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Dynamic Betting Game

Charles Jwo-Yue Lien
Los Altos High School, 201 Almond Ave., Los Altos, CA 94022, USA
Email: charleslien97@gmail.com

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Abstract. Players A and B play a betting game. Player A starts with initial money n. In each of k rounds, player A can wager an integer w between 0 and what he has currently. B then decides whether A wins or loses. If A wins, he receives w money, and if A loses, he loses w money. A total of k rounds are played, but A can only lose r times. What strategy should A use to end with the maximum amount of money, $D(n, k, r)$?

In this paper, we provide a strategy for A to maximize his money and the algorithm to calculate $D(n, k, r)$. We study the periodicity of $D(n+1, k, r) - D(n, k, r)$ relative to n. We will also extend n and w to non-negative real numbers. The maximum amount of money that A can obtain with continuous money is $C(n, k, r)$, and we study the relationship between C and D.

Keywords: Betting game; Dynamic game; Zero-sum game.

1. Introduction

Professor Yeong-Nan Yeh[15] proposed the following problem:

Players A and B play a betting game. Player A starts with initial money n. In each of k rounds, player A can wager an integer w between 0 and what he has currently. B then decides whether A wins or loses. If A wins, he receives w money, and if A loses, he loses w money. A total of k rounds are played, but A can only lose r times. What strategy should A use to end with the maximum amount of money, $D(n, k, r)$?

The strategy of this problem looks like that of a two-player zero-sum game with player A having choice of $n+1$ pure strategies with the w ranging from 0 to n and player B has choice of 2 pure strategies, letting A win or lose. However, in a zero-sum game, A's choice and B's choice are independent. This problem is much more difficult because in this game, B's decision is made in response to A's decision.

When I first started the question, I wrote a computer program to calculate many examples of $D(n,k,r)$. With the data, I could identify the periodic properties of $D(n+1,k,r) - D(n,k,r)$, but I could not find a general equation. Because of this, I decided to look at $C(n,k,r)$, which extends the domains of n and wagers to non-negative real numbers. By doing so, the pattern for $C(n,k,r)$ became very clear and I used the information from $C(n,k,r)$ to help explain certain properties of $D(n,k,r)$.

In section 2, we describe the strategy for player A and the algorithm to calculate $D(n,k,r)$. We also discuss the periodicity of $D(n+1,k,r) - D(n,k,r)$, and properties and some specific examples of D. In section 3, we change the domains of n and the wagers to non-negative real numbers, giving a new maximum money function $C(n,k,r)$. We find A's strategy for C and the equation for C as well as analyze special properties. In section 4, we talk about the relationship between C and D and the formula for calculating the period of $D(n+1,k,r) - D(n,k,r)$.

For further information in dynamic games, combinatorial identities, and number theory, please refer to [16–18].

2. Strategy and Properties of $D(n,k,r)$

In the case of $r = 0$ or $r = k$, B has no choices and A will wager the maximum allowed for $r = 0$ and wagers 0 for $r = k$ at all bets. So we have:

$$D(n,k,0) = n \cdot 2^k, \tag{1}$$
$$D(n,k,k) = n. \tag{2}$$

Lemma 2.1. *For $k > r > 0$, we have:*

$$D(n,k,r) = \max_{0 \le w \le n} \{\min\{D(n-w,k-1,r-1), D(n+w,k-1,r)\}\} \tag{3}$$

Proof. First, we prove that there exists a solution. Let G be the gain matrix for player A. $G_{ij} = D(n-(i-1)\cdot(-1)^j, k-1, r-j+1)$, where $i = w+1$ and $j = 1$ means player A wins and $j = 2$ means player A loses. When player A chooses a pure strategy, he knows that player B will minimize his gain, so player A checks each row for the minimum value and chooses the row that has the highest minimum value. That is, player A chooses a row to maximize $\min_j\{G_{ij}\}$. After player A chooses a row a, player B will choose the column b, to minimize G_{aj} which is $\min_j\{G_{aj}\} = G_{ab}$. That means A's gain is G_{ab}.

Secondly, we prove that the gain G_{ab} is a stable solution. If player B changes his choice to a different column, player A's gain will increase because G_{ab} is the minimum of G_{a1} and G_{a2}. Therefore, player B won't change his choice unless $G_{a1} = G_{a2}$, which will not change the solution. If player A changes his choice to row c, player B will choose the column to minimize the gain to $\min_j\{G_{cj}\}$, which will reduce player A's gain to $\min_j\{G_{cj}\} \le \min_j\{G_{aj}\}$. Therefore, player A will not change his choice unless $\min_j\{G_{cj}\} = \min_j\{G_{aj}\}$, which does not change the gain. So G_{ab} is a stable solution.

Lastly, we show G_{ab} is the maximum value. Assuming there is another solution $G_{de} > G_{ab}$. That is, player A chooses row d. Since player B chose column e, it implies $G_{de} = \min_j\{G_{dj}\} \le \min_j\{G_{aj}\} = G_{ab}$. However, this contradicts our previous assumption that $G_{de} > G_{ab}$. Therefore, G_{ab} is the maximum value. That is, $D(n,k,r) = G_{ab} = \max_{0\le w\le n}\{\min\{D(n-w,k-1,r-1), D(n+w,k-1,r)\}\}$. ■

Lemma 2.2.

(a) *If $n \ge m$, then $D(n,k,r) - D(m,k,r) \ge n-m$;*
(b) *If $k > l$, then $D(n,k,r) \ge D(n,l,r)$;*
(c) *If $r > s$, then $D(n,k,r) \le D(n,k,s)$;*
(a) *If $r = k$ or $n \le r$, then $D(n,k,r) = n$.*

Proof. (a) $D(n,k,r)$ is the maximum money player A can have at the end with initial money n. If player A is required to set aside $n-m$ and only use m as initial money to play the game, he will have maximum amount of money, $n-m+D(m,k,r)$ at the end of the game. Since $D(n,k,r)$ is the upper bound, we have $D(n,k,r) \ge n-m+D(m,k,r)$.

(b) If player A wagers 0 for the initial $k-l$ bets, player B will of course let player A win on those bets. Player A will have equal or less money at the end due to the constraint.

(c) By Lemma 2.1 (a) and (b), we have $D(n,k,r) = \max_{0\le w\le n}\{\min\{D(n-w,k-1,r-1), D(n+w,k-1,r)\}\} \le D(n-w,k-1,r-1) \le D(n,k,r-1)$. Hence it follows from induction that (c) holds.

(d) Since player A has less than $r+1$ money, he will lose the wager if he wagers more than 0. ■

Define $w_d(n,k,r)$ as the wager to maximize $\min\{D(n-w,k-1,r-1), D(n+w,k-1,r)\}$. If there is more than one optimized wager, we will define the smallest wager as $w_d(n,k,r)$. For simplicity, let w_d or $w_d(n)$ denote $w_d(n,k,r)$. That is, $D(n,k,r) = \min\{D(n-w_d,k-1,r-1), D(n+w_d,k-1,r)\}$.

