Southeast Asian Bulletin of Mathematics (2013) 37: 591–600

Southeast Asian
Bulletin of
Mathematics

A Generalization of Hopkins-Levitzki Theorem

Le Phuong-Thao
Department of Mathematics, Mahidol University, Bangkok 10400, Thailand
Can Tho University, Can Tho, Vietnam
Email: lpthao@ctu.edu.vn

Nguyen Van Sanh*
Department of Mathematics, Mahidol University, Center of Excellence in Mathematics, Bangkok, Thailand
Email: frnvs@mahidol.ac.th

Received 18 August 2010
Accepted 29 October 2010

Communicated by K.P. Shum

AMS Mathematics Subject Classification(2000): 16D50, 16D70, 16D80

Abstract. Hopkin-Levitzki theorem said that every right Artinian ring is right Noetherian. It is well-knnown that not every Artinian is Noetherian, for example the Prüfer group $\mathbb{Z}_{p^\infty}$ is Artinian but not Noetherian. In this paper, we prove that if M is an Artinian quasi-projective finitely generated right R-module which is a self-generator, then it is Noetherian. This result can be considered as a generalization of Hopkins-Levitzki Theorem.

Keywords: Prime submodule; Semiprime submodule; Prime radical; Nilpotent submodule.

1. Introduction and Preliminaries

Throughout this paper, all rings are associative with identity and all modules are unitary right R-modules. Let M be a right R-module and $S = \text{End}(M_R)$, its endomorphism ring. We denote $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules. M is called a subgenerator if it generates $\sigma[M]$ and a self-generator if it generates all its submodules only. A

*Corresponding author, partially supported by Center of Excellence in Mathematics, CEM, Thailand, Grant no RG-53-13-1.

right R-module M is retractable if every non-zero submodule contains a non-zero M-generated submodules. It is clear that if M is a subgenerator, then it is a self-generator and every self-generator module is retractable. A submodule X of M is called a *fully invariant* submodule of M if for any $s \in S$, we have $s(X) \subset X$. By the definition, the class of all fully invariant submodules of M is non-empty and closed under intersections and sums. Especially, a right ideal of R is a fully invariant submodule of R_R if it is a two-sided ideal of R.

Let M be a right R-module and X, a fully invariant proper submodule of M. Following [15], the submodule X is called a *prime submodule* of M if for any ideal I of S, and any fully invariant submodule U of M, $I(U) \subset X$ implies that either $I(M) \subset X$ or $U \subset X$. Especially, an ideal P of R is a prime ideal if for any ideals I, J of R, if $IJ \subset P$ then either $I \subset P$ or $J \subset P$. A fully invariant submodule X of a right R-module M is called a *semiprime submodule* if it is an intersection of prime submodules of M. A right R-module M is called a *prime module* if 0 is a prime submodule of M. A ring R is a prime ring if R_R is a prime module. A right R-module M is called a *semiprime module* if 0 is a semiprime submodule of M. Especially, the ring R is a semiprime ring if R_R is a semiprime module. By symmetry, the ring R is a semiprime ring if ${}_RR$ is a semiprime left R-module. From now on, for a submodule X of M, we denote $I_X = \{f \in S \mid f(M) \subset X\}$, the right ideal of S related to X. A submodule X of M is called an M-annihilator if it is of the form $X = \ker A$ for some subset A of S.

Let M be a right R-module. We denote $P(M)$ to be the intersection of all prime submodules of M. It was shown in [15] that for a quasi-projective module M, we have $P(M/P(M)) = 0$, (see [15, Theorem 2.7]) and we call roughly $P(M)$ the *prime radical* of the module M by modifying the notion of the prime radical $P(R)$ of a ring R as the intersection of all prime ideals of R. Following [14, Theorem 1.10], if X is a prime submodule of a right R-module M, then the set I_X is a prime ideal of the endomorphism ring S, and the converse is true if M is a self-generator. Motivating this idea, we will introduce the notions of nilpotent submodules of a given right R-module. We investigate the prime radical and nilpotent submodules of a given right R-module and give a generalization of Hopkins-Levitzki Theorem.

A prime submodule X of a right R-module M is called a minimal prime submodule if there are no prime submodules of M properly contained in X. The following results had been appeared in [17] and we propose them here to use later on.

