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Nonexistence of Solutions of Certain Type of Second Order Generalized α-Difference Equation in $\ell_{2(\alpha(\ell))}$ and $c_{0(\alpha(\ell))}$ Spaces

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Abstract. In this paper, the authors discuss the nonexistence of solutions of second order generalized α-difference equation

$$\Delta^2_{\alpha(\ell)} u(k) + f(k, u(k)) = 0,\ \ k \in [a, \infty),\ \ a > 0,\ \ \alpha > 1. \tag{1}$$

in $\ell_{2(\alpha(\ell))}$ and $c_{0(\alpha(\ell))}$ spaces, where $\Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k)$ and $\ell \in (0, \infty)$.

Keywords: Generalized α-difference equation; Generalized α-difference operator.

1. Introduction

The basic theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k),\ k \in \mathbb{N} = \{0,1,2,3,\cdots\}$. Eventhough many authors ([1, 4, 12, 18]) have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k),\ k \in \mathbb{R},\ \ell \in \mathbb{R} - \{0\}, \tag{2}$$

no significant progress has taken place on this line. But recently, E. Thandapani, M.M.S. Manuel, G.B.A.Xavier [7] considered the definition of Δ as given in (2) and developed the theory of difference equations in a different direction. For convenience, the operator Δ defined by (2) is labelled as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory (see [5]-[7],[10, 9],[16, 17]) were obtained. By extending the study related to the sequences of complex numbers and ℓ to be real, some new qualitative properties of the solutions like rotatory, expanding, shrinking, spiral and weblike of difference equations involving Δ_ℓ were obtained. The results obtained using Δ_ℓ can be found in ([8]). Jerzy Popenda and B.Szmanda ([13],[14]) defined Δ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$ and based on this definition they have studied the qualitative properties of solutions of a particular difference equation and no one else has handled this operator. Here, the generalized definition of the operator is taken as

$$\Delta_{\alpha(\ell)} u(k) \ = \ u(k+\ell) - \alpha u(k). \tag{3}$$

and by defining its inverse, several interesting results on number theory were obtained [11].

ℓ_2 and c_0 solutions of second order difference equation of (1) when $\ell = 1$ and $\alpha = 1$ was discussed in [15]. Nonexistence of solutions of (1) when $\alpha = 1$ was discussed in [5] and [6]. In this paper, we discuss nonexistence of solutions in $\ell_{2(\alpha(\ell))}$ and $c_{0(\alpha(\ell))}$ spaces for the second order generalized α-difference equation (1).

Throughout this paper we use the following notations.
(i) $[k]$ denotes the integer part of k,
(ii) $\mathbb{N} = \{0,1,2,3,\dots\}$, $\mathbb{N}(a) = \{a, a+1, a+2, \dots\}$, for any real a,
(iii)$\mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \dots\}$ and $\mathbb{R}$ is the set of all real numbers.

2. Preliminaries

In this section, we present some basic definitions which will be useful for the subsequent discussion.

Definition 2.1. *Let $u(k), k \in [0,\infty)$ be a real or complex valued function and $\ell \in (0,\infty)$. Then, the generalized α-difference operator $\Delta_{\alpha(\ell)}$ on $u(k)$ is defined as*

$$\Delta_{\alpha(\ell)} u(k) \ = \ u(k+\ell) - \alpha u(k). \tag{4}$$

When $\alpha = 1$, the generalized α-difference operator $\Delta_{\alpha(\ell)}$ becomes the generalized difference operator Δ_ℓ. When $\alpha = 1$ and $\ell = 1$, then $\Delta_{\alpha(\ell)}$ is the usual difference operator Δ.