Lemma 2.3. *$D(x+y,k,r) \ge D(x,k,r) + D(y,k,r)$.*

Proof. For $k=1$, we have $D(x+y,1,0) = 2(x+y) = D(x,1,0) + D(y,1,0)$ and $D(x+y,1,1) = x+y = D(x,1,1) + D(y,1,1)$. Assuming the statement is true for $k \le m$, for $k = m+1$, if $r = m+1$, $D(x+y,m+1,m+1) = D(x,m+1,m+1) + D(y,m+1,m+1)$. If $r \le m$, we have $D(x,m+1,r) =$

$\min\{D(x+w_d(x),m,r), D(x-w_d(x),m,r-1)\}$ and $D(y,m+1,r) = \min(D(y+w_d(y),m,r), D(y-w_d(y),m,r-1))$. From our assumption, we have $D(x+y+w_d(x)+w_d(y),m,r) \geq D(x+w_d(x),m,r) + D(y+w_d(y),m,r) \geq D(x,m+1,r) + D(y,m+1,r)$. Similarly, $D(x+y-w_d(x)-w_d(y),m,r-1) \geq D(x-w_d(x),m,r-1) + D(y-w_d(y),m,r-1) \geq D(x,m+1,r) + D(y,m+1,r)$. Therefore, $D(x+y,m+1,r) \geq D(x,m+1,r) + D(y,m+1,r)$. ■

In order to use the algorithm from Lemma 2.1, we must compare all $n+1$ pairs of values given from $D(n-w,k-1,r-1)$ and $D(n+w,k-1,r)$. We will introduce an integer w_i in Lemma 2.4 such that we only need to compare the values given from two numbers: $D(n-w_i,k-1,r-1)$ and $D(n+w_i,k-1,r)$. w_d can therefore only be either w_i or w_i+1.

Lemma 2.4. *For given* $0 < r < k$ *and* $0 < n$*, the function* $F(w) = D(n-w,k-1,r-1) - D(n+w,k-1,r)$ *is a monotonically decreasing function with respect to* $0 \leq w \leq n$*. Furthermore, there exists a unique integer* $0 \leq w_i < n$ *such that* $F(w_i) \geq 0$ *and* $F(w_i+1) < 0$.

Proof. For $0 \leq w < n$, from Lemma 2.2a, we have $D(n-w,k-1,r-1) \geq D(n-(w+1),k-1,r-1)+1$ and $D(n+(w+1),k-1,r) \geq D(n+w,k-1,r)$. Hence, $F(w) = D(n-w,k-1,r-1) - D(n+w,k-1,r) \geq D(n-(w+1),k-1,r-1) - D(n+(w+1),k-1,r) + 2 > F(w+1)$. Therefore, $F(w) = D(n,k-1,r-1) - D(n+w,k-1,r)$ is a monotonically decreasing function with respect to $0 \leq w \leq n$. Moreover, since $F(0) = D(n,k-1,r-1) - D(n,k-1,r) \geq 0$ by Lemma 2.2c and $F(n) = D(0,k-1,r-1) - D(2n,k-1,r) < 0$, there exists a unique integer $0 \leq w_i < n$ such that $F(w_i) \geq 0$ and $F(w_i+1) < 0$. ■

Lemma 2.5.
(a) $D(n,k,r) = \max\{D(n+w_i,k-1,r), D(n-w_i-1,k-1,r-1)\}$;
(b) *If* $D(n+w_i,k-1,r) = D(n-w_i,k-1,r-1)$*, then* $w_d = w_i$ *and* $D(n,k,r) = D(n+w_d,k-1,r) = D(n-w_d,k-1,r-1)$;
(c) *If* $D(n+w_i,k-1,r) \neq D(n-w_i,k-1,r-1)$*, then:*
  (i) *If* $D(n+w_i,k-1,r) > D(n-w_i-1,k-1,r-1)$*, then* $w_d = w_i$ *and* $D(n,k,r) = D(n+w_d,k-1,r)$;
  (ii) *If* $D(n+w_i,k-1,r) < D(n-w_i-1,k-1,r-1)$*, then* $w_d = w_i+1$ *and* $D(n,k,r) = D(n-w_d,k-1,r-1)$.

Proof. (a) From Lemma 2.4, there exists a unique integer $0 \leq w_i < n$ such that $F(w_i) \geq 0$ and $F(w_i+1) < 0$. Hence for $0 \leq w \leq w_i$, $D(n-w,k-1,r-1) \geq D(n+w,k-1,r)$. For $w_i+1 \leq w \leq n$, $D(n-w,k-1,r-1) < D(n+w,k-1,r)$. Therefore, Lemma 2.1, we have that $D(n,k,r) = \max_{0\leq w\leq n}\{\min\{D(n-w,k-1,r-1), D(n+w,k-1,r)\}\} = \max\{\max_{0\leq w\leq w_i}\{\min\{D(n-w,k-1,r-1), D(n+w,k-1,r)\}\}, \max_{w_i+1\leq w\leq n}\{\min\{D(n-w,k-1,r-1), D(n+w,k-1,r)\}\}\} = \max\{\max_{0\leq w\leq w_i}\{D(n+w,k-1,r)\}, \max_{w_i+1\leq w\leq n}\{D(n-w,k-1,r-1)\}\} = \max\{D(n+w_i,k-1,r), D(n-w_i-1,k-1,r-1)\}$.

(b) If $D(n+w_i,k-1,r) = D(n-w_i,k-1,r-1)$, then $D(n,k,r) = D(n+w_i,k-1,r)$ and $w_d = w_i$. This is because $D(n-w_i,k-1,r-1) > D(n-w_i-1,k-1,r-1)$.

(c) (i) If $D(n+w_i,k-1,r) > D(n-w_i-1,k-1,r-1)$, then $D(n,k,r) = D(n+w_d,k-1,r)$ and $w_d = w_i$.

(c) (ii) If $D(n+w_i,k-1,r) < D(n-w_i-1,k-1,r-1)$, then $w_d = w_i+1$ and $D(n,k,r) = D(n-w_d,k-1,r-1)$. ■

Lemmas 2.4 and 2.5 allowed us to write a simple computer program to find $D(n,k,r)$. We then used this computer program to find many values of $D(n,k,r)$. From the data, we observed properties such as the periodicity of $D(n+1,k,r) - D(n,k,r)$ with respect to n, but it is difficult to find a pattern for the period. We will, however, be able to prove the periodicity and find the period in Section 4. Some special cases are provided such as $r = 1$, $r = 2$, and $r = k-1$. In these cases, we will use Lemmas 2.4 and 2.5 to find w_i, w_d, and $D(n,k,r)$. The periodicity of $D(n+1,k,r) - D(n,k,r)$ is also observed in these cases.

