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A Short Note on Approximation Properties of q-Baskakov-Szász-Stancu Operators

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Abstract. In this paper, we introduce the q-analogue of Baskakov-Szász-Stancu operators. First, we use properties of q-calculus to estimate moments of these operators. We have estimated some approximation properties and asymptotic formula for these operators. In last section, we give better error estimations for said operators.

Keywords: Baskakov-Szász-Stancu operators; q-calculus; Asymptotic formula; Rate of convergence of weighted spaces.

1. Introduction

The approximation of functions by linear positive operators is an important research topic in general mathematics and it also provides powerful tools to application areas such as computer-aided geometric design, numerical analysis, and solutions of differential equations. q-Calculus is a generalization of many subjects, such as hypergeometric series, complex analysis and particle physics. Currently it continues being an important subject of study. It has been shown that linear positive operators constructed by q-numbers are quite effective as far as the rate of convergence is concerned and we can have some unexpected results, which are not observed for classical case. This type of construction was first used to generate Bernstein operators. In 1987, Lupaş [13] defined a q-analogue of Bernstein operators and studied some approximation properties of them. In 1997, Phillips [25] introduced another generalization of Bernstein operators based on the q-integers called q-Bernstein operators. Research results show that q-Bernstein operators possess good convergence and approximation properties in $C[0,1]$. Very recently Aral [1] introduced the q-Szász-Mirakyan operators. Aral and Gupta [2] extended the study and established some approximation properties for q-Szász Mirakyan operators.

After the paper of Phillips [25] who generalized the classical Bernstein polynomials based on q-integers, many generalizations of well-known positive linear operators, based on q-integers were introduced and studied by several authors.

For $f \in C[0,\infty)$, a new type of Baskakov-Szász type operators proposed by Gupta and Srivastava [8] which is defined as

$$\mathcal{D}_n(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} s_{n,k}(t)f(t)dt,\ \ x\in[0,\infty) \tag{1}$$

where $p_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$ and $s_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}$.
It is observed from [8] that these operators reproduce only the constant functions. In the last decade lots of work has been done on q-operators and approximation by different types of summability operators. We refer the recent work in this direction due to Aral[1], Khan [11], Beg [3, 32], Mursaleen [15, 16, 17], Wafi [33], H.M. Srivastava [26]-[30], Mishra *et al*. [18]-[24] etc.

In 2010, Ibrahim [9] introduced Stancu-Chlodowsky polynomials and investigated convergence and approximation properties of operators

$$S_n^{\alpha}(f,x) = \sum_{k=0}^{n} p'_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right), 0\le x\le 1, \tag{2}$$

where $p'_{n,k}(x)$ is the Bernstein basis function.

In [7] Gupta introduced the q-Baskakov-Szász type operators $\mathcal{D}_n^q(f,x)$, which was generalization of (1).

$$\mathcal{D}_n^q(f,x) = [n]_q\sum_{k=0}^{\infty} p_{n,k}^q(x)\int_0^{q/1-q^n} q^{-k-1}s_{n,k}^q(t)f\big(t\,q^{-k}\big)d_qt, \tag{3}$$

where $x\in[0,\infty)$ and

$$p_{n,k}^q(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k-1)/2}\frac{x^k}{(1+x)_q^{n+k}}, \tag{4}$$

and

$$s_{n,k}^q(t) = E(-[n]_q t)\frac{([n]_q t)^k}{[k]_q!}. \tag{5}$$

In case $q = 1$, the above operators reduce to the operators (1). Motivated by such types of operators, we generalize Stancu type generalization of the Baskakov-Szász type operators as follows

$$\mathcal{D}_{n,q}^{(\alpha,\beta)}(f,x) = [n]_q \sum_{k=0}^{\infty} p_{n,k}^q(x) \int_0^{q/1-q^n} q^{-k-1} s_{n,k}^q(t) f\left(\tfrac{[n]_q t q^{-k}+\alpha}{[n]_q+\beta}\right) d_q t \qquad (6)$$

where $p_{n,k}^q(x)$ and $s_{n,k}^q(t)$ are defined as above. The operators $\mathcal{D}_{n,q}^{(\alpha,\beta)}(f,x)$ in (6) are called q-Baskakov-Szász-Stancu operators. For $\alpha = 0$, $\beta = 0$ the operators (6) reduce to the operators (3).

During last decade, q-Calculus was extensively used for constructing various generalization of many classical approximation operators.