Proposition 1.1. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then we have the following:*

(1) *If X is a minimal prime submodule of M, then I_X is a minimal prime ideal of S.*
(2) *If P is a minimal prime ideal of S, then $X := P(M)$ is a minimal prime submodule of M and $I_X = P$.*

Theorem 1.2. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Let X be a fully invariant submodule of M. Then the following conditions are equivalent:*

(1) *X is a semiprime submodule of M;*
(2) *If J is any ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;*
(3) *If J is any ideal of S properly containing X, then $J^2(M) \not\subset X$;*
(4) *If J is any right ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;*
(5) *If J is any left ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$.*

From Theorem 1.2, we have the following lemma:

Proposition 1.3. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a semiprime submodule of M. If I is a right or left ideal of S such that $I^n(M) \subset X$ for some positive integer n, then $I(M) \subset X$.*

2. Prime Submodules and Semiprime Submodules

It was shown in [8, Theorem 2.4] that there exist only finitely many minimal prime ideals in a right Noetherian ring R. Using this result we can prove the following theorem.

Theorem 2.1. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a Noetherian module, then there exist only finitely many minimal prime submodules.*

Proof. Since M is a quasi-projective Noetherian module which is a self-generator, it would imply that S is a right Noetherian ring. Indeed, suppose that we have an ascending chain of right ideals of S, $I_1 \subset I_2 \subset \cdots$ says. Then we have $I_1(M) \subset I_2(M) \subset \cdots$ is an ascending chain of submodules of M. Since M is a Noetherian module, there is an integer n such that $I_n(M) = I_k(M)$, for all $k > n$. Then by [18, 18.4], we have $I_n = Hom(M, I_n(M)) = Hom(M, I_k(M)) = I_k$. Thus the chain $I_1 \subset I_2 \subset \cdots$ is stationary, so S is a right Noetherian ring. By [8, Theorem 2.4], S has only finitely many minimal prime ideals, $P_1, \ldots, P_t$ says. By Proposition 1.1, $P_1(M), \ldots, P_t(M)$ are the only minimal prime submodules of M. ∎

Lemma 2.2. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a simple submodule of M. Then I_X is a minimal right ideal of S.*

Proof. Let I be a right ideal of S such that $0 \neq I \subset I_X$. Then $I(M)$ is a nonzero submodule of M and $I(M) \subset X$. Thus $I(M) = X$ and it follows from [18. 18.4] that $I = I_X$ since M is a self-generator. ∎

Proposition 2.3. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Let X be a simple submodule of M. Then either $I_X^2 = 0$ or $X = f(M)$ for some idempotent $f \in I_X$.*

Proof. Since X is a simple submodule of M, by Lemma 2.2, I_X is a minimal right ideal of S. Suppose that $I_X^2 \neq 0$. Then there is a $g \in I_X$ such that $gI_X \neq 0$. Since gI_X is a right ideal of S and $gI_X \subset I_X$, we have $gI_X = I_X$ by the minimality of I_X. Hence there exists $f \in I_X$ such that $gf = g$. The set $I = \{h \in I_X \mid gh = 0\}$ is a right ideal of S and I is properly contained in I_X since $f \notin I$. By the minimality of I_X, we must have $I = 0$. It follows that $f^2 - f \in I_X$ and $g(f^2 - f) = 0$, and hence $f^2 = f$. Note that $f(M) \subset X$ and $f(M) \neq 0$, and from this we have $f(M) = X$. ∎

Corollary 2.4. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a simple submodule of M. If M is a semiprime module, then $X = f(M)$ for some idempotent $f \in I_X$.*

Proof. Since M is a semiprime module, it follows from [14, Theorem 2.9] that S is a semiprime ring and hence $I_X^2 \neq 0$. Thus $X = f(M)$ for some idempotent $f \in I_X$ by Proposition 2.3, proving our corollary. ∎

Proposition 2.5. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then $Z_r(S)(M) \subset Z(M)$ where $Z_r(S)$ is the singular right ideal of S and $Z(M)$ is a singular submodule of M.*

Proof. Let $f \in Z_r(S)$ and $x \in M$. We will show that $f(x) \in Z(M)$. Since $f \in Z_r(S)$, there exists an essential right ideal K of S such that $fK = 0$. It would imply that $fK(M) = 0$. Note that K is an essential right ideal of S, we can see that $K(M)$ is an essential submodule of M, and hence the set $x^{-1}K(M) = \{r \in R \mid xr \in K(M)\}$ is an essential right ideal of R, and therefore $f(x)(x^{-1}K(M)) = f(x(x^{-1}K(M)) \subset fK(M) = 0$, proving that $f(x) \in Z(M)$. ∎

The following Corollary is a direct consequence of the above proposition.