Case 1 ($r = 1$):

For $k = 2$, $D(n-w_i,1,0) \geq D(n+w_i,1,1)$ implies $2(n-w_i) \geq n+w_i$. This means $w_i \leq \frac{n}{3}$. Similarly, $D(n-w_i-1,1,0) < D(n+w_i+1,1,1)$ implies $w_i+1 > \frac{n}{3}$, so we have $w_i = \lfloor\frac{n}{3}\rfloor$. $D(3m+s,2,1) = \max\{D(2m+s-1,1,0), D(4m+s,1,1)\} = \max\{2(2m+s-1), 4m+s\} = 4m+s = \lfloor\frac{4n}{3}\rfloor$, where $0 \leq s \leq 2$ and $w_d = m$. Therefore, $D(n,2,1) = \lfloor\frac{4n}{3}\rfloor$ and $w_d = \lfloor\frac{n}{3}\rfloor$. We have observed that for $n \geq 0$, $D(n+1,2,1) - D(n,2,1)$ is a periodic sequence with a period of 3: 1, 1, 2, 1, 1, 2, 1, 1, 2, ... We define $P(2,1) = 3$.

For $k = 3$, $D(n-w_i,2,0) \geq D(n+w_i,2,1)$ and $D(n-w_i-1,2,0) < D(n+w_i+1,2,1)$ imply $4(n-w_i) \geq \lfloor\frac{4(n+w_i)}{3}\rfloor$ and $4(n-w_i-1) < \lfloor\frac{4(n+w_i+1)}{3}\rfloor$. Therefore, we have $w_i = \lfloor\frac{n}{2}\rfloor$, so $D(2m,3,1) = D(m,2,0) = D(3m,2,1) = 4m = 2n$ and $w_i = m$. $D(2m+1,3,1) = \max\{D(m,2,0), D(3m+1,2,1)\} = \max\{4m, \lfloor\frac{4(3m+1)}{3}\rfloor\} = \lfloor\frac{4(3m+1)}{3}\rfloor = 4m+1 = n+2\lfloor\frac{n}{2}\rfloor$ and $w_d = m$. Therefore, $D(n,3,1) = n+2\lfloor\frac{n}{2}\rfloor$ and $w_d = \lfloor\frac{n}{2}\rfloor$. We have observed that for $n \geq 0$, $D(n+1,3,1) - D(n,3,1)$ is a periodic sequence with a period of 2: 1, 3, 1, 3, 1, 3, ... We define $P(3,1) = 2$.

For $k = 4$, $D(n-w_i,3,0) \geq D(n+w_i,3,1)$ and $D(n-w_i-1,3,0) < D(n+w_i+1,3,1)$ imply $8(n-w_i) \geq n+w_i+2\lfloor\frac{n+w_i}{2}\rfloor$ and $8(n-w_i-1) < n+w_i+1+2\lfloor\frac{n+w_i+1}{2}\rfloor$. If $n = 5m+s$, where $0 \leq s < 5$, then $w_i = 3m+\lfloor\frac{s}{2}\rfloor = \lfloor\frac{3n}{5}\rfloor$. $D(5m+s,4,1) = \max\{D(2m+s-\lfloor\frac{s}{2}\rfloor-1,3,0), D(8m+s+\lfloor\frac{s}{2}\rfloor,3,1)\} = \max\{8(2m+s-\lfloor\frac{s}{2}\rfloor-1), 8m+s+\lfloor\frac{s}{2}\rfloor+2\lfloor\frac{m+s+\lfloor\frac{s}{2}\rfloor}{2}\rfloor\} = 8m+s+\lfloor\frac{s}{2}\rfloor+2\lfloor\frac{8m+s+\lfloor\frac{s}{2}\rfloor}{2}\rfloor = \lfloor\frac{8n}{5}\rfloor+2\lfloor\frac{4n}{5}\rfloor$ and $w_d = w_i = \lfloor\frac{3n}{5}\rfloor$. We have observed that for $n \geq 0$, $D(n+1,4,1) = D(n,4,1)$ is a periodic sequence with a period of 5: 1, 4, 3, 4, 4, 1, 4, 3, 4, 4, 1, 4, 3, 4, 4, ... We define $P(4,1) = 5$.

Examples of $D(n,k,1)$ are shown in Table 1. Later on, we will prove that if k is even, $P(k,1) = k+1$, if k is odd, $P(k,1) = \frac{k+1}{2}$, and $D(n+1,k,r) - D(n,k,r)$ is a periodic sequence with a period of $P(k,r)$.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11
2	1	2	4	5	6	8	9	10	12	13	14
3	1	4	5	8	9	12	13	16	17	20	21
4	1	5	8	12	16	17	21	24	28	32	33
5	1	8	16	17	24	32	33	40	48	49	56
6	1	16	24	32	40	49	64	65	80	88	96
7	1	24	40	64	65	88	104	128	129	152	168
8	1	40	65	104	128	152	192	216	256	257	296
9	1	65	128	192	256	257	321	384	448	512	513
10	1	128	256	321	448	512	577	704	769	896	1024

Table 1. Values of $D(n, k, 1)$

Case 2 ($r = 2$):

For $k = 3$, $D(n - w_i, 2, 1) \geq D(n + w_i, 2, 2)$ and $D(n - w_i - 1, 2, 1) < D(n + w_i + 1, 2, 2)$ imply $\lfloor \frac{4(n-w_i)}{3} \rfloor \geq n + w_i$ and $\lfloor \frac{4(n-w_i-1)}{3} \rfloor < n + w_i + 1$, so $w_i = \lfloor \frac{n}{7} \rfloor$. $D(7m+s, 3, 2) = \max\{D(6m+s, 2, 1), D(8m+s, 2, 2)\} = \max\{\lfloor \frac{4(6m+s-1)}{3} \rfloor, 8m+s\} = 8m + s = \lfloor \frac{8n}{7} \rfloor$ and $w_d = w_i = \lfloor \frac{n}{7} \rfloor$ for $0 \leq s < 7$. We have observed that for $n \geq 0$, $D(n+1, 3, 2) - D(n, 3, 2)$ is a periodic sequence with a period of 7: 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 2, ... We define $P(3, 2) = 7$.

For $k = 4$, $D(n - w_i, 3, 1) \geq D(n + w_i, 3, 2)$ and $D(n - w_i - 1, 3, 1) < D(n + w_i + 1, 3, 2)$ imply $n - w_i + 2\lfloor \frac{n-w_i}{2} \rfloor \geq \lfloor \frac{8(n+w_i)}{7} \rfloor$ and $n - w_i - 1 + 2\lfloor \frac{n-w_i-1}{2} \rfloor < \lfloor \frac{8(n+w_i+1)}{7} \rfloor$.So we have $w_i = \lfloor \frac{n}{11} \rfloor + \lfloor \frac{2n+6}{11} \rfloor$. If $n = 11m + s$, let $w_i = 3m + a$. Then, $D(11m + s, 4, 2) = \max\{D(8m + s - a - 1, 3, 1), D(14m + s + a, 3, 2)\} = \max\{8m + s - a - 1 + 2\lfloor \frac{8m+s-a-1}{2} \rfloor, \lfloor \frac{8(14m+s+a)}{7} \rfloor\} = 16m + s + a + \lfloor \frac{s+a}{7} \rfloor$ and $w_d = w_i$, where $0 \leq s < 11$. Therefore, $D(n, 4, 2) = \lfloor \frac{12n}{11} \rfloor + \lfloor \frac{4n+1}{11} \rfloor$ and $w_d = \lfloor \frac{n}{11} \rfloor + \lfloor \frac{2n+6}{11} \rfloor$. We have observed that for $n \geq 0$, $D(n+1, 4, 2) - D(n, 4, 2)$ is a periodic sequence with a period of 11: 1, 1, 2, 1, 1, 2, 1, 2, 1, 1, 3, 1, 1, 2, 1, 1, 2, 1, 2, 1, 1, 3, ... We define $P(4, 2) = 11$.