The aim of this paper is to study the approximation properties of a new generalization of the Baskakov-Szász-Stancu operators based on q-integers. We estimate moments for these operators. Also, we study asymptotic formula for these operators. Finally, we give better error estimations for operators $\mathcal{D}_{n,q}^{(\alpha,\beta)}$.

First, we recall some definitions and notations of q-calculus. Such notations can be found in ([5],[10]). We consider q as a real number satisfying $0 < q < 1$.

For

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases}$$

and

$$[n]_q! = \begin{cases} [n]_q[n-1]_q[n-2]_q...[1]_q, & n = 1, 2, ..., \\ 1, & n = 0. \end{cases}$$

We observe that

$$(1+x)_q^n = (-x;q)_n = \begin{cases} (1+x)(1+qx)(1+q^2x)...(1+q^{n-1}x), & n = 1, 2, ..., \\ 1, & n = 0. \end{cases}$$

Also, for any real number α, we have

$$(1+x)_q^{\alpha} = \frac{(1+x)_q^{\infty}}{(1+q^{\alpha}x)_q^{\infty}}.$$

In special case, when α is a whole number, this definition coincides with the above definition.

The q-binomial coefficients are given by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \ \ 0 \leq k \leq n.$$

The q-derivative $D_q f$ of a function f is given by

$$D_q(f(x)) = \frac{f(x)-f(qx)}{(1-q)x}.$$

The q-Jackson integral and q-improper integral defined as

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n$$

and

$$\int_0^{\infty/A} f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f\left(\tfrac{q^n}{A}\right)\tfrac{q^n}{A},$$

provided sum converges absolutely.
The q-analogues of the exponential function e^x (see [10]), used here is defined as

$$E_q(z) = \prod_{j=0}^{\infty}(1+(1-q)q^jz) = \sum_{k=0}^{\infty} q^{k(k-1)/2}\tfrac{z^k}{[k]_q!} = (1+(1-q)z)_q^{\infty},\ |q|<1,$$

where $(1-x)_q^{\infty} = \prod_{j=0}^{\infty}(1-q^jx)$.
De Sole and Kac [31] define q-analogue of beta function of second kind

$$B(t,s) = \int_0^{\infty} \tfrac{x^{t-1}}{(1+x)^{t+s}}dx$$

as follows

$$B(t,s) = K(A,t)\int_0^{\infty/A} \tfrac{x^{t-1}}{(1+x)_q^{t+s}}d_qx,$$

where $K(x,t) = \frac{1}{1+x}x^t\left(1+\frac{1}{x}\right)_q^t(1+x)_q^{1-t}$. This function is q-constant in x i.e. $K(qx,t) = K(x,t)$.

In particular for any positive integer n, we have

$$K(x,n) = q^{\frac{n(n-1)}{2}},\ K(x,0) = 1,\ B_q(t,s) = \tfrac{[t-1]_q![s-1]_q!}{[t+s-1]_q!}.$$

2. Moment estimates

Lemma 2.1. [7] *The following hold:*

1. $\mathcal{D}_n^q(1,x) = 1$,
2. $\mathcal{D}_n^q(t,x) = x + \frac{q}{[n]_q}$,
3. $\mathcal{D}_n^q(t^2,x) = \left(1+\frac{1}{q[n]_q}\right)x^2 + \frac{x}{[n]_q}(1+q(q+2)) + \frac{q^2(1+q)}{[n]_q^2}$.

Lemma 2.2. *The following hold:*

1. $\mathcal{D}_{n,q}^{(\alpha,\beta)}(1,x) = 1$,

2. $\mathcal{D}_{n,q}^{(\alpha,\beta)}(t,x) = \frac{[n]_q x+q+\alpha}{[n]_q+\beta}$,
3. $\mathcal{D}_{n,q}^{(\alpha,\beta)}(t^2,x) = \left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2}\right)x^2 + \left(\frac{(1+q(q+2))[n]_q+2\alpha[n]_q}{([n]_q+\beta)^2}\right)x + \frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}$.

