definitions and cite some known results, see [21].

Definition 4.1. *An intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ is called an intuitionistic fuzzy relation on any set X if α_A and β_A are fuzzy relations on X.*

Definition 4.2. *Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be intuitionstic fuzzy sets on a set X. If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy relation on X , then $A = (\alpha_A, \beta_A)$ is called an intuitionistic (S, T)-fuzzy relation on $B = (\alpha_B, \beta_B)$ if $\alpha_A(x, y) \leq T(\alpha_B(x), \alpha_B(y))$ and $\beta_A(x, y) \geq S(\beta_B(x), \beta_B(y))$, for all $x, y \in X$.*

Definition 4.3. *The intuitionistic (S, T)-Cartesian product of A and B, denoted by $A \times B$, is an intuitionistic fuzzy set on X, which is defined by $A \times B = (\alpha_A, \beta_A) \times (\alpha_B, \beta_B) = (\alpha_A \times \alpha_B, \beta_A \times \beta_B)$, where $(\alpha_A \times \alpha_B)(x, y) = T(\alpha_A(x), \alpha_B(y))$ and $(\beta_A \times \beta_B)(x, y) = S(\beta_A(x), \beta_B(y))$ hold for all $x, y \in X$.*

Lemma 4.4. *If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic fuzzy sets on a set X. Then we have*

(i) *$A \times B$ is an intuitionistic (S, T)-fuzzy relation on X;*

(ii) *$U(\alpha_A \times \alpha_B; \alpha) = U(\alpha_A; \alpha) \times U(\alpha_B; \alpha)$ and $U(\beta_A \times \beta_B; \alpha) = U(\beta_A; \alpha) \times U(\beta_B; \alpha)$ for all $\alpha \in [0, 1]$.*

Definition 4.5. *If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic fuzzy sets on a set X, the strongest intuitionistic (S, T)-fuzzy relation on X is defined by $A_B = (\alpha_{A_{\alpha_B}}, \beta_{A_{\beta_B}})$, where $\alpha_{A_{\alpha_B}}(x, y) = T(\alpha_B(x), \alpha_B(y))$ and $\beta_{A_{\beta_B}}(x, y) = S(\beta_B(x), \beta_B(y))$ for all $x, y \in X$.*

Lemma 4.6. *For the intuitionistic fuzzy sets $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ on a set X, let A_B be the strongest intuitionistic (S, T)-fuzzy relation on X. Then for any $\alpha \in [0, 1]$, we have $U(\alpha_{A_{\alpha_B}}; \alpha) = U(\alpha_B; \alpha) \times U(\alpha_B; \alpha)$ and $L(\beta_{A_{\beta_B}}; \alpha) = L(\beta_B; \alpha) \times L(\beta_B; \alpha)$.*

Lemma 4.7. [18] *For all $\alpha, \beta, \delta, \gamma \in [0, 1]$, we have $T(T(\alpha, \beta), T(\gamma, \delta)) = T(T(\alpha, \gamma),$*

$(\beta, \delta)); S(S(\alpha, \beta), S(\gamma, \delta)) = S(S(\alpha, \gamma), S(\beta, \delta))$.

By using the above lemmas, we have the following theorem.

Theorem 4.8. *If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic (S,T)-fuzzy H_v-submodules of M. Then $A \times B$ is an intuitionistic (S,T)-fuzzy H_v-submodule of $M \times M$.*

Corollary 4.9. *If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are imaginable intuitionistic (S,T)-fuzzy H_v-submodules of M. Then $A \times B$ is an imaginable intuitionistic (S,T)-fuzzy H_v-submodule of $M \times M$.*

The following theorem characterizes the imaginable intuitionistic (S,T)-fuzzy H_v-submodules on H_v-modules.

