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On Serial Artinian Modules and Their Endomorphism Rings

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Abstract. It was shown that for an Artinian serial right R-module M, if M is quasi-p-injective and a progenerator in $\sigma[M]$, then the Endomorphism ring $S = End(M_R)$ is serial Artinian (two-sided). As a corollary, every right serial right Artinian ring which is right self-p-injective is left serial left Artinian.

Keywords: Serial module; Indecomposable module; Quasi-injective module; Quasi-projective module; Artinian module.

1. Introduction and Preliminaries

Throughout this paper, R is an associative ring with identity and Mod-R is the category of unitary right R-modules. The notations M_R or M will usually be a unitary right R-module and $S = End_R(M)$, its endomorphism ring. A right R-module M is called *uniserial* if the lattice of its submodules is linearly ordered

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by inclusion, i.e., if A and B are submodules of M, either $A \subset B$ or $B \subset A$. A ring R is *right uniserial* if it is uniserial as a right R-module. Note that left and right uniserial rings are in particular local rings.

A right R-module M is called a *serial module* if it is a direct sum of uniserial modules. Note that submodules and factor modules of serial modules need not to be serial. A ring R is *right serial* if it is serial as a right R-module. So a ring R is right serial if there are orthogonal idempotents $e_1, \ldots, e_n$ of R such that $R = e_1R \oplus \cdots \oplus e_nR$ and each e_iR is uniserial as a right R-module. We say R is *serial* if R is left and right serial.

A right R-module N is called *M-generated* if there exists an epimorphism $M^{(I)} \longrightarrow N$ for some index set I. If I is finite, then N is called *finitely M-generated.* Factor modules of M are called *M- cyclic.* We denote $\sigma[M]$, the full subcategory of Mod-R, whose objects are submodules of M-generated modules. M is a *subgenerator* if it generates all $N \in \sigma[M]$, and a *self-generator* if it generates all its submodules.

Following [10], for a given right R-module M, a right R-module N is called *M-p-injective* if every homomorphism from an M-cyclic submodule of M to N can be extended to one from M to N. M is called *quasi-p-injective* if it is M-p-injective. A ring R is right *self-p-injective* if R_R is quasi-p-injective as a right R-module. For more details of finite injectivity, we can refer to [6, 10].

2. Results

First we need some Lemmas.

Lemma 2.1. [12, Theorem 31.11] *Let M be an Artinian right R-module which is finitely generated and quasi-projective. Then the endomorphism ring $S = \mathrm{End}(M_R)$ is right Artinian.*

Lemma 2.2. [12, Theorem 55.2] *Let M be a finitely generated, quasi-projective module. If M is serial, then $S = \mathrm{End}(M_R)$ is right serial.*

Lemma 2.3. *Let M be a serial right R-module. If M is finitely generated and quasi-p-injective, then the endomorphism ring $S = \mathrm{End}(M_R)$ is left serial. In particular, a right serial, right self-p-injective ring is left serial.*

Proof. Since M is finitely generated and serial, we can write $M = \bigoplus_{i=1}^n U_i$ where each U_i is uniserial. Since every R-homomorphism f from U_i to M can be considered as an element of S and $\mathrm{End}(M_R) = \mathrm{End}(\bigoplus_{i=1}^n U_i) \simeq \bigoplus_{j=1}^n \mathrm{Hom}(U_j, M)$, it is enough to show that $\mathrm{Hom}(U_j, M)$ is uniserial as a left ideal of S.

Take any $f, g \in \mathrm{Hom}(U_j, M)$. Since M_j is uniserial, we can suppose that $\mathrm{Ker}(g) \subset \mathrm{Ker}(f)$. Considering $g : U_j \to g(U_j)$ as an epimorphism, we can find $\varphi : g(U_j) \to M$ such that $\varphi g = f$. Since M is quasi-p-injective and $g(U_j)$ is an M-cyclic submodule of M, there is an R-homomorphism $\psi \in \mathrm{End}(M_R)$ such that $\psi_{|_{g(U_j)}} = \varphi$. It means that for any $x \in U_j, \psi(g(x)) = \varphi(g(x)) = f(x)$. Hence $f = \psi g \in Sg$, proving that $Sf \subset Sg$ and therefore $\text{Hom}(U_j, M)$ is uniserial. ■

Theorem 2.4. *Let M be an Artinian right R-module which is quasi-p-injective and a projective subgenerator with $S = \text{End}(M_R)$. Then the following conditions are equivalent:*

(1) *M is serial;*
(2) *S is Artinian and serial;*
(3) *Every indecomposable module in $\sigma[M]$ is quasi-injective;*
(4) *Every indecomposable module in $\sigma[M]$ is quasi-projective;*
(5) *Every indecomposable quasi-injective module in $\sigma[M]$ is quasi-projective;*
(6) *Every indecomposable quasi-projective module in $\sigma[M]$ is quasi-injective.*

Proof. Since M is a projective generator in $\sigma[M]$, we can see that the category $\sigma[M]$ is equivalent to the category Mod-S of right S-modules, by [12, Theorem 46.2].