Definition 2.2. [7] *Let $u(k), k \in [0, \infty)$ be a real or complex valued function and $\ell \in (0, \infty)$. Then, the inverse operator Δ_ℓ^{-1} is defined as follows.*

$$\text{If } \Delta_\ell v(k) \ = \ u(k), \ \text{then } v(k) \ = \ \Delta_\ell^{-1} u(k) + c_j, \tag{5}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - \left[\frac{k}{\ell}\right]\ell$.
If $\lim_{k\to\infty} u(k) = 0$, then we can take $c_j = 0$.

Definition 2.3. *The inverse of the Generalized α-difference operator denoted by $\Delta^{-1}_{\alpha(\ell)}$ on $u(k)$ is defined as, if $\Delta_{\alpha(\ell)} v(k) \ = \ u(k)$, then*

$$\Delta^{-1}_{\alpha(\ell)} u(k) \ = \ v(k) - \alpha^{\left[\frac{k}{\ell}\right]} c_j. \tag{6}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - \left[\frac{k}{\ell}\right]\ell$.

Definition 2.4. [5] *A function $u(k)$, $k \in [a, \infty)$ is said to be in $\ell_{2(\ell)}$-space if*

$$\sum_{\gamma=0}^{\infty} |u(a + j + \gamma\ell)|^2 < \infty \ \text{for all } j \in [0, \ell). \tag{7}$$

If $\lim_{r\to\infty} |u(a + j + r\ell)| = 0$ for all $j \in [0, \ell)$, then $u(k)$ is said to be in the $c_{0(\ell)}$-space.

Definition 2.5. [7] *Generalized polynomial factorial for $\ell > 0$ is defined as*

$$k_\ell^{(n)} = k(k - \ell)(k - 2\ell)\cdots(k - (n-1)\ell). \tag{8}$$

Theorem 2.6. *For $\ell > 0$, if $\lim_{k\to\infty} u(k) = 0$, then*

$$\Delta_\ell^{-1} u(k) = -\sum_{r=0}^{\infty} u(k + r\ell), \ \text{for all} \quad k \in [0, \infty). \tag{9}$$

Proof. Let $z(k) = \sum_{r=0}^{\infty} u(k + r\ell)$.
$\Delta_\ell z(k) = z(k + \ell) - z(k) = \sum_{r=0}^{\infty} u(k + \ell + r\ell) - \sum_{r=0}^{\infty} u(k + r\ell)$.
Since $\lim_{k\to\infty} u(k) = 0$, we get $\Delta_\ell z(k) = -u(k)$ and the proof follows from Definition 2.2. ∎

Theorem 2.7. *If* $\lim_{k\to\infty} \frac{u(k)}{\alpha^{(r+1)}} = 0$ *and* $\ell > 0$*, then*

$$\Delta^{-1}_{\alpha(\ell)} u(k) = -\sum_{r=0}^{\infty} \frac{u(k+r\ell)}{\alpha^{(r+1)}}, \text{ for all } \quad k \in [0,\infty), \alpha > 1. \tag{10}$$

Proof. Assume $z(k) = \sum_{r=0}^{\infty} \frac{u(k+r\ell)}{\alpha^{(r+1)}}$.
Then, $\Delta_{\alpha(\ell)} z(k) = z(k+\ell) - \alpha z(k) = \sum_{r=0}^{\infty} \frac{u(k+\ell+r\ell)}{\alpha^{(r+1)}} - \sum_{r=0}^{\infty} \frac{u(k+r\ell)}{\alpha^{r}} = -u(k)$.
Now, the proof follows from $\lim_{k\to\infty} u(k) = 0$ and Definition 2.3. ∎

Lemma 2.8. *Let* $u(k)$ *and* $v(k)$ *be any two functions. Then,* $\forall\ k \in [a,\infty)$

$$\begin{aligned} &\Delta_{\alpha(\ell)}\{u(k)v(k)\} \\ =&u(k+\ell)\Delta_{\alpha(\ell)}v(k) + u(k+\ell)v(k)(\alpha-1) + v(k)\Delta_{\alpha(\ell)}u(k) \\ =&v(k+\ell)\Delta_{\alpha(\ell)}u(k) + v(k+\ell)u(k)(\alpha-1) + u(k)\Delta_{\alpha(\ell)}v(k). \end{aligned} \tag{11}$$