Proof. The operator $\mathcal{D}_{n,q}^{(\alpha,\beta)}$ is well defined for $1, t, t^2$ by Lemma 2.1, and $x \in [0,\infty)$, we have

$$\mathcal{D}_{n,q}^{(\alpha,\beta)}(1,x) = \mathcal{D}_n^q(1,x) = 1,$$

$$\mathcal{D}_{n,q}^{(\alpha,\beta)}(t,x) = \frac{[n]_q}{[n]_q+\beta}\mathcal{D}_n^q(t,x) + \frac{\alpha}{[n]_q+\beta}\mathcal{D}_n^q(1,x) = \frac{[n]_q x+q+\alpha}{[n]_q+\beta}.$$

Finally,

$$\begin{aligned}
\mathcal{D}_{n,q}^{(\alpha,\beta)}(t^2,x) &= \frac{[n]_q^2}{([n]_q+\beta)^2}\mathcal{D}_n^q(t^2,x) + \frac{2[n]_q\alpha}{([n]_q+\beta)^2}\mathcal{D}_n^q(t,x) + \frac{\alpha^2}{([n]_q+\beta)^2}\mathcal{D}_n^q(1,x) \\
&= \frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2}x^2 + \left(\frac{(1+q(q+2))[n]_q+2\alpha[n]_q)}{([n]_q+\beta)^2}\right)x \\
&\quad + \frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}.
\end{aligned}$$

■

Remark 2.3. If we put $q = 1$ and $\alpha = \beta = 0$, we get the moments of Baskakov-Szász operators (1) as $\mathcal{D}_n(t,x) = \frac{nx+1}{n}$ and $\mathcal{D}_n(t^2,x) = \frac{1}{n^2}[n(n+1)x^2+4nx+2]$.

Lemma 2.4. *If we define the central moments as* $\mu_{n,m}^q(x) = \mathcal{D}_{n,q}^{(\alpha,\beta)}((t-x)^m,x)$, $m \in \mathbb{N}$. *Then*

$$\begin{aligned}
\mu_{n,1}^q(x) &= \mathcal{D}_{n,q}^{(\alpha,\beta)}(t-x,x) = \frac{q+\alpha-\beta x}{[n]_q+\beta}, \\
\alpha_n(q;x) = \mu_{n,2}^q(x) &= \mathcal{D}_{n,q}^{(\alpha,\beta)}((t-x)^2,x) = \left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2} + 1 - \frac{2[n]_q}{[n]_q+\beta}\right)x^2 \\
&\quad + \left(\frac{[n]_q+q^2[n]_q-2\alpha\beta-2q\beta}{([n]_q+\beta)^2}\right)x + \frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}.
\end{aligned}$$

Proof. Notice that

$$\begin{aligned}
\mu_{n,1}^q(x) &= \mathcal{D}_{n,q}^{(\alpha,\beta)}((t-x),x) \\
&= \mathcal{D}_{n,q}^{(\alpha,\beta)}(t,x) - x\mathcal{D}_{n,q}^{(\alpha,\beta)}(1,x) = \frac{q+[n]_q x+\alpha}{[n]_q+\beta} - x = \frac{q+\alpha-\beta x}{[n]_q+\beta}.
\end{aligned}$$

$$\begin{aligned}
\mu_{n,2}^q(x) &= \mathcal{D}_{n,q}^{(\alpha,\beta)}((t-x)^2,x) = \mathcal{D}_{n,q}^{(\alpha,\beta)}(t^2,x) - 2x\mathcal{D}_{n,q}^{(\alpha,\beta)}(t,x) + x^2\mathcal{D}_{n,q}^{(\alpha,\beta)}(1,x) \\
&= \frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2}x^2 + \left(\frac{(1+q(q+2))[n]_q+2\alpha[n]_q}{([n]_q+\beta)^2}\right)x \\
&\quad + \frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2} - 2x\left(\frac{[n]_q x+\alpha+q}{[n]_q+\beta}\right) + x^2 \\
&= \left(\frac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2} + 1 - \frac{2[n]_q}{[n]_q+\beta}\right)x^2 \\
&\quad + \left(\frac{[n]_q+q^2[n]_q-2\alpha\beta-2q\beta}{([n]_q+\beta)^2}\right)x + \frac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}.
\end{aligned}$$

■

Remark 2.5. For all $m \in \mathbb{N}$, $0 \leq \alpha \leq \beta$; we have the following recursive relation for the images of the monomials t^m under $\mathcal{D}_{n,q}^{\alpha,\beta}(t^m, x)$ in terms of $\mathcal{D}_n^q(t^j, x)$, $j = 0, 1, 2, ..., m$ as

$$\mathcal{D}_{n,q}^{(\alpha,\beta)}(t^m, x) = \sum_{j=0}^{m} \binom{m}{j} \frac{[n]_q^j \alpha^{m-j}}{([n]_q+\beta)^m} \mathcal{D}_n^q(t^j, x).$$