(1) $\Rightarrow$ (2). Since M is serial, finitely generated and quasi-projective, the endomorphism ring $S = \text{End}(M_R)$ is right serial by Lemma 2.2 and left serial by Lemma 2.3.

Using Lemma 2.1, we see that S is right Artinian. We now show that S is left Artinian. Since S is left serial, $S = \bigoplus_{i=1}^{k} S_i$ with each S_i is a uniserial left ideal. Therefore every finitely generated left ideal I which is contained in S_i is cyclic and hence I is isomorphic to $\text{Hom}(M/\text{Ker}I, M)$ since $\text{Ker}I = \text{Ker}(f)$ for some $f \in S$ and M is quasi-p-injective. Now let $I_1 \subset I_2 \subset I_3 \subset ... \subset I_n \subset ...$ be an ascending chain of finitely generated left ideals contained in S_i. Then $\text{Ker}I_1 \supset \text{Ker}I_2 \supset ... \supset \text{Ker}I_n \supset ...$ is a descending chain of submodules of M. This chain is stationary since M is Artinian. Therefore there is an integer n_0 such that $KerI_n = KerI_{n+k}$ for all $n \geq n_0$. This would imply that $I_n = I_{n+k}$ for any $n \geq n_0$ and hence S_i is a Noetherian left ideal of S. Thus S is left Noetherian. Since S is right Artinian and left Noetherian, it follows that S is left Artinian. Hence S is serial and Artinian, proving (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Applying [5, Theorem 5.4], we see that every indecomposable left (right) S-module is uniserial. Using the equivalence between $\sigma[M]$ and Mod-S, we can see that every indecomposable module in $\sigma[M]$ is uniserial. Since $M = \bigoplus_{i=1}^{n} M_i$, with each M_i is indecomposable, we infer that M is a serial right R-module.

(2) $\Leftrightarrow$ (3). Using [5, Theorem 5.3], the condition (2) is equivalent to the fact that every indecomposable right S-module is quasi-injective. Since $\sigma[M]$ is equivalent to Mod-S, we infer that every indecomposable module in $\sigma[M]$ is quasi-injective and vice versa.

(2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). By a similar argument as that of (2) $\Leftrightarrow$ (3). ■

Corollary 2.5. *For a right Artinian right self-p-injective, the following conditions are equivalent:*

(1) *R is right serial;*
(2) *R is Artinian and serial;*
(3) *Every indecomposable right R-module is quasi-projective;*
(4) *Every indecomposable right R-module is quasi-injective;*
(5) *Every indecomposable quasi-injective right R-module is quasi-projective;*
(6) *Every indecomposable quasi-projective right R-module is quasi-injective.*

Corollary 2.6. *Let R be a right Artinian right serial ring. If R is right self-p-injective, then R is left Artinian and left serial.*

By applying the Theorem 2.5 we can prove the following:

Theorem 2.7. *Let M be an Artinian right R-module which is quasi-p-injective and a projective generator in $\sigma[M]$. Then the following conditions are equivalent:*

(1) *M is serial;*
(2) *Every indecomposable module in $\sigma[M]$ is both quasi-projective and quasi-injective;*
(3) *Every indecomposable module in $\sigma[M]$ is uniserial.*

References

[1] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, Heidelberg, Berlin, 1974.

[2] J. Clark, Ch. Lomp, Na. Vanaja, R. Wisbauer, *Lifting Modules Supplementes and Projectivity in Module Theory*, Basel, Boston, Berlin, 2006.

[3] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, *Extending Modules*, 1996.

[4] C. Faith, *Algebra: Rings, Modules and Categories II*, Springer, 1973.

[5] K.R. Fuller, On indecomposable injectives over Artinian rings, *Pacific J. Math.* **29** (1) (1969) 115–135.

[6] P. Jampachon, J. Ittharat, N.V. Sanh, On finite injectivity, *Southeast Asian Bull. Math.* **24** (4) (2000) 559–564.

[7] K.F. Kasch, *Modules and Rings,* London Mathematical Society monograph, Academic Press, London, New York, Paris, 1982.

[8] S.H. Mohamed, B.J. Muller, *Continuous and Discrete Modules*, London Mathematical Society, Cambridge University Press, Cambridge, New York, 1990.

[9] B.L. Osofsky, Rings all of whose finitely generated modules are injective, *Pacific. J. Math.* **14** (1982) 646–650.

[10] N.V. Sanh, K.P. Shum, S. Dhompongsa, S. Wongwai, On quasi-principally injective modules, *Algebra Colloq.* **6** (3) (1999) 269–276.

[11] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, Heidelberg, New York, 1975.

[12] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Tokyo, 1991.