Theorem 2.9. [5] *For all* $(k,u) \in [a,\infty)\times\mathbb{R}$ *the function* $f(k,u)$ *be defined and*

$$|f(k,u)| \le \frac{\ell^2}{2} k^{-2} |u|. \tag{12}$$

Then, if $u(k) \in \ell_{2(\ell)}$ *is a solution of (1), there exists* $k_1 \ge a$*,* $(a \ge 2\ell)$ *such that* $u(k) = 0$ *for all* $k \in [k_1,\infty)$*.*

3. Main Results

In this section, we present the condition for nonexistence of nontrivial solutions of (1).

Definition 3.1. *A function* $u(k)$*,* $k \in [a,\infty)$ *is said to be in* $\ell_{2(\alpha(\ell))}$ *space if*

$$\sum_{r=0}^{\infty} \left| \frac{u(a+j+r\ell)}{\alpha^{(r+1)}} \right|^2 < \infty, \text{ for all } j \in [0,\ell). \tag{13}$$

If $\lim_{r\to\infty} \frac{|u(a+j+r\ell)|}{\alpha^{(r+1)}} = 0$ *for all* $j \in [0,\ell)$ *and* $a \in [0,\infty)$*, then* $u(k)$ *is said to be in the* $c_{0(\alpha(\ell))}$ *space.*

Example 3.2. For $n \in \mathbb{N}(1)$, k^n and $k_\ell^{(n)}$ are in $\ell_{2(\alpha(\ell))}$ and $c_{0(\alpha(\ell))}$ spaces.

Lemma 3.3. *For* $k \in (0,\infty), \ell > 0$, $\sum_{r=0}^{\infty} (k+r\ell)^{-2} \le \frac{1}{\ell(k-\ell)}$.

Proof. $\Delta_\ell \frac{1}{k-\ell} = -\frac{\ell}{(k-\ell)k}$ yields $\Delta_\ell^{-1}\frac{-1}{(k-\ell)k} = \frac{1}{\ell(k-\ell)}$. Now, the proof follows from Theorem 2.6 and $\frac{1}{(k-r\ell)^2} \le \frac{1}{(k+(r-1)\ell)(k+r\ell)}$. ■

Lemma 3.4. *Let $a \ge 2\ell,\ \alpha > 1,\ k \in [a,\infty)$ and $r(k) = \frac{4}{\left(\sqrt{k+\ell}+\sqrt{k}\right)\left(\sqrt{k}+\sqrt{k-\ell}\right)}$. Then $kr(k)\alpha^2 > 1$.*

Proof. Multiplying and dividing $r(k)$ by $\left(\sqrt{k+\ell}-\sqrt{k}\right)\left(\sqrt{k}-\sqrt{k-\ell}\right)$, we get

$$\begin{aligned}
r(k) =&\frac{4}{\ell^2}\sqrt{k}\sqrt{k}\left[\left(1+\frac{\ell}{k}\right)^{\frac{1}{2}}-1\right]\left[1-\left(1-\frac{\ell}{k}\right)^{\frac{1}{2}}\right]\\
=&\frac{4k}{\ell^2}\left[1+\frac{1}{2}\frac{\ell}{k}-\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3-\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4+\cdots-1\right]\\
&\times\left[1-\left(1-\frac{1}{2}\frac{\ell}{k}-\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2-\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3-\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4-\cdots\right)\right]. \qquad (14)
\end{aligned}$$

We notice that, in the first expression of the above equation the sum of each pair-wise positive and its consecutive negative terms yields a positive value. Hence we obtain.