3. Direct result and asymptotic formula

Let the space $C_B[0,\infty)$ of all continuous and bounded functions be endowed with the norm $\|f\| = sup\{|f(x)| : x \in [0,\infty)\}$. Further let us consider the following K-functional:

$$K_2(f,\delta) = \inf_{g \in W_\infty^2} \{\|f-g\| + \delta\|g''\|\}, \tag{7}$$

where $\delta > 0$ and $W_\infty^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$.
By the method as given in ([4], p.177, Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}), \tag{8}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h<\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)| \tag{9}$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. Also we set

$$\omega(f,\sqrt{\delta}) = \sup_{0<h<\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|. \tag{10}$$

Theorem 3.1. *Let $f \in C_B[0,\infty)$ and $0 < q < 1$. Then for all $x \in [0,\infty)$, there exists an absolute constant $M > 0$ such that*

$$|\mathcal{D}_{n,q}^{\alpha,\beta}(f,x) - f(x)| \leq M\omega_2\left(f, \sqrt{\mu_{n,2}^q + (\mu_{n,1}^q)^2}\right) + \omega(f, \mu_{n,1}^q). \tag{11}$$

Proof. Let $g \in W_\infty^2$ and $x, t \in [0,\infty)$. By Taylor's expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du. \tag{12}$$

Define

$$\tilde{\mathcal{D}}_{n,q}^{(\alpha,\beta)}(f,x) = \mathcal{D}_{n,q}^{(\alpha,\beta)}(f,x) + f(x) - f\left(\frac{[n]_q x + q + \alpha}{[n]_q + \beta}\right). \tag{13}$$

Now, we have $\tilde{\mathcal{D}}_{n,q}^{(\alpha,\beta)}(t-x,x)=0$, $t\in[0,\infty)$. Applying $\widetilde{\mathcal{D}}_{n,q}^{\alpha,\beta}$ on both sides of (12), we get

$$\begin{aligned}\widetilde{\mathcal{D}}_{n,q}^{(\alpha,\beta)}(g,x)-g(x) &= g'(x)\tilde{\mathcal{D}}_{n,q}^{(\alpha,\beta)}((t-x),x)+\tilde{\mathcal{D}}_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du,x\right)\\ &= \mathcal{D}_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du,x\right)\\ &\quad+\int_x^{x+\frac{\alpha-\beta x+q}{[n]_q+\beta}}\left(x+\tfrac{\alpha-\beta x+q}{[n]_q+\beta}-u\right)g''(u)du,\end{aligned}$$

on the other hand $\left|\int_x^t (x-u)g''(u)du\right|\le\|g''\|(t-x)^2$ and

$$\begin{aligned}\int_x^{x+\frac{\alpha-\beta x+q}{[n]_q+\beta}}\left(x+\tfrac{\alpha-\beta x+q}{[n]_q+\beta}-u\right)g''(u)du &\le \left(x+\tfrac{\alpha-\beta x+q}{[n]_q+\beta}-x\right)^2\|g''\|\\ &= \|g''\|\left(\mathcal{D}_{n,q}^{(\alpha,\beta)}(t-x,x)\right)^2\\ &= \|g''\|\left(\mu_{n,1}^q(x)\right)^2.\end{aligned}$$

One can do this

$$\begin{aligned}\left|\tilde{\mathcal{D}}_{n,q}^{(\alpha,\beta)}(g,x)-g(x)\right| &\le \left|\mathcal{D}_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du,x\right)\right|\\ &\quad+\left|\int_x^{x+\frac{\alpha-\beta x+q}{[n]_q+\beta}}\left(x+\tfrac{\alpha-\beta x+q}{[n]_q+\beta}-u\right)g''(u)du\right|\\ &\le \|g''\|\mathcal{D}_{n,q}^{(\alpha,\beta)}\left((t-x)^2,x\right)+\|g''\|\left(\mu_{n,1}^q(x)\right)^2\\ &\le \|g''\|\left[\mu_{n,2}^q(x)+\left(\mu_{n,1}^q(x)\right)^2\right].\end{aligned}$$

We observe that,

$$\begin{aligned}\left|\mathcal{D}_{n,q}^{(\alpha,\beta)}(f,x)-f(x)\right| &\le \left|\tilde{\mathcal{D}}_{n,q}^{(\alpha,\beta)}(f-g,x)-(f-g)(x)\right|\\ &\quad+\left|\tilde{\mathcal{D}}_{n,q}^{(\alpha,\beta)}(g,x)-g(x)\right|+\left|f(x)-f\left(x+\tfrac{\alpha-\beta x+q}{[n]_q+\beta}\right)\right|\\ &\le 4\|f-g\|+\|g''\|\left[\mu_{n,2}^q(x)+\left(\mu_{n,1}^q(x)\right)^2\right]+\omega(f,\mu_{n,1}^q(x)).\end{aligned}$$