Table 2 shows examples of $D(n, k, 2)$.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
2	1	2	3	4	5	6	7	8	9	10
3	1	2	3	4	5	6	8	9	10	11
4	1	2	4	5	6	8	9	11	12	13
5	1	2	5	6	8	11	12	16	17	18
6	1	2	6	8	12	16	17	21	24	27
7	1	2	8	16	21	24	27	32	34	40
8	1	2	16	24	27	34	44	49	59	64
9	1	2	24	34	44	59	65	76	88	104
10	1	2	34	59	65	88	112	128	145	168

Table 2. Values of $D(n, k, 2)$

From the above discussions and analyses, for small k and r, we have the explicit formula for $D(n,k,r)$.

Corollary 2.6. $D(n,2,1) = \lfloor \frac{4n}{3} \rfloor$; $D(n,3,1) = n + 2\lfloor \frac{n}{2} \rfloor$; $D(n,4,1) = \lfloor \frac{8n}{5} \rfloor + 2\lfloor \frac{4n}{5} \rfloor$; $D(n,3,2) = \lfloor \frac{8n}{7} \rfloor$; $D(n,4,2) = \lfloor \frac{12n}{11} \rfloor + \lfloor \frac{4n+1}{11} \rfloor$.

Case 3 ($r = k-1$):
For $k=3$, we have $D(n,2,1) = \lfloor \frac{4n}{3} \rfloor$, $w_d = \lfloor \frac{n}{3} \rfloor$, and $P(2,1) = 3$.
For $k=4$, we have $D(n,3,2) = \lfloor \frac{8n}{7} \rfloor$, $w_d = \lfloor \frac{n}{7} \rfloor$, and $P(3,2) = 7$.
For $k=5$, we have $D(n,4,3) = \lfloor \frac{16n}{15} \rfloor$, $w_d = \lfloor \frac{n}{15} \rfloor$, and $P(4,3) = 15$.

We observe and will later prove that $D(n,k,k-1) = \lfloor \frac{n \cdot 2^k}{2^k - 1} \rfloor$, $w_d = \lfloor \frac{n}{2^k - 1} \rfloor$, and $P(k,k-1) = 2^k - 1$.

Table 3 shows examples of $P(k,r)$.

$k \backslash r$	0	1	2	3	4	5	6	7	8
1	1	(1)							
2	1	2^2-1	(1)						
3	1	2^1	2^3-1	(1)					
4	1	5	2^4-5	2^4-1	(1)				
5	1	3	2^3	2^4-3	2^5-1	(1)			
6	1	7	11	2^5-11	2^6-7	2^6-1	(1)		
7	1	4	29	2^4	2^7-29	2^6-4	2^7-1	(1)	
8	1	9	37	93	2^8-93	2^8-37	2^8-9	2^8-1	(1)
9	1	5	23	65	2^7	2^8-65	2^8-23	2^8-5	2^9-1
10	1	11	14	44	193	2^9-193	2^8-44	2^8-14	$2^{10}-11$
11	1	6	67	29	281	2^8	$2^{10}-281$	2^8-29	$2^{11}-67$
12	1	13	79	299	397	793	$2^{11}-793$	$2^{11}-397$	$2^{12}-299$
13	1	7	46	189	1093	595	2^{10}	$2^{11}-595$	$2^{13}-1093$
14	1	15	53	235	1471	3473	1619	$2^{12}-1619$	$2^{14}-3473$
15	1	8	121	144	1941	2472	9949	2^{11}	$2^{15}-9949$

Table 3. Values $P(k,r)$

3. Solutions of $C(n,k,r)$

Let the initial money n and wagers be non-negative real numbers and call the maximum money player A can have after all k rounds $C(n,k,r)$.

Lemma 3.1.

(a) $C(1,k,0) = 2^k$;
(b) $C(1,k,k) = 1$;
(c) $D(n,k,r) \le C(n,k,r)$;
(d) $C(n,k,r) = n \cdot C(1,k,r)$.

Proof. (a) If Player A wagers the maximum money allowed at each bet, he will win all the bets and yield part (a).

(b) Player A cannot wager any amount of money other than 0 at each bet or else he will lose the bet.

(c) Since $D(n,k,r)$ is the money at the end with the constraint of integer wagers and $C(n,k,r)$ is the maximum money at the end without the constraint, so $D(n,k,r) \leq C(n,k,r)$.

(d) Converting n dollars into 1 unit and changing the unit back to dollars at the very end yields part (d). Therefore, $C(n,k,r)$ is a continuous function with respect to n. ■

Lemma 3.2. *For given $0 < r < k$, there exists a unique non-negative real number $w_c(k,r) \leq 1$, which will be written as w_c for simplicity, such that $C(1-w_c, k-1, r-1) = C(1+w_c, k-1, r)$.*

Proof. As w increases from 0 to n, $C(1-w,k-1,r-1) - C(1+w,k-1,r)$ is monotonically decreasing from $C(1,k-1,r-1) - C(1,k-1,r) \geq 0$ to $C(0,k-1,r-1) - C(2,k-1,r) = C(2,k-1,r) < 0$. Since $C(1-w,k-1,r-1) - C(1+w,k-1,r)$ is continuous, there exists a unique number, w_c, such that $C(1-w_c,k-1,r-1) - C(1+w_c,k-1,r) = 0$. ■

Similar to the integer case, we have that, for continuous wagers, $C(1,k,r) = \max_{0 \leq w \leq 1}\{\min\{C(1-w,k-1,r-1), C(1+w,k-1,r)\}\}$.

Lemma 3.3. *For $k > r > 0$,*

$$C(1,k,r) = C(1-w_c,k-1,r-1) = C(1+w_c,k-1,r) \tag{4}$$

Proof. If $w < w_c$, we have $C(1+w,k-1,r) < C(1+w_c,k-1,r) = C(1-w_c,k-1,r-1) < C(1-w,k-1,r-1)$. Therefore $\min\{C(1-w_c,k-1,r-1), C(1+w_c,k-1,r)\} > \min\{C(1-w,k-1,r-1), C(1+w,k-1,r)\}$. Similarly, If $w > w_c$, we have $C(1-w,k-1,r-1) < C(1-w_c,k-1,r-1) = C(1+w_c,k-1,r) < C(1+w,k-1,r)$. Therefore $\min\{C(1-w_c,k-1,r-1), C(1+w_c,k-1,r)\} > \min\{C(1-w,k-1,r-1), C(1+w,k-1,r)\}$. Since $C(1,k,r) = \max_{0 \leq w \leq 1}\{\min\{C(1-w,k-1,r-1), C(1+w,k-1,r)\}\}$ we have $C(1,k,r) = C(1-w_c,k-1,r-1) = C(1+w_c,k-1,r)$. ■

That is, for $k > r > 0$, player A chooses a wager at every possible bet such that the maximum money at the very end is independent of whether player B let him win or lose.