$$\begin{aligned}
r(k) >& \frac{4k}{\ell^2}\left[\frac{1}{2}\frac{\ell}{k}-\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2\right]\left[\frac{1}{2}\frac{\ell}{k}+\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3+\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4+\cdots\right]\\
=& \frac{4}{\ell^2}\left[\frac{\ell}{2}-\frac{\ell}{2}\frac{1}{4}\frac{\ell}{k}\right]\left[\frac{1}{2}\frac{\ell}{k}+\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3+\cdots\right]\\
=& \frac{4}{\ell^2}\frac{\ell}{2}\left[\frac{1}{2}\frac{\ell}{k}+\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3+\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4+\cdots\right]\\
&-\frac{4}{\ell^2}\frac{\ell}{2}\frac{1}{4}\frac{\ell}{k}\left[\frac{1}{2}\frac{\ell}{k}+\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3+\cdots\right]\\
=& \frac{1}{k}+\frac{2}{\ell}\left[\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3+\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4+\cdots\right]\\
&-\frac{2}{\ell}\left[\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{2!}\frac{1}{4}\frac{1}{4}\left(\frac{\ell}{k}\right)^3+\frac{1}{3!}\frac{1}{4}\frac{1}{4}\frac{1}{4}\left(\frac{\ell}{k}\right)^4+\cdots\right]\\
=& \frac{1}{k}+\frac{2}{4\ell}\left[\frac{1}{3!}\left(\frac{3}{2}-\frac{3}{4}\right)\left(\frac{\ell}{k}\right)^3+\frac{1}{4!}\frac{3}{2}\left(\frac{5}{2}-\frac{4}{4}\right)\left(\frac{\ell}{k}\right)^4+\cdots\right].
\end{aligned}$$

Since second term of above is positive, we obtain $r(k) > \frac{1}{k}$. Now, the proof is obvious. ■

Lemma 3.5. *Let $a \ge 2\ell,\ k \in [a,\infty)$ and $d(k) = \frac{\sqrt{k+\ell}}{\sqrt{k}} - \frac{\sqrt{k}}{\sqrt{k+\ell}+\sqrt{k-\ell}}$. Then $d(k) < 1$.*

Proof. Multiplying and dividing the 2^{nd} term of $d(k)$ by $\sqrt{k+\ell}-\sqrt{k-\ell}$ and from the Binomial theorem for rational index, we find

$$\begin{aligned}
d(k) =&1+\frac{1}{2}\frac{\ell}{k}-\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3-\cdots\infty\\
&-\frac{k}{2\ell}\left[1+\frac{1}{2}\frac{\ell}{k}-\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3-\cdots\infty\right.\\
&\left.-\left(1-\frac{1}{2}\frac{\ell}{k}-\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2-\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3-\cdots\infty\right)\right]
\end{aligned}$$

$$=1+\frac{1}{2}\frac{\ell}{k}-\frac{1}{2!}\frac{1}{4}\Big(\frac{\ell}{k}\Big)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\Big(\frac{\ell}{k}\Big)^3-\cdots\infty$$
$$-\frac{k}{2\ell}\Big[\frac{\ell}{k}+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\Big(\frac{\ell}{k}\Big)^3+\cdots\infty\Big].$$

In the first expression of the above equation, each sum of negative term and the consecutive positive term of $d(k)$ is negative. Hence, we obtain $d(k)<1+\frac{1}{2}\frac{\ell}{k}-\frac{1}{2}=\frac{1}{2}+\frac{1}{2}\frac{\ell}{k}<1$, which completes the proof. ∎

Lemma 3.6. *Let $a\geq 2\ell$, $k\in[a+\ell,\infty)$ and $j=k-a-\left[\frac{k-a}{\ell}\right]\ell$. If*

$$\Delta_{\alpha(\ell)}z(k)\leq\gamma(k)+\alpha\beta(k)z(k) \tag{15}$$

and $\frac{-\ell}{k}<\beta(k)<\frac{-\ell^2}{k^2}$ for all $k\in[a,\infty)$, then

$$\Delta_{\alpha(\ell)}\left(z(k)\prod_{r=0}^{\left[\frac{k-a}{\ell}\right]-1}(1+\beta(j+a+r\ell))^{-1}\right)\leq\gamma(k)\prod_{r=0}^{\left[\frac{k-a}{\ell}\right]}(1+\beta(j+a+r\ell))^{-1} \tag{16}$$

where $j=k-a-\left[\frac{k-a}{\ell}\right]\ell$.