Corollary 2.6. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a nonsingular module, then S is a right nonsingular ring.*

Following [4, Lemma 1.16], for a semiprime ring R with ACC on annihilators, there are a finite number of minimal prime ideals and moreover, a prime ideal is minimal if and only if it is an annihilator ideal, we generalize this result to right R-module as in the following proposition.

Proposition 2.7. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a semiprime module with the ACC for M-annihilators, then M has only a finite number of minimal prime submodules. If $P_1, \ldots, P_n$ are minimal prime submodules of M, then $P_1 \cap \cdots \cap P_n = 0$. Also a prime submodule P of M is minimal if and only if I_P is an annihilator ideal of S.*

Proof. It follows from [14, Theorem 2.9] that S is a semiprime ring. Since M satisfies the ACC for M-annihilators, we can see that S satisfies the ACC for right annihilators (cf. [16, Lemma 3.2]). By [4, Lemma 1.16], S has only a finite number of minimal prime ideals. Therefore M has only finite number of minimal prime submodules, by Proposition 1.1. If $P_1, \ldots, P_n$ are minimal prime submodules of M, then $I_{P_1}, \ldots, I_{P_n}$ are minimal prime ideals of S. Thus $I_{P_1} \cap \cdots \cap I_{P_n} = 0$, by [4, Lemma 1.16]. From $I_{P_1} \cap \cdots \cap I_{P_n} = I_{P_1 \cap \cdots \cap P_n}$, we have $P_1 \cap \cdots \cap P_n = 0$. Finally, a prime submodule P of M is minimal if and only if I_P is a minimal prime ideal of S. It is equivalent to that fact that I_P is an annihilator ideal of S. ■

Proposition 2.8. *Let M be a quasi-projective right R-module and X be a fully invariant submodule of M. Then the following are equivalent:*

(1) *X is a semiprime submodule of M;*
(2) *M/X is a semiprime module.*

Proof. (1) $\Rightarrow$ (2). Let $X = \bigcap_{P_i \in \mathcal{F}} P_i$ be a semiprime submodule of M, where each P_i is a prime submodule of M. It follows from [14, Lemma 2.5] that each P_i/X is a prime submodule of M/X. Thus, $\bigcap_{P_i \in \mathcal{F}} (P_i/X) = 0$, proving that M/X is semiprime.

(2) $\Rightarrow$ (1). Suppose that 0 is a semiprime submodule of M/X. Then we can write $0 = \bigcap_{Q_i \in \mathcal{K}} Q_i$, where each Q_i is a prime submodule of M/X. Then $X = \nu^{-1}(0) = \nu^{-1}(\bigcap_{Q_i \in \mathcal{K}} Q_i) = \bigcap_{Q_i \in \mathcal{K}} \nu^{-1}(Q_i)$. Since each Q_i is a prime submodule of M/X, we see that $\nu^{-1}(Q_i)$ is a prime submodule of M by [14, Lemma 2.6]. Therefore X is a semiprime submodule of M. ■

3. A Generalization of Hopkins-Levitzki Theorem

In this section, we introduce the notion of nilpotent submodules and study the properties of prime radical together with nilpotent submodules of a given right R-module M.

Definition 3.1. *Let M be a right R-module and X, a submodule of M. We say that X is a nilpotent submodule of M if I_X is a nilpotent right ideal of S.*

From the definition we see that X is a nilpotent fully invariant submodule of

M if and only if I_X is a nilpotent two-sided ideal of S. First, we get a property of semiprime submodules similar to that of semiprime ideals.