Theorem 4.10. *If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are imaginable intuitionistic fuzzy sets of M and A_B is the strongest intuitionistic (S,T)-fuzzy relation on M. Then $B = (\alpha_B, \beta_B)$ is an imaginable intuitionistic (S,T)-fuzzy H_v-submodule of M if and only if A_B is an imaginable intuitionistic (S,T)-fuzzy H_v-submodule of $M \times M$.*

Proof. Let $B = (\alpha_B, \beta_B)$ be an imaginable intuitionistic (S,T)-fuzzy H_v-submodule of M. Then we can verify the following conditions:

(ISTFH1) Let $x = (x_1, x_2), y = (y_1, y_2) \in M \times M$. For any $z = (z_1, z_2) \in x + y$, we have

$$\begin{aligned}
\inf_{z \in x+y} \alpha_{A_{\alpha_B}}(z) &= \inf_{(z_1,z_2)\in(x_1,x_2)+(y_1,y_2)} \alpha_{A_{\alpha_B}}(z_1, z_2) \\
&= \inf_{(z_1,z_2)\in(x_1+y_1,x_2+y_2)} \{T(\alpha_B(z_1), \alpha_B(z_2))\} \\
&= T(\inf_{z_1\in x_1+y_1} \alpha_B(z_1), \inf_{z_2\in x_2+y_2} \alpha_B(z_2))) \\
&\geq T(T(\alpha_B(x_1), \alpha_B(y_1)), T(\alpha_B(x_2), \alpha_B(y_2))) \\
&= T(T(\alpha_B(x_1), \alpha_B(x_2)), T(\alpha_B(y_1), \alpha_B(y_2))) \\
&= T(\alpha_{A_{\alpha_B}}(x_1, x_2), \alpha_{A_{\alpha_B}}(y_1, y_2)) \\
&= T(\alpha_{A_{\alpha_B}}(x), \alpha_{A_{\alpha_B}}(y)).
\end{aligned}$$

Similarly, we have $\sup_{z\in x+y} \beta_{A_{\beta_B}}(z) \leq S(\beta_{A_{\beta_B}}(x), \beta_{A_{\beta_B}}(y))$.

(ISTFH2) for all $x = (x_1, x_2), a = (a_1, a_2) \in M \times M$. Then $y_1, y_2 \in M$ such that $x_1 \in a_1 + y_1$ and $x_2 \in a_2 + y_2$, and thus $(x_1, x_2) \in (a_1 + y_1, a_2 + y_2) = (a_1, a_2) + (y_1, y_2)$. Moreover, we have

$$\begin{aligned}
\alpha_{A_{\alpha_B}}(y) = \alpha_{A_{\alpha_B}}(y_1, y_2) &= T(\alpha_B(y_1), \alpha_B(y_2)) \\
&\geq T(T(\alpha_B(a_1), \alpha_B(x_1)), T(\alpha_B(a_2), \alpha_B(x_2)))
\end{aligned}$$

$$
\begin{aligned}
&= T(T(\alpha_B(a_1),\alpha_B(a_2)),T(\alpha_B(x_1),\alpha_B(x_2)))\\
&= T(\alpha_{A_{\alpha_B}}(a_1,a_2),\alpha_{A_{\alpha_B}}(x_1,x_2))\\
&= T(\alpha_{A_{\alpha_B}}(a),\alpha_{A_{\alpha_B}}(x))
\end{aligned}
$$

Similarly, $\beta_{A_{\beta_B}}(y) \le S(\beta_{A_{\beta_B}}(a),\beta_{A_{\beta_B}}(x))$.

(ISTFH3) is similar to (ISTFH2).

(IFH4) let $x = (x_1,x_2) \in M \times M$ and $r = (r_1,r_2) \in R \times R$. For any $z = (z_1,z_2) \in (r_1,r_2)\cdot(x_1,x_2)$, we have

$$
\begin{aligned}
\inf_{z\in r\cdot x} \alpha_{A_{\alpha_B}}(z) &= \inf_{(z_1,z_2)\in(r_1,r_2)\cdot(x_1,x_2)} \alpha_{A_{\alpha_B}}(z_1,z_2)\\
&= \inf_{(z_1,z_2)\in(r_1\cdot x_1,r_2\cdot x_2)} T(\alpha_B(z_1),\alpha_B(z_2))\\
&\ge T(\inf_{z_1\in r_1\cdot x_1}\alpha_B(z_1),\inf_{z_2\in r_2\cdot x_2}\alpha_B(z_2))\\
&\ge T(\alpha_B(x_1),\alpha_B(x_2))\\
&= \alpha_{A_{\alpha_B}}(x_1,x_2) = \alpha_{A_{\alpha_B}}(x)
\end{aligned}
$$

Similarly, $\sup\limits_{z\in r\cdot x} \beta_{A_{\beta_B}}(z) \le \beta_{A_{\beta_B}}(x)$.