Lemma 3.4. $G(k,r) = \sum_{j=0}^{r} \binom{k}{j}$ *is the solution to the initial conditions $G(k,0) = 1$, $G(k,k) = 2^k$, and the recurrence formula $G(k,r) = G(k-1,r-1) + G(k-1,r)$,* *where* $0 < r < k$.

Proof. It is easy to see that $G(k,r)$ meets the initial conditions: $G(k,0) = \binom{k}{0} = 1$ and $G(k,k) = \sum_{j=0}^{k} \binom{k}{j} = 2^k$. For $0 < r < k$, we have $G(k,r) = \sum_{j=1}^{r} \binom{k}{j} + \binom{k}{0} = \sum_{j=1}^{r} (\binom{k-1}{j} + \binom{k-1}{j-1}) + \binom{k-1}{0} = G(k-1,r) + G(k-1,r-1)$. Therefore, $G(k,r) - G(k-1,r) = G(k-1,r-1)$. ■

Lemma 3.5. *If* r *and* k *are integers such that* $0 \le r \le k$, *then:*

(a) $C(1,k,r) = \frac{2^k}{G(k,r)}$;

(b) $w_c(k,r) = \frac{\binom{k-1}{r}}{G(k,r)}$.

Proof. (a) The cases for $r = 0$ and $r = k$ are trivial. For $0 < r < k$, from Lemma 3.3, we have $C(1+w_c, k-1, r) = C(1-w_c, k-1, r-1)$, so $(1+w_c) \cdot C(1,k-1,r) = (1-w_c) \cdot C(1,k-1,r-1)$ and $w_c = \frac{C(1,k-1,r-1) - C(1,k-1,r)}{C(1,k-1,r-1) + C(1,k-1,r)}$. Therefore, $\frac{1}{C(1,k,r)} = \frac{1}{2} \cdot (\frac{1}{C(1,k-1,r)} + \frac{1}{C(1,k-1,r-1)})$.

Let $F(k,r) = \frac{2^k}{C(1,k,r)}$. Then, we have $F(k,r) = F(k-1,r) + F(k-1,r-1)$ for $0 < r < k$, $F(k,0) = \frac{2^k}{C(1,k,0)} = 1$, and $F(k,k) = \frac{2^k}{C(1,k,k)} = 2^k$. Therefore, from Lemma 3.4, we have that $F(k,r) = G(k,r)$. We therefore have $C(1,k,r) = \frac{2^k}{G(k,r)}$.

(b) $w_c(k,r) = \frac{G(k-1,r) - G(k-1,r-1)}{G(k,r)} = \frac{\binom{k-1}{r}}{G(k,r)}$. ■

Lemma 3.6. $\frac{1}{C(1,k,r)} + \frac{1}{C(1,k,k-r-1)} = 1$.

Proof. $\frac{2^k}{C(1,k,r)} + \frac{2^k}{C(1,k,k-r-1)} = G(k,r) + G(k,k-r-1) = \sum_{j=0}^{r} \binom{k}{j} + \sum_{j=0}^{k-r-1} \binom{k}{j} = \sum_{j=0}^{r} \binom{k}{j} + \sum_{j=r+1}^{k} \binom{k}{j} = 2^k$. Therefore, $\frac{1}{C(1,k,r)} + \frac{1}{C(1,k,k-r-1)} = 1$. ■

Theorem 3.7. $C(n,k,r) = \frac{n \cdot 2^k}{\sum_{j=0}^{r} \binom{k}{j}}$.

Proof. From Lemmas 3.1 (d) and 3.5 (a), we have that $C(n,k,r) = \frac{n \cdot 2^k}{\sum_{j=0}^{r} \binom{k}{j}}$. ■

It is obvious that $C(n,k,r)$ has the following monotonic properties:

(a) If $n > m$, then $C(n,k,r) > C(m,k,r)$;

(b) If $k > m$ and $n > 0$, then $C(n,k,r) > C(n,m,r)$;

(c) If $r > m$ and $n > 0$, then $C(n, k, r) < C(n, k, m)$.

All the possible bets of $C(1, k, r)$ form a binary rooted tree, called the $C(1, k, r)$ tree. Each possible bet is a node which is labeled as (l, s, u), where l is how many bets are left over, s is how many times player A can still lose, and if there are more than one node for the same l and s, we use a natural number u to label them, where smaller u means earlier wins. Let the money owned at node (l, s, u) be $n(l, s, u)$. Then, the optimal wager at node (l, s, u) is $n(l, s, u) \cdot w_c(l, s)$. This is because w_c is the optimal wager when $n = 1$. The optimal wager at (l, s, u) is a function of n, k, r, l, s, and u. For our application, we normally fix k and r. For simplicity, we will call the optimal wagers of the $C(n, k, r)$ tree "wagers" and define them as $w_n(l, s, u)$ from now on. The wager of the root node $(k, r, 1)$ is $w_1(k, r, 1) = 1 \cdot w_c(k, r) = \frac{\binom{k-1}{r}}{G(k,r)}$ and the money owned for the root node is $n(k, r, 1) = 1$. $(0, 0, u)$, where $1 \leq u \leq \binom{k}{r}$, are leaf nodes which have no wagers and its $n(0, 0, u) = C(1, k, r)$.

The $C(n, k, r)$ tree has the same structure as that of the $C(1, k, r)$ tree except at each node, the optimal and the money owned are exactly n times of the corresponding wager and money owned of the $C(1, k, r)$ tree, respectively.

Lemma 3.8. *The $C(n, k, r)$ tree has the following properties:*

(a) *$C(n(l, s, u), l, s) = C(n, k, r)$;*

(b) *Money owned, $n(l, s, u)$, and wager, $w_n(l, s, u)$ at node (l, s, u) are independent of u.*

Proof. We will use k as the variable for induction. (a) Nodes of the $C(n, 1, 0)$ tree are $(1, 0, 1)$ and $(0, 0, 1)$. $n(1, 0, 1) = n$, $w_n(1, 0, 1) = n$, and $n(0, 0, 1) = 2n$. $C(n(1, 0, 1), 1, 0) = 2n = C(n(0, 0, 1), 0, 0) = C(n, 1, 0)$. Assuming the statement is true for $k \leq m$, for $k = m+1$, when $r = m+1$, the nodes of the $C(n, m+1, m+1)$ tree are $(0, 0, 1)$ and $(l, l, 1)$, where $0 < l \leq m + 1$. $n(0, 0, 1) = n(l, l, 1) = n$, and $w_n(l, l, 1) = 0$. $C(n(0, 0, 1), 0, 0) = C(n(l, l, 1), l, l) = n(l, l, 1) = n = C(n, m + 1, m + 1)$. For $0 \leq r \leq m$, non-root nodes (l, s, u) of the $C(n, m + 1, r)$ tree are nodes of the $C(n \cdot (1 + w_c), m, r)$ tree or $C(n \cdot (1 - w_c), m, r - 1)$ tree. $C(n(l, s, u), l, s) = C(n \cdot (1 + w_c), m, r)$ or $C(n \cdot (1 - w_c), m, r - 1)$. Since $C(n \cdot (1 + w_c), m, r) = C(n \cdot (1 - w_c), m, r - 1) = C(n, m + 1, r)$, we have proven part (a).