Proposition 3.2. *Let M be a quasi-projective finitely generated right R-module which is a self-generator and N be a semiprime submodule of M. Then N contains all nilpotent submodules of M.*

Proof. Let X be a nilpotent submodule of M. Then I_X is a nilpotent right ideal of S. Thus, $I_X^n = 0$ for some positive integer n, and therefore $I_X^n(M) = 0 \subset N$. Since N is a semiprime submodule of M which is a self-generator, $X = I_X(M) \subset N$, proving our proposition. ∎

Corollary 3.3. *Let M be a quasi-projective finitely generated right R-module which is a self-generator. Let $P(M)$ be the prime radical of M. Then $P(M)$ contains all nilpotent submodules of M.*

Proposition 3.4. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M satisfies the ACC on fully invariant submodules, then $P(M)$ is nilpotent.*

Proof. We first claim that if M satisfies the ACC on fully invariant submodules, then S satisfies the ACC for two-sided ideals. Indeed, if $I_1 \subset I_2 \subset \cdots$ is an ascending chain of two-sided ideals of S, then $I_1(M) \subset I_2(M) \subset \cdots$ is an ascending chain of fully invariant submodules of M. By assumption, there exists a positive integer n such that $I_n(M) = I_k(M)$ for all $k > n$. Thus $I_n = I_k$ for all $k > n$, showing that S satisfies the ACC for two-sided ideals. It follows from [18, Proposition XV.1.4] that $P(S)$ is nilpotent. Since M is a self-generator, $P(S) = I_{P(M)}$, showing that $P(M)$ is a nilpotent submodule of M. ∎

It is well-known that a semiprime ring contains no nilpotent right ideals, the following theorem gives us a similar result for semiprime modules.

Theorem 3.5. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then M is a semiprime module if and only if M contains no nonzero nilpotent submodules.*

Proof. By hypothesis, 0 is a semiprime submodule of M. If X is a nilpotent submodule of M, then $I_X^n = 0$ for some positive integer n, and hence $I_X^n(M) = 0$. Note that $I_X(M) = 0$ by Corollary 1.3, we can see that $X = 0$.

Conversely, suppose that M contains no nonzero nilpotent submodules. Let I be an ideal of S such that $I^2(M) = 0$. Then we can write $I = I_{I(M)}$ and hence $I_{I(M)}^2 = 0$. It follows that $I(M)$ is a nilpotent submodule of M and we get $I(M) = 0$. Thus 0 is a semiprime submodule of M by Theorem 1.2, showing that M is a semiprime module. ∎

Proposition 3.6. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and $P(M)$ be the prime radical of M. If M is a Noetherian module, then $P(M)$ is the largest nilpotent submodule of M.*

Proof. Let $\mathcal{F}$ be the family of all minimal submodules of M. Then we can we write $P(M) = \bigcap_{X \in \mathcal{F}} X$. By Corollary 3.3, $P(M)$ contains all nilpotent submodules of M. By Proposition 1.1, $I_{P(M)} = \bigcap_{X \in \mathcal{F}} I_X = P(S)$. Note that from our assumption we can see that S is a right Noetherian ring. By [8, Theorem 2.4], there exist only finitely many minimal prime ideals of S and there is a finite product of them which is 0, says $P_1 \cdots P_n = 0$. Since $I_{P(M)}$ is contained in each $P_i, i = 1, \ldots, n$, we have $I^n_{P(M)} = 0$. Thus $P(M)$ is nilpotent. ∎

In the following, we write $X \subsetneq^{*} M$ to indicate that X is essential in M.

Lemma 3.7. *Let M be a right R-module and X, a fully invariant submodule of M. Then the following statements hold:*

(1) *$l_S(I_X) \subsetneq^{*} S_S$ if and only if $l_S(I^n_X) \subsetneq^{*} S_S$ for any $n \geq 1$.*
(2) *If X is nilpotent, then $l_S(I_X) \subsetneq^{*} S_S$.*

Proof. (1) Note that $l_S(I_X) \subset l_S(I^n_X)$. It follows that if $l_S(I_X) \subsetneq^{*} S_S$ then $l_S(I^n_X) \subsetneq^{*} S_S$. Conversely, it suffices to show that if $l_S(I^2_X) \subsetneq^{*} S_S$, then $l_S(I_X) \subsetneq^{*} S_S$. Indeed, let f be any nonzero element of S. Since $l_S(I^2_X) \subsetneq^{*} S_S$, there exists an element $g \in S$ such that $0 \neq fg \in l_S(I^2_X)$. If $fgI_X = 0$, then $fg \in l_S(I_X)$, and hence $fS \cap l_S(I_X) \neq 0$ and we are done. If $fgI_X \neq 0$, there is an element $h \in I_X$ such that $fgh \neq 0$. Then $fghI_X \subset fgI^2_X = 0$. This shows that $fgh \in l_S(I_X)$ and hence $fS \cap l_S(I_X) \neq 0$. Thus $l_S(I_X) \subsetneq^{*} S_S$.