This shows that A_B is an intuitionistic (S,T)-fuzzy H_v-submodule of $M\times M$.

Now, for any $x = (x_1,x_2) \in M \times M$, we can easily show that $T(\alpha_{A_{\alpha_B}}(x),\alpha_{A_{\alpha_B}}(x)) = \alpha_{A_{\alpha_B}}(x)$ and $S(\beta_{A_{\beta_B}}(x),\beta_{A_{\beta_B}}(x)) = \beta_{A_{\beta_B}}(x)$.

Hence, A_B is an imaginable intuitionistic (S,T)-fuzzy H_v-submodule of $M \times M$.

To prove the converse of the theorem, we need prove the conditions (ISTFH1)-(IFH4) hold.

(ISTFH1) Let $x,y \in M$. Then we have

$$
\begin{aligned}
\inf_{z\in x+y}\alpha_B(z) &= \inf_{z\in x+y} T(\alpha_B(z),\alpha_B(z)) = \inf_{z\in x+y}\alpha_{A_{\alpha_B}}(z,z)\\
&= \inf_{(z,z)\in(x,x)+(y,y)} \alpha_{A_{\alpha_B}}(z,z)\\
&\ge T(\alpha_{A_{\alpha_B}}(x,x),\alpha_{A_{\alpha_B}}(y,y))\\
&\ge T(T(\alpha_B(x),\alpha_B(x)),T(\alpha_B(y),\alpha_B(y)))\\
&= T(\alpha_B(x),\alpha_B(y)).
\end{aligned}
$$

Similarly, we have $\sup_{z\in x+y}\beta_B(z) \le S(\beta_B(x),\beta_B(y))$.

(ISTFH2) for all $x,a \in M$, and thus $(x,x),(a,a) \in M\times M$. Then there exists $(y,y) \in M$ such that $(x,x) \in (a,a)+(y,y) = (a+y,a+y)$. That is, $x \in a+y$. Moreover, we have

$$
\begin{aligned}
\alpha_B(y) &= T(\alpha_B(y),\alpha_B(y)) = \alpha_{A_{\alpha_B}}(y,y)\\
&\ge T(\alpha_{A_{\alpha_B}}(a,a),\alpha_{A_{\alpha_B}}(x,x))\\
&= T(T(\alpha_B(a),\alpha_B(a)),T(\alpha_B(x),\alpha_B(x)))\\
&= T(\alpha_B(a),\alpha_B(x)).
\end{aligned}
$$

Similarly, $\beta_B(y) \leq S(\beta_B(a), \beta_B(x))$.
(ISTFH3) is similar to (ISTFH2).
(IFH4) let $x \in M$ and $r \in R$, we have

$$\begin{aligned}\inf_{z \in r \cdot x} \alpha_B(z) &= \inf_{z \in r \cdot x} T(\alpha_B(z), \alpha_B(z)) = \inf_{(z,z) \in (r,r)\cdot(x,x)} \alpha_{A_{\alpha_B}}(z,z) \\ &\geq \alpha_{A_{\alpha_B}}(x,x) = T(\alpha_B(x), \alpha_B(x)) \\ &= \alpha_B(x).\end{aligned}$$

Similarly, $\sup_{z \in r \cdot x} \beta_B(z) \leq \beta_{A_{\beta_B}}(x)$.