(b) From part (a), we have $C(n(l, s, u_1), l, s) = C(n, k, r) = C(n(l, s, u_2), l, s)$. This implies $n(l, s, u_1) = n(l, s, u_2)$. Since $w_n(l, s, u) = n(l, s, u) \cdot w_c(l, s)$, both $n(l, s, u)$ and $w_n(l, s, u)$ are independent of u. ■

We will use $n(l, s)$ and $w_n(l, s)$ instead of $n(l, s, u)$ and $w_n(l, s, u)$. Note that $w_n(l, s) = n(l, s) \cdot w_c(l, s)$.

Lemma 3.9.

(a) *Wagers of the $C(G(k, r), k, r)$ tree, $w_{G(k,r)}(k-m+1, r-j) = 2^{m-1} \cdot \binom{k-m}{r-j}$, where $1 \leq m \leq k$ and $\max\{0, r + m - k - 1\} \leq j \leq \min\{r, m - 1\}$;*

(b) *All the wagers of the* $C(G(k,r),k,r)$ *tree are integers.*

Proof. (a) Please refer to Tables 4 and 5 for visual aid. At the 1st bet (level 1) - the root node $(k,r,1)$, the money owned $n(k,r) = G(k,r)$ and wager $w_{G(k,r)}(k,r) = \binom{k-1}{r}$.

At the 2nd bet (level 2), there are 2 possible nodes:

Node $(k-1,r,1)$: $n(k-1,r) = G(k,r)+\binom{k-1}{r} = 2 \cdot G(k-1,r)$ and $w_{G(k,r)}(k-1,r) = 2 \cdot \binom{k-2}{r}$

Node $(k-1,r-1,1)$: $n(k-1,r-1) = 2 \cdot G(k-1,r-1)$ and $w_{G(k,r)}(k-1,r-1) = 2 \cdot \binom{k-2}{r-1}$

At the m^{th} bet (level m), nodes are $(k-m+1, r-j, u)$, where $\max\{0, r+m-k-1\} \leq j \leq \min\{r, m-1\}$.

Therefore, $n(k-m+1, r-j) = 2^{m-1} \cdot G(k-m+1, r-j)$, and $w_{G(k,r)}(k-m+1, r-j) = 2^{m-1} \cdot \binom{k-m}{r-j}$.

(b) Since $\binom{k-m}{r-j}$ are all integers, all the wagers are integers. ■

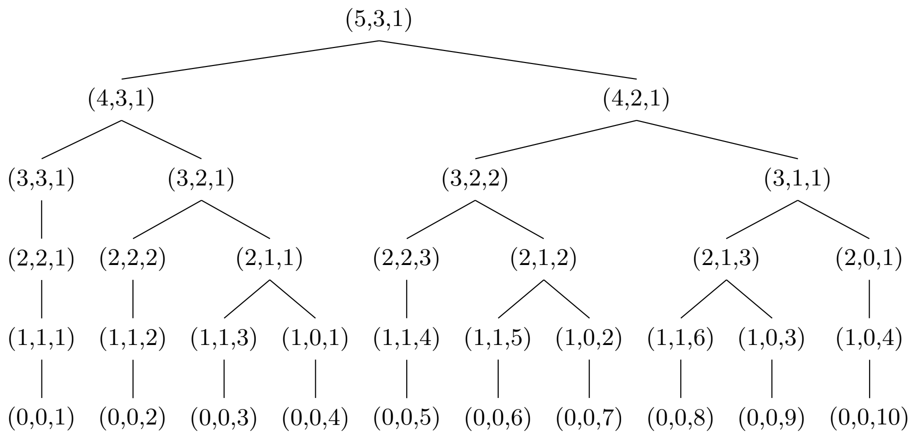

Table 4. The $C(n,5,3)$ tree.

Definition 3.10. *For any natural number* k *and any non-negative integer* $r < k$, *we define* $g(k,k) = 2^k$ *and* $g(k,r)$ *is the greatest common divisor of all* $w_{G(k,r)}(l,s)$ *of the* $C(G(k,r),k,r)$ *tree.*

Let $h(k,r,m)$ denote $2^{m-1} \cdot \gcd(\binom{k-m}{r}, \binom{k-m}{r-1}, \ldots, \binom{k-m}{r-m+1})$, where $0 < m \leq \min\{k-1, k-r, r+1\}$. Note that $h(k,r,m)$ is the greatest common factor of all the wagers at the m^{th} bet of the $C(G(k,r),k,r)$ tree. Since $w_{G(k,r)}(r+1,r) = 2^{k-r-1}$, we have $h(k,r,k-r) = 2^{k-r-1}$.

$l\backslash s$	8	7	6	5	4	3	2	1	0
13	$\binom{12}{8}$								
12	$2^1\cdot\binom{11}{8}$	$2^1\cdot\binom{11}{7}$							
11	$2^2\cdot\binom{10}{8}$	$2^2\cdot\binom{10}{7}$	$2^2\cdot\binom{10}{6}$						
10	$2^3\cdot\binom{9}{8}$	$2^3\cdot\binom{9}{7}$	$2^3\cdot\binom{9}{6}$	$2^3\cdot\binom{9}{5}$					
9	2^4	$2^4\cdot\binom{8}{7}$	$2^4\cdot\binom{8}{6}$	$2^4\cdot\binom{8}{5}$	$2^4\cdot\binom{8}{4}$				
8	0	2^5	$2^5\cdot\binom{7}{6}$	$2^5\cdot\binom{7}{5}$	$2^5\cdot\binom{7}{4}$	$2^5\cdot\binom{7}{3}$			
7		0	2^6	$2^6\cdot\binom{6}{5}$	$2^6\cdot\binom{6}{4}$	$2^6\cdot\binom{6}{3}$	$2^6\cdot\binom{6}{2}$		
6			0	2^7	$2^7\cdot\binom{5}{4}$	$2^7\cdot\binom{5}{3}$	$2^7\cdot\binom{5}{2}$	$2^7\cdot\binom{5}{1}$	
5				0	2^8	$2^8\cdot\binom{4}{3}$	$2^8\cdot\binom{4}{2}$	$2^8\cdot\binom{4}{1}$	2^8
4					0	2^9	$2^9\cdot\binom{3}{2}$	$2^9\cdot\binom{3}{1}$	2^9
3						0	2^{10}	$2^{10}\cdot\binom{2}{1}$	2^{10}
2							0	2^{11}	2^{11}
1								0	2^{12}
0									

Table 5. $w_{G(13,8)}(l,s)$ of the $C(G(13,8),13,8)$ tree

Lemma 3.11. *For $r<k$, $g(k,r)=\gcd(h(k,r,1),h(k,r,2),\ldots,h(k,r,min(k-1,k-r,r+1)))$.*