(2) If X is nilpotent, then $I^n_X = 0$ for some positive integer n. We have $l_S(I^n_X) = l_S(0) = S \subsetneq^{*} S_S$. By (1), we can see that $l_S(I_X) \subsetneq^{*} S_S$, proving our lemma. ∎

Proposition 3.8. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and assume that every ideal right essential in S contains a right regular element of S. Then M is a semiprime module.*

Proof. Let K be any ideal of S such that $K^2(M) = 0$. We wish to show that $K(M) = 0$. Since $K^2(M) = 0$ and $K = I_{K(M)}$ we see that $X := K(M)$ is nilpotent. Thus $l_S(I_X) = l_S(K)$ is an ideal which is right essential in S by Lemma 2.7. By assumption, $l_S(J)$ contains a right regular element f of S. Now from $fK = 0$ and f is right regular, we have $K = 0$. Thus $K(M) = 0$, proving that 0 is a semiprime submodule of M. ∎

Proposition 3.9. *Let M be a right R-module and X, a nilpotent submodule of M. If N is a fully invariant submodule of M with $M = N + I_X(M)$, then $M = N$. In particular, if $M \neq 0$, then $M \neq I_X(M)$.*

Proof. We show by induction on $i \geq 1$ that $M = N + I_X^i(M)$. The case $i = 1$ is true by the hypothesis. Suppose that $M = N + I_X^i(M)$. Then $I_X(M) = I_X(N) + I_X^{i+1}(M)$. Thus $M = N + I_X(M) = N + I_X(N) + I_X^{i+1}(M) = N + I_X^{i+1}(M)$. Since X is nilpotent, I_X is nilpotent by definition. Then $I_X^n = 0$ for some positive integer n, and so $I_X^n(M) = 0$. Thus $M = N$. In particular, 0 is fully invariant submodule of M and if $M = I_X(M)$, then $M = 0$, proving our proposition. ■

Lemma 3.10. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then the following statements hold:*

(1) *If X is a maximal submodule of M, then I_X is a maximal right ideal of S;*
(2) *If P is a maximal right ideal of S, then $X := P(M)$ is a maximal submodule of M and $P = I_X$.*

Proof. (1) Let K be a right ideal of S and suppose that I_X is properly contained in K. Then X is properly contained in $K(M)$. Thus $K(M) = M$ by the maximality of X, proving that I_X is maximal in S.

(2) Since M is a self-generator, we have $P = I_X$, where $X = P(M)$. If N is a submodule of M and properly contains X, then I_X is properly contained in I_N. Therefore $I_N = S$ by the maximality of P. Thus $N = M$, showing that $P(M)$ is a maximal submodule of M. ■

Let M be a right R-module. The *radical* of M, denote by $Rad(M)$ or $J(M)$, is the intersection of all maximal submodules of M. The Jacobson radical $J(R)$ of the ring R is the intersection of all maximal right (left) ideals of R.

Recall that if R is right Artinian, then $Rad(R)$ is the unique largest nilpotent right, left or two-sided ideal of R, (see [9, Corollary 9.3.10]). Hopkins-Levitzki Theorem says that if the ring R is right Artinian, then R is right Noetherian and the Jacobson radical $J(R)$ is nilpotent (cf. [8, Theorem 3.15]). Note that not every Artinian module is Noetherian, for example, the Prüfer group $\mathbb{Z}_{p^\infty}$ is Artinian but not Noetherian. Therefore, it is natural to ask a question that when is an Artinian module Noetherian? The following theorem can be considered as a generalization of Hopkins-Levitzki Theorem.