Now, taking infimum on the right-hand side over all $g\in C_B^2[0,\infty)$ and from (8), we get

$$\begin{aligned}\left|\mathcal{D}_{n,q}^{(\alpha,\beta)}(f,x)-f(x)\right| &\le 4K_2\left(f,\mu_{n,2}^q(x)+\left(\mu_{n,1}^q(x)\right)^2\right)+\omega(f,\mu_{n,1}^q(x))\\ &\le M\omega_2\left(f,\sqrt{\mu_{n,2}^q(x)+\left(\mu_{n,1}^q(x)\right)^2}\right)+\omega(f,\mu_{n,1}^q(x)),\end{aligned}$$

which proves the theorem. ∎

Theorem 3.2. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For each $f \in C_x^2[0,\infty)$ and $\gamma > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)|}{(1+x^2)^{1+\gamma}} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)|}{(1+x^2)^{1+\gamma}} &\leq \sup_{x\leq x_0} \frac{|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)|}{(1+x^2)^{1+\gamma}} + \sup_{x\geq x_0} \frac{|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)|}{(1+x^2)^{1+\gamma}} \\ &\leq \|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(f) - f\|_{C[0,x_0]} + \|f\|_{x^2} \sup_{x\geq x_0} \frac{|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(1+t^2,x)|}{(1+x^2)^{1+\gamma}} \\ &\quad + \sup_{x\geq x_0} \frac{|f(x)|}{(1+x^2)^{1+\gamma}}.\end{aligned}$$

The first term of the above inequality tends to zero from Theorem 3.1. For some calculation, it is easily seen that for any fixed $x_0 > 0$ and $\sup_{x\geq x_0} \frac{|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(1+t^2,x)|}{(1+x^2)^{1+\gamma}}$ tends to zero as $n \to \infty$. We can choose $x_0 > 0$ so large that the last part of the above inequality can be made small enough. Thus the proof is completed. ∎

4. Voronovskaja type theorem

Our Next Result in this section is the Voronovskaja type asymptotic formula: Let $B_{x^2}[0,\infty) = \{f :$ for every $x \in [0,\infty), |f(x)| \leq M_f(1+x^2)\}$, M_f being a constant depending on f. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous function belonging to $B_{x^2}[0,\infty)$. Also, $C^*_{x^2}[0,\infty)$ is subspace of all function $f \in C_{x^2}[0,\infty)$ for which $\lim_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}$.

Lemma 4.1. *Assume that $q_n \in (0,1)$, $q \to 1$ as $n \to \infty$. Then, for every $x \in [0,\infty)$, we have*

$$\lim_{n\to\infty} [n]_{q_n} \mathcal{D}_{n,q_n}^{(\alpha,\beta)}(t-x,x) = \alpha - \beta x + 1,$$

and

$$\lim_{n\to\infty} [n]_{q_n} \mathcal{D}_{n,q_n}^{(\alpha,\beta)}((t-x)^2,x) = x(x+2).$$

Theorem 4.2. *Let f be bounded and integrable on the interval $[0,\infty)$. First and second derivatives of f exists at a fixed point $x \in [0,\infty)$ and $q = q_n \in (0,1)$ such that $q = q_n \to 1$ as $n \to \infty$, then*

$$\lim_{n\to\infty} [n]_{q_n} \left[\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\right] = (\alpha - \beta x + 1)f'(x) + x(x+2)/2f''(x).$$

Proof. Using Taylor's expansion of f, we can write

$$f(t)-f(x)=(t-x)f'(x)+\tfrac{(t-x)^2}{2!}f''(x)+r(t,x)(t-x)^2,$$

where $r(t,x)$ is Peano form of the remainder and $\lim_{t\to x} r(t,x)=0$. Applying the operator $\mathcal{D}_{n,q_n}^{(\alpha,\beta)}$ to above relation, we get

$$\begin{aligned}\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x) &= f'(x)\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(t-x,x)+\tfrac{f''(x)}{2}\mathcal{D}_{n,q_n}^{(\alpha,\beta)}((t-x)^2,x)\\ &\quad+\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(r(t,x)(t-x)^2,x)\\ &= f'(x)\mu_{n,1}^{q_n}(x)+\tfrac{f''(x)}{2}\mu_{n,2}^{q_n}(x)+\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(r(t,x)(t-x)^2,x),\end{aligned}$$

where $\mu_{n,1}^{q_n}$ and $\mu_{n,2}^{q_n}$ are defined in Lemma 2.4.
Using Cauchy-Schwarz inequality, we have

$$\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(r(t,x)(t-x)^2,x)\le\sqrt{\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(r^2(t,x),x)}\sqrt{\mathcal{D}_{n,q_n}^{(\alpha,\beta)}((t-x)^4,x)}. \tag{14}$$