Proof. Since $w_{G(k,r)}(r+1,r)=2^{k-r-1}$, we have $g(k,r)=2^g$ where $g\le k-r-1$. Also, $2^{k-r}\mid w_{G(k,r)}(k-m,s)$ for $m>k-r$. Therefore, those cases do not need to be included in the calculation of the gcd. That is, $g(k,r)=\gcd(h(k,r,1),h(k,r,2),\ldots,h(k,r,min(k-1,k-r,r+1)))$. ■

Lemma 3.12. $g(k,r)=g(k,k-r-1)$ *for $r<k$.*

Proof. $h(k,k-r-1,m)=2^{m-1}\cdot\gcd(\binom{k-m}{k-r-1},\binom{k-m}{k-r-2},\ldots,\binom{k-m}{k-r-m})=2^{m-1}\cdot\gcd(\binom{k-m}{r-m+1},\binom{k-m}{r-m+2},\ldots,\binom{k-m}{r})=h(k,r,m)$. $g(k,k-r-1)=\gcd(h(k,k-r-1,1),h(k,k-r-1,2),\ldots,h(k,k-r-1,\min\{k-1,r+1,k-2\}))=g(k,r)$. ■

It is easy to see that $g(k,0)=1=g(k,k-1)$ since $\binom{k-1}{0}=1=\binom{k-1}{k-1}$.

Lemma 3.13.

(a) *For $0\le r<2^m$, we have $g(2^m,r)=1$.*
(b) *For $0<r<2^m$, we have $g(2^m+1,r)=2$.*

Proof. (a) $h(k,r,1)=\binom{k-1}{r}$, which is odd when $k=2^m$[8]. Therefore, $g(2^m,r)=$

1, where $r < 2^m$.

(b) $h(k,r,2) = 2\cdot\gcd(\binom{k-2}{r}, \binom{k-2}{r-1}) = 2$ when $k = 2^m+1$[8]. $h(k,r,1) = \binom{k-1}{r}$, which is even when $k = 2^m + 1$[8]. Therefore, $g(2^m+1, r) = 2$, where $r < 2^m$. ∎

Theorem 3.14. *The wagers of the $C(n,k,r)$ tree are all integers if and only if $\frac{G(k,r)}{g(k,r)} \mid n$, where $a \mid b$ means a divides b.*

Proof. For $k = r$, the theorem is true since both $\frac{G(k,r)}{g(k,r)} \mid n$ and the wagers of the $C(n,k,r)$ tree are all integers.

For $k > r$, from Lemma 3.9b and the definition of $g(k,r)$, we know that $w_{\frac{G(k,r)}{g(k,r)}}(l,s) = \frac{w_{G(k,r)}(l,s)}{g(k,r)}$ are all integers and are relatively prime. Therefore, the gcd of all $w_{\frac{G(k,r)}{g(k,r)}}(l,s)$ is 1. $w_n(l,s) = \frac{n}{\frac{G(k,r)}{g(k,r)}} \cdot w_{\frac{G(k,r)}{g(k,r)}}(l,s)$. If $n = m \cdot \frac{G(k,r)}{g(k,r)}$, then every $w_n(l,s)$ is an integer. Conversely, if every $w_n(l,s)$ is an integer, then $\frac{n}{\frac{G(k,r)}{g(k,r)}}$ is an integer. ∎

$k\backslash r$	0	1	2	3	4	5	6	7	8
1	1	(2)							
2	1	1	(2^2)						
3	1	2	1	(2^3)					
4	1	1	1	1	(2^4)				
5	1	2	2	2	1	(2^5)			
6	1	1	2	2	1	1	(2^6)		
7	1	2	1	2^2	1	2	1	(2^7)	
8	1	1	1	1	1	1	1	1	(2^8)
9	1	2	2	2	2	2	2	2	1
10	1	1	2^2	2^2	2	2	2^2	2^2	1
11	1	2	1	2^3	2	2^2	2	2^3	1
12	1	1	1	1	2	2	2	2	1
13	1	2	2	2	1	2^2	2^2	2^2	1
14	1	1	2	2	1	1	2^2	2^2	1
15	1	2	1	2^2	1	2	1	2^3	1

Table 6. Values of $g(k,r)$

4. Further Properties of $D(n,k,r)$

Theorem 4.1. *For any $0 \le r \le k$ and $n > 0$, $C(n,k,r) = D(n,k,r)$ if and only if the wagers at each node of the $C(n,k,r)$ tree are integers.*

Proof. For $k = 1$, $C(n,1,0) = 2n = D(n,1,0)$ and the wager is n, which is an integer. $C(n,1,1) = n = D(n,1,1)$ and the wager is 0, which is an integer, so for $0 \le r \le k$, we have $C(n,1,r) = D(n,1,r)$ if and only if all possible wagers in the $C(n,1,r)$ tree are integers.

Assuming the statement is true for all $k \le m$, we want to prove it is true for $k = m+1$. $C(n,m+1,m+1) = n = D(n,m+1,m+1)$ and all the wagers are 0 so the statement is true for $r = m+1$. $C(n,m+1,0) = n \cdot 2^{m+1} = D(n,m+1,0)$ and all the wagers are $n, n \cdot 2, \ldots, n \cdot 2^m$, so the statement is true for $r = 0$.

For $0 < r < m+1$, Let us first prove that if $D(n,m+1,r) = C(n,m+1,r)$, then all wagers of the $C(n,m+1,r)$ tree are integers. We know that the first wager of $C(n,m+1,r)$ is $n \cdot w_c$ and $C(n,m+1,r) = C(n \cdot (1+w_c), m, r) = C(n \cdot (1-w_c), m, r-1)$. Let the first wager of $D(n,m+1,r)$ be w_d. Then, $D(n,m+1,r) = \min\{D(n+w_d,m,r), D(n-w_d,m,r-1)\}$. If $w_d \le n \cdot w_c$, $D(n,m+1,r) \le D(n+w_d,m,r) \le C(n+w_d,m,r) \le C(n+n\cdot w_c,m,r) = C(n,m+1,r)$. The inequalities become equalities and $w_d = n \cdot w_c$. Similarly, if $w_d \ge n \cdot w_c$, $D(n,m+1,r) \le D(n-w_d,m,r-1) \le C(n-w_d,m,r-1) \le C(n-n\cdot w_c,m,r-1) = C(n,m+1,r)$. The equalities hold and $w_d = n \cdot w_c$. Therefore, $w_d = n \cdot w_c$ and $D(n,m+1,r) = D(n\cdot(1+w_c),m,r-1) = C(n,m+1,r) = C(n\cdot(1+w_c),m,r)$. $D(n\cdot(1+w_c),m,r) = C(n\cdot(1+w_c),m,r)$ implies all the wagers of the $C(n\cdot(1+w_c),m,r)$ tree are integers and similarly, all the wagers of the $C(n\cdot(1-w_c),m,r-1)$ tree are integers. Therefore, all the wagers of the $C(n,m+1,r)$ tree are integers.