Theorem 3.11. *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is Artinian, then M is Noetherian, $Rad(M)$ is nilpotent and $Rad(M) = P(M)$.*

Proof. By Lemma 3.10, a submodule X of M is maximal if and only if I_X is a maximal right ideals of S, it is easy to see that $Rad(S) = I_{Rad(M)}$. Since M is Artinian, from our hypothesis, the ring S is a right Artinian ring. Thus S is right Noetherian and $Rad(S)$ is nilpotent (by [8, Theorem 3.15]). Let $N_1 \subset N_2 \subset \cdots$ be an ascending chain of submodules of M. Then $I_{N_1} \subset I_{N_2} \subset \cdots$ is an ascending chain of right ideals of S. Since S is right Noetherian, there exists a positive integer n such that $I_{N_n} = I_{N_k}$ for all $k > n$, and it would imply that

$N_n = N_k$ for all $k > n$, since M is a self-generator. This shows that M is a Noetherian module. Since $Rad(S) = I_{Rad(M)}$ and $Rad(S)$ is nilpotent, $Rad(M)$ is a nilpotent submodule of M.

Note that S is a right Artinian ring, we have $Rad(S) = P(S)$ (see [8, Corollary 3.16]). But $Rad(S) = I_{Rad(M)}$ and $P(S) = I_{P(M)}$, proving that $Rad(M) = P(M)$. ■

We remark here that the concept of quasi-projectivity on a finitely generated right R-module has been generalized to the quasi-rp-injectivity of a Kasch module in [10], it is natural to ask whether the above results related to quasi-projectivity on a finitely generated R-module can be extended to a Kasch module? As we have mentioned in the Introduction that every self-generator right R-module is retractble. We now propose the question. Can we replace the condition of self-generator by retractable property in almost all of our results?

References

[1] M.M. Ali, Idempotent and nilpotent submodules of multiplication modules, *Comm. Algebra* **36** (12) (2008) 4620–4642.

[2] R. Ameri, On the prime submodules of multiplication modules, *IJMMS* **27** (2003) 1715–1724.

[3] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, Berlin, Heidelberg, New York, 1992.

[4] A.W. Chatters, C.R. Hajarnavis, *Rings With Chain Conditions*, Pitman Advanced Publishing Program, 1980.

[5] J. Dauns, Prime submodules, *J. Reine Angrew. Math.* **298** (1978) 156–187.

[6] A. Gaur, A.K. Maloo, A. Parkask, Prime submodules in multiplication modules, *Inter. J. Algebra* **1** (8) (2007) 375–380.

[7] A. Gaur, A. Maloo, Minimal prime submodules, *Inter. J. Algebra* **2** (20) (2008) 953–956.

[8] K.R. Goodearl, R.B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, Cambridge Uni. Press, Cambridge, 1989.

[9] F.Kasch, *Modules and Rings*, London Math. Society, 1982.

[10] H.D. Hai, N.V. Sanh, A. Sudprasert, A weaker form of p-injectivity, *Southeast Asian Bull. Math.* **33** (6) (2009) 1063–1069.

[11] T.Y Lam, *A First Course in Noncommutative Rings*, Springer-Verlag, New York, 1991.

[12] C.P. Lu, Prime submodules of modules, *Comment. Mat. Univ. St. Pal.* **33** (1) (1984) 61–69.

[13] R.L. McCasland, P.F. Smith, Prime submodules of Noetherian modules, *Rocky Mountain J. Math.* **23** (1993) 1041–1062.

[14] D.S. Passman, *A Course in Ring Theory*, AMS Chelsea Publishing, Amer. Math. Society-Providence, Rhode Island, 2004.

[15] N.V. Sanh, N.A. Vu, K.F.U. Ahmed, S. Asawasamrit, L.P. Thao, Primeness in module category, *Asian-Eur. J. Math.* **3** (1) (2010) 145–154.

[16] N.V. Sanh, S. Asawasamrit, K.F.U. Ahmed, L.P. Thao, On prime and semiprime Goldie modules, *Asian-Eur. J. Math.* **4** (2) (2011) 321–334.

[17] N.V. Sanh, K.F.U. Ahmed, L.P. Thao, On semiprime modules with chain conditions, *Algebra Colloq.* (to appear).

[18] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, Heidelberg, New York, 1975.

[19] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Tokyo, E.A., 1991.