Proof. From the inequality (15) and $1+\beta(k)>0$, we find $\frac{z(k+\ell)}{1+\beta(k)}-\alpha z(k)\leq\frac{\gamma(k)}{1+\beta(k)}$, which yields,

$$\frac{z(k+\ell)}{1+\beta(k)}\prod_{r=0}^{\left[\frac{k-a}{\ell}\right]-1}(1+\beta(j+a+r\ell))^{-1}-\alpha z(k)\prod_{r=0}^{\left[\frac{k-a}{\ell}\right]-1}(1+\beta(j+a+r\ell))^{-1}$$
$$\leq\frac{\gamma(k)}{1+\beta(k)}\prod_{r=0}^{\left[\frac{k-a}{\ell}\right]-1}(1+\beta(j+a+r\ell))^{-1}$$

Now (16) follows by assigning $j+a+\left[\frac{k-a}{\ell}\right]\ell=k$. ∎

The following theorem gives the condition for nonexistence of nontrivial solutions of (1).

Theorem 3.7. *Let for all $(k,u)\in[a,\infty)\times\mathbb{R}$ and $\alpha>1$ the function $f(k,u)$ be defined and*

$$|f(k,u)|\leq\frac{\ell^2}{2}k^{-2}|u|. \tag{17}$$

Then, if $u(k)\in\ell_{2(\alpha(\ell))}$ is a solution of (1), there exists a real $k_1\geq a(a\geq 2\ell)$ such that $u(k)=0$ for all $k\in[k_1,\infty)$.

Proof. Since $u(k)$ is a solution of (1) and satisfies Definition 3.1, we find,

$$\lim_{k\to\infty}\Delta_{\alpha(\ell)}\frac{u(k)}{\alpha^{(\lceil\frac{k+\ell}{\ell}\rceil)}}=\lim_{k\to\infty}\Delta^2_{\alpha(\ell)}\frac{u(k)}{\alpha^{(\lceil\frac{k+2\ell}{\ell}\rceil)}}=0. \tag{18}$$

Hence, taking $\Delta^{-1}_{\alpha(\ell)}$ on equation (1) and using Theorem 2.6, we find

$$\Delta_{\alpha(\ell)}u(k) = \sum_{r=0}^{\infty} \frac{f(k+r\ell, u(k+r\ell))}{\alpha^{(r+1)}}. \tag{19}$$

Again taking $\Delta^{-1}_{\alpha(\ell)}$ and by Theorem 2.6, we obtain

$$u(k) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{f(k+r\ell+s\ell, u(k+r\ell+s\ell))}{\alpha^{(r+s+2)}}, \tag{20}$$

which yields

$$u(k) = -\sum_{r=0}^{\infty}(r+1)\frac{f(k+r\ell, u(k+r\ell))}{\alpha^{(r+2)}},\ k \in [a,\infty). \tag{21}$$

Therefore, from (17), we obtain

$$|u(k)| \le \frac{\ell^2}{2}v(k), \tag{22}$$

where

$$v(k) = \sum_{r=0}^{\infty}(r+1)(k+r\ell)^{-2}\left|\frac{u(k+r\ell)}{\alpha^{(r+2)}}\right| \text{ for all } k \in [a,\infty). \tag{23}$$