This shows that conditions (ISTFH1)-(IFH4) hold and hence $B = (\alpha_B, \beta_B)$ is an imaginable intuitionistic (S,T)-fuzzy H_v-submodule of M. ∎

Definition 4.11. *If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are imaginable intuitionistic fuzzy sets on any set X, then the intuitionistic (S,T)-product of A and B, denoted by $[A \cdot B]_{(S,T)}$, is defined by*

$$\begin{aligned}[A \cdot B]_{(S,T)} &= [(\alpha_A, \beta_A) \cdot (\alpha_B, \beta_B)]_{(S,T)} \\ &= ([\alpha_A \cdot \alpha_B], [\beta_A \cdot \beta_B])_{(S,T)} \\ &= ([\alpha_A \cdot \alpha_B]_T, [\beta_A \cdot \beta_B]_S),\end{aligned}$$

where $[\alpha_A \cdot \alpha_B]_T(x) = T(\alpha_A(x), \alpha_B(x))$ and $[\beta_A \cdot \beta_B]_S(x) = S(\beta_A(x), \beta_B(x))$, for all $x \in X$.

Theorem 4.12. *If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are imaginable intuitionistic (S,T)-fuzzy H_v-submodules of M. If T^*(resp. S^*) is a t-norm (resp. s-norm) which dominates T (resp. S), that is, $T^*(T(\alpha,\beta), T(\gamma,\delta)) \geq T(T^*(\alpha,\gamma), T^*(\beta,\delta))$ and $S^*(S(\alpha,\beta), S(\gamma,\delta)) \leq S(S^*(\alpha,\gamma), S^*(\beta,\delta))$ for all α, $\beta, \gamma, \delta \in [0,1]$. Then for the intuitionistic (S^*,T^*)-product of A and B, $[A \cdot B]_{(S*,T*)}$ is an intuitionistic (S,T)-fuzzy H_v-submodule of M.*

Proof. In proving this theorem, we only need verify that the conditions (ISTFH1)-(IFH4) hold. The verification is montioned and we omit the details. ∎

Let $f : M \to M'$ be a strong epimorphism of H_v-modules. Let T (resp. S) and T^* (resp. S^*) be the t-norms (resp. s-norms) such that T^* (resp. S^*) dominates T (resp. S). If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are imaginable intuitionistic fuzzy H_v-submodules of M', then the intuitionistic (S^*,T^*)-product of A and B, we have $[A \cdot B]_{(S*,T*)}$ is an intuitionistic (S,T)-fuzzy H_v-submodule of M'. Since every strong epimorphic inverse image of an intuitionistic (S,T)-fuzzy H_v-submodule is an intuitionistic (S,T)-fuzzy H_v-submodule, the inverse images $f^{-1}(A), f^{-1}(B)$, and $f^{-1}([A \cdot B]_{(S*,T*)})$ are also intuitionistic (S,T)-fuzzy H_v-submodules of M. In the next theorem, we described that the relation between $f^{-1}([A \cdot B]_{(S*,T*)})$ and intuitionistic (S^*,T^*)-product $[f^{-1}(A) \cdot f^{-1}(B)]_{(S*,T*)}$ of $f^{-1}(A)$ and $f^{-1}(B)$.

Based on the above discussion, we have:

Theorem 4.13. *Let $f : M \to M'$ be a strong epimorphism of H_v-modules. Let T^* (resp. S^*) be a t-norm (resp. s-norm) such that T^* (resp. S^*) dominates T (resp. S). If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic (S,T)-fuzzy H_v-submodule of M'. Then for the intuitionistic (S^*, T^*)-product $[A \cdot B]_{(S*,T*)}$ of A and B and the intuitionistic (S^*, T^*)-product $[f^{-1}(A) \cdot f^{-1}(B)]_{(S*,T*)}$ of $f^{-1}(A)$ and $f^{-1}(B)$, we have $f^{-1}([A \cdot B]_{(S*,T*)}) = [f^{-1}(A) \cdot f^{-1}(B)]_{(S*,T*)}$*

5. On Fundamental Modules

The main tools in the theory of an H_v-structures are fundamental relations. Consider an H_v-module M over an H_v-ring R. The relation γ^* is the smallest equivalence relation on R such that the quotient R/γ^* is a ring. We know that γ^* is called the fundamental equivalence relation on R and R/γ^* is called the fundamental ring, see [14,16]. The fundamental relation ϵ^* on M over R is the smallest equivalence relation such that M/γ^* is a module over the ring R/γ^*.