We observe that $r^2(x,x)=0$ and $r^2(\cdot,x)\in C_{x^2}^*[0,\infty)$. Then it follows that

$$\lim_{n\to\infty}[n]_{q_n}\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(r^2(t,x),x)=r^2(x,x)=0, \tag{15}$$

uniformly with respect to $x\in[0,A]$, where $A>0$. Now from (15) and (14), we obtain

$$\lim_{n\to\infty}[n]_{q_n}\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(r(t,x)(t-x)^2,x)=0.$$

Thus, we have

$$\begin{aligned}\lim_{n\to\infty}[n]_{q_n}(\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)) &= \lim_{n\to\infty}[n]_{q_n}f'(x)\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(t-x,x)\\ &\quad+\lim_{n\to\infty}[n]_{q_n}\tfrac{f''(x)}{2}\mathcal{D}_{n,q_n}^{(\alpha,\beta)}((t-x)^2,x).\end{aligned}$$

Using Lemma 4.1, we get

$$\lim_{n\to\infty}[n]_{q_n}(\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x))=(\alpha-\beta x+1)f'(x)+x(2+x)/2f''(x),$$

which completes the proof. ∎

Now we discuss the weighted approximation theorem, when the approximation formula holds true on the interval $[0,\infty)$.

Theorem 4.3. *Let $q=q_n$ satisfies $0<q_n<1$ and let $q_n\to 1$ as $n\to\infty$. For each $f\in C_{x^2}^*[0,\infty)$, we have*

$$\lim_{n\to\infty}\|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)\|_{x^2}=0.$$

Proof. Using the theorem in [6] and [14], we see that it is sufficient to verify that the following three conditions

$$\lim_{n\to\infty} \|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(t^r,x)-x^r\|_{x^2}=0,\ r=0,1,2. \tag{16}$$

Since, $\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(1,x)=1$, the first condition of (16) is satisfied for $r=0$. Now,

$$\begin{aligned}
\|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(t,x)-x\|_{x^2} &= \sup_{x\in[0,\infty)} \tfrac{|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(t,x)-x|}{1+x^2}\\
&\le \sup_{x\in[0,\infty)} \left|\tfrac{[n]_q x+q+\alpha}{[n]_q+\beta}-x\right| \times \tfrac{1}{1+x^2}\\
&\le \left|\tfrac{[n]_q}{([n]_q+\beta)}\right| \sup_{x\in[0,\infty)} \tfrac{x}{1+x^2} + \left|\tfrac{q+\alpha}{([n]_q+\beta)}\right| \sup_{x\in[0,\infty)} \tfrac{1}{1+x^2}\\
&\quad - \sup_{x\in[0,\infty)} \tfrac{x}{1+x^2}\\
&\to 0 \ as\ [n]_q \to \infty.
\end{aligned}$$

Therefore, condition (16) holds for $r=1$. Similarly, we can write

$$\begin{aligned}
\|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(t^2,x)-x^2\|_{x^2} &= \sup_{x\in[0,\infty)} \tfrac{|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(t^2,x)-x^2|}{1+x^2}\\
&\le \left|\left(\tfrac{[n]_q(q[n]_q+1)}{q([n]_q+\beta)^2}-1\right)\right| \sup_{x\in[0,\infty)} \tfrac{x^2}{1+x^2}\\
&\quad + \left|\tfrac{(1+q(q+2))[n]_q+2\alpha[n]_q)}{([n]_q+\beta)^2}\right| \sup_{x\in[0,\infty)} \tfrac{x}{1+x^2} +\\
&\quad \left|\tfrac{q^2(1+q)+2q\alpha+\alpha^2}{([n]_q+\beta)^2}\right| \sup_{x\in[0,\infty)} \tfrac{1}{1+x^2}
\end{aligned}$$

which implies that $\|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(t^2,x)-x^2\|_{x^2} \to 0$ as $n\to\infty$. Thus the proof is completed. ∎

5. Rate of convergence

Now we give a rate of convergence theorem for the operator $\mathcal{D}_{n,q}^{(\alpha,\beta)}$.