Obviously $v(k) \ge 0$ for all $k \in [a,\infty)$ and $\lim\limits_{k\to\infty} v(k) = 0$ by Definition 3.1. If $v(k+j) = 0$, for all $j \in [0,\ell)$, for some $k = k_1 \ge a$, then

$$(r+1)(k+j+r\ell)^{-2}\Big(\frac{u(k+j+r\ell)}{\alpha^{(r+2)}}\Big) = 0, \text{ for all } r = 0,1,2,\dots.$$

Hence $u(k) = 0$, for all $k \ge k_1$. In this case the proof is complete. Now, we suppose that $v(k) > 0$, for all $k \in [a,\infty)$. From (23) we obtain, $\Delta_{\alpha(\ell)}v(k) = -\sum\limits_{r=0}^{\infty}(k+r\ell)^{-2}\left|\frac{u(k+r\ell)}{\alpha^{(r+1)}}\right|$ and $\Delta^2_{\alpha(\ell)}v(k) = k^{-2}|u(k)|$. From (22), we find

$$\Delta^2_{\alpha(\ell)}v(k) \le \frac{\ell^2}{2}k^{-2}v(k), \quad \text{for all } k \in [a,\infty). \tag{24}$$

From the definition of $v(k)$, $a \ge 2\ell$, $\frac{r+1}{\alpha(k+r\ell)} \le \frac{1}{\ell}$ and Schwartz's inequality, we obtain

$$\begin{aligned} v(k) &\le \ell^{-1}\sum_{r=0}^{\infty}(k+r\ell)^{-1}\left|\frac{u(k+r\ell)}{\alpha^{(r+1)}}\right| \\ &\le \ell^{-1}\left(\sum_{r=0}^{\infty}(k+r\ell)^{-2}\right)^{\frac{1}{2}}\left(\sum_{r=0}^{\infty}\left|\frac{u(k+r\ell)}{\alpha^{(r+1)}}\right|^2\right)^{\frac{1}{2}}. \end{aligned}$$

By Lemma 3.3, we get $v(k) \leq \ell^{-\frac{3}{2}} \frac{1}{\sqrt{k-\ell}} \left(\sum_{r=0}^{\infty} \left| \frac{u(k+r\ell)}{\alpha^{(r+1)}} \right|^2 \right)^{\frac{1}{2}}$.
Thus it follows that

$$w(k) = \ell^{\frac{3}{2}} \sqrt{k-\ell} v(k) \leq \left(\sum_{r=0}^{\infty} \left| \frac{u(a+j+r\ell)}{\alpha^{(r+1)}} \right|^2 \right)^{\frac{1}{2}}. \tag{25}$$

Hence we have

$$w(k) \to 0 \text{ and } w(k) > 0 \text{ for all } \ k \in [a, \infty). \tag{26}$$

Applying Lemma 2.8 to (25) twice, we arrive at

$$\begin{aligned} \Delta^2_{\alpha(\ell)} w(k) =& \ell^{\frac{3}{2}} \big(\sqrt{k+\ell} \Delta^2_{\alpha(\ell)} v(k) + 2(\alpha-1) \sqrt{k+\ell} \Delta_{\alpha(\ell)} v(k) \\ & + 2\Delta_{\alpha(\ell)} v(k) \Delta_{\alpha(\ell)} \sqrt{k} + + \sqrt{k+\ell} v(k) (\alpha-1)^2 \\ & + 2(\alpha-1) \Delta_{\alpha(\ell)} \sqrt{k} v(k) + v(k) \Delta^2_{\alpha(\ell)} \sqrt{k-\ell} \big). \end{aligned} \tag{27}$$

Again from Lemma 2.8 and (25), we get

$$\Delta_{\alpha(\ell)} v(k) = \ell^{-\frac{3}{2}} \Big(\frac{1}{\sqrt{k}} \Delta_{\alpha(\ell)} w(k) + \frac{(\alpha-1)}{\sqrt{k}} \Delta_{\alpha(\ell)} w(k) + w(k) \Delta_{\alpha(\ell)} \frac{1}{\sqrt{k-\ell}} \Big). \tag{28}$$