Lastly, we will prove that if all wagers of the $C(n,m+1,r)$ tree are integers, then $D(n,m+1,r) = C(n,m+1,r)$. The $C(n\cdot(1+w_c),m,r)$ tree is a sub-tree of the $C(n,m+1,r)$ tree. Therefore all wagers of the $C(n\cdot(1+w_c),m,r)$ tree are integers. This implies that $C(n\cdot(1+w_c),m,r) = D(n\cdot(1+w_c),m,r)$. Similarly, $C(n\cdot(1-w_c),m,r-1) = D(n\cdot(1-w_c),m,r)$. Since $n \cdot w_c$ is an integer, we have $D(n,m+1,r) \ge \min\{D(n\cdot(1+w_c),m,r), D(n\cdot(1-w_c),m,r-1)\} = C(n,m+1,r)$. Since $C(n,m+1,r) \ge D(n,m+1,r)$, we have $C(n,m+1,r) = D(n,m+1,r)$. ■

Lemma 4.2. *For any $0 \le r \le k$ and $n > 0$, we have:*

(a) *Existence: There exists a positive integer n such that $C(n,k,r) = D(n,k,r)$;*

(b) *Homogeneity: If $C(n,k,r) = D(n,k,r)$, then $C(n\cdot m,k,r) = D(n\cdot m,k,r)$;*

(c) *Additivity: If $C(n_1,k,r) = D(n_1,k,r)$ and $C(n_2,k,r) = D(n_2,k,r)$ where $n_2 > n_1$, then $C(n_2-n_1,k,r) = D(n_2-n_1,k,r)$.*

Proof. (a) From Lemma 3.9 (b) and Theorem 4.1, we have $C(G(k,r),k,r) = D(G(k,r),k,r)$.

(b) If $C(n,k,r) = D(n,k,r)$, then all wagers of the $C(n,k,r)$ tree are integers. That implies that all wagers of the $C(n\cdot m,k,r)$ tree are integers.

(c) If $C(n_1,k,r) = D(n_1,k,r)$ and $C(n_2,k,r) = D(n_2,k,r)$, then all wagers of the $C(n_1,k,r)$ tree and the $C(n_2,k,r)$ tree are integers. Since the wagers of the $C(n,k,r)$ tree are n times the corresponding wagers of the $C(1,k,r)$ tree, wagers of $C(n_2-n_1,k,r)$ tree are the difference of the corresponding wagers of the $C(n_1,k,r)$ tree and the $C(n_2,k,r)$ tree. ■

Lemma 4.3. *For every* $0 \le r \le k$*, let* $P(k,r) = \frac{G(k,r)}{g(k,r)}$*. Then,* $C(n,k,r) = D(n,k,r)$ *if and only if* $P(k,r) \mid n$*.*

Proof. From Theorem 3.14, we have $P(k,r) \mid n$ if and only if all the wagers of the $C(n,k,r)$ tree are integers. From Theorem 4.1, we have that all the wagers of the $C(n,k,r)$ tree are integers if and only if $C(n,k,r) = D(n,k,r)$. Therefore, $C(n,k,r) = D(n,k,r)$ if and only if $P(k,r) \mid n$. ∎

Lemma 4.4. $P(k,k-r-1) = \frac{2^k}{g(k,r)} - P(k,r)$.

Proof. $P(k,k-r-1) = \frac{G(k,k-r-1)}{g(k,k-r-1)} = \frac{2^k - G(k,r)}{g(k,r)} = \frac{2^k}{g(k,r)} - P(k,r)$. ∎

Theorem 4.5.

(a) $D(n,k,r) = C(m,k,r) + D(y,k,r) = \frac{m \cdot 2^k}{G(k,r)} + D(y,k,r)$ *where* $m = \lfloor \frac{n}{P(k,r)} \rfloor \cdot P(k,r)$ *and* $y = n - m$*;*

(b) $D(n+1,k,r) - D(n,k,r)$ *is a periodic function of* n *with a period of* $P(k,r)$*.*

Proof. (a) For $D(y,k,r)$, we can use the $C(1,k,r)$ tree structure and attach each node with wagers and money owned of $D(y,k,r)$. The wagers and money owned are increasing functions of y. There exists an optimized path from the root node to a leaf node of the $C(1,k,r)$ tree such that the leaf node has money owned equal to $D(y,k,r)$. $D(m,k,r)$ and $C(m,k,r)$ have the same tree and values attached at each node and the optimized $D(m,k,r)$ path can be any path of the $C(m,k,r)$ tree. If we use the optimized $D(y,k,r)$ path for $D(y+m,k,r)$, at each node the wager and money owned for $D(y+m,k,r)$ are the sum of those of $D(y,k,r)$ and $C(m,k,r)$, so we have $D(y+m,k,r) \ge D(y,k,r) + C(m,k,r)$. If we use the optimized $D(y+m,k,r)$ path for $D(y,k,r)$ at each node, the wager and money owned for $D(y,k,r)$ are the difference of those of $D(y+m,k,r)$ and $C(m,k,r)$. We then have $D(y,k,r) \ge D(m+y,k,r) - C(m,k,r)$. Therefore, $D(m+y,k,r) = C(m,k,r) + D(y,k,r)$. This implies $D(n,k,r) = C(m,k,r) + D(y,k,r) = \frac{m \cdot 2^k}{G(k,r)} + D(y,k,r)$.

(b) $D(n+1,k,r) - D(n,k,r) = D(y+1,k,r) - D(y,k,r)$ for $0 \le y < P(k,r)$, so it is a periodic function of n with period $P(k,r)$. ∎

Corollary 4.6.

(a) $D(n,k,k-1) = \lfloor \frac{n \cdot 2^k}{2^k - 1} \rfloor$*;*

(b) $D(\frac{m \cdot G(k,r)}{g(k,r)}, k, r) = C(\frac{m \cdot G(k,r)}{g(k,r)}, k, r) = \frac{m \cdot 2^k}{g(k,r)}$.

Proof. (a) $P(k,k-1) = \frac{G(k,k-1)}{g(k,k-1)} = 2^k - 1$. For non-negative integers $y < 2^k - 1$, $y \le D(y,k,k-1) \le C(y,k,k-1) = \frac{y \cdot 2^k}{2^k - 1} = y + \frac{y}{2^k - 1}$. This implies $D(y,k,k-1) = y$. From Theorem 4.5, we have that $D(n,k,k-1) = \lfloor \frac{n}{P(k,r)} \rfloor \cdot$

$P(k,r) \cdot \frac{2^k}{G(k,r)} + D(y,k,r)$, where $y = n - \lfloor \frac{n}{P(k,r)} \rfloor \cdot P(k,r)$, so $D(n,k,k-1) = \lfloor \frac{n}{2^k-1} \rfloor \cdot 2^k + y = \lfloor \frac{n}{2^k-1} \rfloor \cdot 2^k + n - \lfloor \frac{n}{2^k-1} \rfloor \cdot (2^k - 1) = n + \lfloor \frac{n}{2^k-1} \rfloor = \lfloor \frac{n \cdot 2^k}{2^k-1} \rfloor$.

(b) From Lemma 4.3, we have that $D(\frac{m \cdot G(k,r)}{g(k,r)}, k, r) = C(\frac{m \cdot G(k,r)}{g(k,r)}, k, r) = \frac{m \cdot 2^k}{g(k,r)}$. ■

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