Let $\mathcal{U}$ be the set of all expressions consisting of finite hyperoperations of either on R and M or the external hyperoperation applied on finite sets of R and M. Then a relation ϵ can be defined on M whose transitive closure is the fundamental relation ϵ^*. The relation ϵ is as follows: $x\epsilon y$ iff $\{x, y\} \subseteq u$ for some $u \in \mathcal{U}$.

Theorem 5.1. [15] *The fundamental relation ϵ^* is the transitive closure of the relation ϵ.*

Suppose $\gamma^*(r)$ is the equivalence class containing $r \in R$ and $\epsilon^*(x)$ is the equivalence class containing $x \in M$. On M/ϵ^* the sum $\bigoplus$ and the external product $\bigodot$ using the γ^* classes in R, are defined as follows:

$$\epsilon^*(x) \bigoplus \epsilon^*(y) = \epsilon^*(c) \quad \text{for all } c \in \epsilon^*(x) + \epsilon^*(y),$$
$$\gamma^*(r) \bigodot \epsilon^*(x) = \epsilon^*(d) \quad \text{for all } d \in \gamma^*(r) \cdot \epsilon^*(x).$$

The kernel of the canonical map $\phi : M \to M/\epsilon^*$ is called the core of M and is denoted by ω_M. Here we also denote by ω_M the unit element of the group $(M/\epsilon^*, \bigoplus)$.

Now, we introduce the following definitions:

Definition 5.2. *Let M be a module over a ring R. An intuitionistic fuzzy set $A = (\alpha_A, \beta_A)$ in M is called an intuitionistic (S,T)-fuzzy submodule of M if the following axioms hold:*

(IF1) $\alpha_A(0) = 1$ *and* $\beta_A(0) = 0$*;*

(ISIF2) $T(\alpha_A(x), \alpha_A(y)) \leq \alpha_A(x-y)$ *and* $S(\beta_A(x), \beta_A(y)) \geq \beta_A(x-y)$ *for all* $x, y \in M$;

(IF3) $\alpha_A(x) \leq \alpha_A(r \cdot x)$ *and* $\beta_A(x) \geq \beta_A(r \cdot x)$ *for all* $x \in M$ *and* $r \in R$.

Definition 5.3. *Let* M *be an* H_v*-submodule over an* H_v*-ring* R *and let* $A = (\alpha_A, \beta_A)$ *be an intuitionistic fuzzy set of* M. *The intuitionistic fuzzy set* $A_{\epsilon^*} = (\alpha_{A_{\epsilon^*}}, \beta_{A_{\epsilon^*}})$ *is defined as follows:*

$$\alpha_{A_{\epsilon^*}} : M/\epsilon^* \to [0,1] \quad \text{and} \quad \beta_{A_{\epsilon^*}} : M/\epsilon^* \to [0,1]$$

such that

$$\alpha_{A_{\epsilon^*}}(\epsilon^*(x)) = \begin{cases} \sup\limits_{a \in \epsilon^*(x)} \alpha_A(a) & \text{if } \epsilon^*(x) \neq \omega_M, \\ 1 & \text{otherwise,} \end{cases}$$

$$\beta_{A_{\epsilon^*}}(\epsilon^*(x)) = \begin{cases} \inf\limits_{a \in \epsilon^*(x)} \beta_A(a) & \text{if } \epsilon^*(x) \neq \omega_M, \\ 0 & \text{otherwise,} \end{cases}$$

From the above discussion, we can show that the following result and omit the details.

Theorem 5.4. *Let* M *be an* H_v*-module over an* H_v*-ring* R *and let* $A = (\alpha_A, \beta_A)$ *be an intuitionistic* (S,T)*-fuzzy* H_v*-submodule of* M. *Then* $A_{\epsilon^*} = (\alpha_{A_{\epsilon^*}}, \beta_{A_{\epsilon^*}})$ *is an intuitionistic* (S,T)*-fuzzy submodule of the module* M/ϵ^*.

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