Theorem 5.1. *Let $f\in C_{x^2}[0,\infty)$, $q=q_n\in(0,1)$ such that $q_n\to 1$ as $n\to\infty$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1]\subset[0,\infty)$, where $a>0$. Then, we have*

$$|\mathcal{D}_{n,q_n}^{(\alpha,\beta)}(f)-f| \le 6M_f(1+a^2)\alpha_n(q_n,x)+2\omega_{a+1}\left(f,\sqrt{\alpha_n(q_n,x)}\right).$$

Proof. For $x \in [0, a]$ and $t > a + 1$. Since $t - x > 1$, we have

$$\begin{aligned} |f(t) - f(x)| &\leq M_f(2 + x^2 + t^2) \\ &\leq M_f(2 + 3x^2 + 2(t - x)^2) \\ &\leq 3M_f(1 + x^2 + (t - x)^2) \end{aligned}$$

$$|f(t) - f(x)| \leq 6M_f(1 + a^2)(t - x)^2. \tag{17}$$

For $x \in [0, a]$ and $t \leq a + 1$, we have

$$|f(t) - f(x)| \leq \omega_{a+1}(f, |t - x|) \leq \left(1 + \frac{|t-x|}{\delta}\right) \omega_{a+1}(f, \delta) \tag{18}$$

with $\delta > 0$.
From (17) and (18), we get

$$|f(t) - f(x)| \leq 6M_f(1 + a^2)(t - x)^2 + \left(1 + \frac{|t-x|}{\delta}\right) \omega_{a+1}(f, \delta).$$

For $x \in [0, a]$ and $t \geq 0$,

$$\begin{aligned} |\mathcal{D}^{(\alpha,\beta)}_{n,q_n}(f, x) - f(x)| &\leq \mathcal{D}^{(\alpha,\beta)}_{n,q_n}(|f(x) - f(t)|, x) \\ &\leq 6M_f(1 + a^2)\mathcal{D}^{(\alpha,\beta)}_{n,q_n}((t - x)^2, x) \\ &\quad + \omega_{a+1}(f, \delta) \left(1 + \frac{1}{\delta}[\mathcal{D}^{(\alpha,\beta)}_{n,q_n}((t - x)^2, x)]^{\frac{1}{2}}\right). \end{aligned}$$

Hence, by Schwarz's inequality and Lemma 2.4, for every $q_n \in (0, 1)$ and $x \in [0, a]$

$$|\mathcal{D}^{(\alpha,\beta)}_{n,q_n}(f, x) - f(x)| \leq 6M_f(1 + a^2)\alpha_n(q_n, x) + \omega_{a+1}(f, \delta) \left(1 + \frac{1}{\delta}\sqrt{\alpha_n(q_n, x)}\right).$$

By taking $\delta = \sqrt{\alpha_n(q_n, x)}$, we get the assertion of our theorem. ■

6. Better estimation

It is well know that the operators preserve constant as well as linear functions. To make the convergence faster, King [12] proposed an approach to modify the classical Bernstein polynomials, so that this sequence preserves two test functions e_0 and e_1. After this several researchers have studied that many approximating operators L, possess these properties i.e. $L(e_i, x) = e_i(x)$ where $e_i(x) = x^i (i = 0, 1)$, for examples Bernstein, Baskakov and Baskakov-Durrmeyer-Stancu operators.

As the operators $\mathcal{D}^{(\alpha,\beta)}_{n,q}$ introduced in (6) preserve only the constant functions so further modification of these operators is proposed to be made so that the modified operators preserve the constant as well as linear functions, for this purpose the modification of $\mathcal{D}^{(\alpha,\beta)}_{n,q}$ as follows:

$$\mathcal{D}^{*(\alpha,\beta)}_{n,q}(f, x) = = [n]_q \sum_{k=0}^{\infty} p^q_{n,k}(r_n(x)) \int_0^{q/1-q^n} q^{-k-1} s^q_{n,k}(t) f\left(\frac{[n]_q t q^{-k} + \alpha}{[n]_q + \beta}\right) d_q t$$

where $r_n(x) = \frac{([n]_q+\beta)x-(\alpha+q)}{[n]_q}$ and $x \in I_n = \left[\frac{\alpha+q}{[n]_q+\beta}, \infty\right)$.

Lemma 6.1. *For each $x \in I_n$, we have*

$$D^{*(\alpha,\beta)}_{n,q}(1,x) = 1,\ \mathcal{D}^{*(\alpha,\beta)}_{n,q}(t,x) = x,$$
$$\mathcal{D}^{*(\alpha,\beta)}_{n,q}(t^2,x) = \left[\frac{[n]_q}{q([n]_q+\beta)^2} + 1\right]x^2 + \left[\frac{[n]_q(1+q^2)}{([n]_q+\beta)^2}\right]x + \left[\frac{q^3}{([n]_q+\beta)^2}\right].$$

Lemma 6.2. *For $x \in I_n$, the following holds,*

$$\tilde{\mu}^q_{n,1}(x) = \mathcal{D}^{*(\alpha,\beta)}_{n,q}(t-x,x) = 0,$$

$$\tilde{\mu}^q_{n,2}(x) = \mathcal{D}^{*(\alpha,\beta)}_{n,q}((t-x)^2,x) = \frac{[n]_q}{q([n]_q+\beta)^2}x^2 + \left[\frac{[n]_q(1+q^2)}{([n]_q+\beta)^2}\right]x + \left[\frac{q^3}{([n]_q+\beta)^2}\right].$$

Theorem 6.3. *Let $f \in C_B(I_n)$, $x \in I_n$ and $0 < q < 1$. Then, for $n > 1$, there exist an absolute constant $C > 0$ such that*

$$\left|\mathcal{D}^{*(\alpha,\beta)}_{n,q}(f,x) - f(x)\right| \le C\omega_2\left(f, \sqrt{\tilde{\mu}^q_{n,2}(x)}\right).$$

Proof. Let $g \in C_B(I_n)$ and $x, t \in I_n$. By Taylor's expansion we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du. \tag{19}$$

Applying $\mathcal{D}^{*(\alpha,\beta)}_{n,q}$, we get

$$\mathcal{D}^{*(\alpha,\beta)}_{n,q}(g,x) - g(x) = g'(x)\mathcal{D}^{*(\alpha,\beta)}_{n,q}((t-x),x) + \mathcal{D}^{*(\alpha,\beta)}_{n,q}\left(\int_x^t (t-u)g''(u)du, x\right).$$

Obviously we have $\left|\int_x^t (t-x)g''(u)du\right| \le (t-x)^2\|g''\|$,

$$\left|\mathcal{D}^{*(\alpha,\beta)}_{n,q}(g,x) - g(x)\right| \le \mathcal{D}^{*(\alpha,\beta)}_{n,q}((t-x)^2,x)\|g''\| = \tilde{\mu}^q_{n,2}\|g''\|.$$

Since $\left|\mathcal{D}^{*(\alpha,\beta)}_{n,q}(f,x)\right| \le \|f\|$,

$$\begin{aligned}\left|\mathcal{D}^{*(\alpha,\beta)}_{n,q}(f,x) - f(x)\right| &\le \left|\mathcal{D}^{*(\alpha,\beta)}_{n,q}(f-g,x) - (f-g)(x)\right| + \left|\mathcal{D}^{*(\alpha,\beta)}_{n,q}(g,x) - g(x)\right| \\ &\le 2\|f-g\| + \tilde{\mu}^q_{n,2}\|g''\|.\end{aligned}$$

Taking infimum overall $g \in C^2(I_n)$, we obtain

$$\left|\mathcal{D}^{*(\alpha,\beta)}_{n,q}(f,x) - f(x)\right| \le K_2(f, \tilde{\mu}^q_{n,2}).$$

In view of (8), we have

$$\left|\mathcal{D}^{*(\alpha,\beta)}_{n,q}(f,x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\tilde{\mu}^q_{n,2}}\right),$$

which proves the theorem. ■

Theorem 6.4. *Assume that $q_n \in (0,1)$, $q_n \to 1$ as $n \to \infty$. Then for any $f \in C^*_{x^2}(I_n)$ such that $f', f'' \in C^*_{x^2}(I_n)$, we have*

$$\lim_{n\to\infty}[n]_{q_n}\left[\mathcal{D}^{*(\alpha,\beta)}_{n,q_n}(f,x) - f(x)\right] = x(1+x)/2f''(x)$$

for every $x \in I_n$.

Proof. The proof of above Theorem is in similar manner as Theorem 4.2. ■

7. Conclusion

The results of our lemmas and theorems are more general rather than the results of any other previous proved lemmas and theorems, which will be enrich the literate of Applications of quantum calculus in operator theory and convergence estimates in the theory of approximations by linear positive operators. The researchers and professionals working or intend to work in areas of analysis and its applications will find this research article to be quite useful. Consequently, the results so established may be found useful in several interesting situation appearing in the literature on Mathematical Analysis, Applied Mathematics and Mathematical Physics.

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