From (27), (28) and by Lemma 2.8, we find that

$$\begin{aligned} & \Delta_{\alpha(\ell)} \left(\frac{1}{k-\ell} \Delta_{\alpha(\ell)} w(k) \right) \\ =& \frac{1}{k} \Delta^2_{\alpha(\ell)} w(k) + \frac{(\alpha-1)}{k} \Delta_{\alpha(\ell)} w(k) + \Delta_{\alpha(\ell)} \frac{1}{k-\ell} \Delta_{\alpha(\ell)} w(k) \\ =& \frac{\ell^{\frac{3}{2}}}{k} \Big\{ \sqrt{k+\ell} \Delta^2_{\alpha(\ell)} v(k) + 2(\alpha-1) \sqrt{k+\ell} \Delta_{\alpha(\ell)} v(k) + 2\Delta_{\alpha(\ell)} v(k) \Delta_{\alpha(\ell)} \sqrt{k} \\ & + \sqrt{k+\ell} v(k) (\alpha-1)^2 + 2(\alpha-1) \Delta_{\alpha(\ell)} \sqrt{k} v(k) + v(k) \Delta^2_{\alpha(\ell)} \sqrt{k-\ell} \Big\} \\ & + \frac{(\alpha-1)}{k} \Delta_{\alpha(\ell)} w(k) + \Big(\frac{k(1-\alpha)-\ell}{k(k-\ell)} \Big) \Delta_{\alpha(\ell)} w(k) \\ =& \frac{\ell^{\frac{3}{2}}}{k} \Big\{ \sqrt{k+\ell} \Delta^2_{\alpha(\ell)} v(k) + 2\ell^{\frac{-3}{2}} \big((\alpha-1) \sqrt{k+\ell} \\ & + \Delta_{\alpha(\ell)} \sqrt{k} \big) \Big[\frac{1}{\sqrt{k}} \Delta_{\alpha(\ell)} w(k) + \frac{(\alpha-1)}{\sqrt{k}} w(k) + w(k) \Delta_{\alpha(\ell)} \frac{1}{\sqrt{k-\ell}} \Big] \\ & + \sqrt{k+\ell} v(k) (\alpha-1) v(k) (\alpha-1)^2 + 2(\alpha-1) \Delta_{\alpha(\ell)} \sqrt{k} v(k) \\ & + v(k) \Delta^2_{\alpha(\ell)} \sqrt{k-\ell} \Big\} + \left\{ \frac{\alpha-1}{k} + \frac{k(1-\alpha)-\ell}{k(k-\ell)} \right\} \Delta_{\alpha(\ell)} w(k) \\ \leq& \frac{\ell^{\frac{3}{2}}}{k} \left\{ \frac{\ell^2 \sqrt{k+\ell}}{2k^2} v(k) + \frac{2\alpha^2}{\sqrt{k}} (\sqrt{k+\ell} - \sqrt{k})(\sqrt{k-\ell} - \sqrt{k}) v(k) \right. \end{aligned}$$

$$+ (\alpha-1)^2\sqrt{k+\ell}v(k) + 2(\alpha-1)\Delta_{\alpha(\ell)}\sqrt{k}v(k) + v(k)\Delta^2_{\alpha(\ell)}\sqrt{k-\ell}\Big\}$$
$$+\alpha\left(\frac{2(k-\ell)}{k\sqrt{k}}(\sqrt{k+\ell}-\sqrt{k})-\frac{\ell}{k}\right)\frac{1}{k-\ell}\Delta_{\alpha(\ell)}w(k)$$

which in view of (24), (26) gives

$$\Delta_{\alpha(\ell)}z(k) \le \gamma(k) + \alpha\beta(k)z(k) \tag{29}$$

where

$$z(k) = \frac{1}{k-\ell}\Delta_{\alpha(\ell)}w(k) \tag{30}$$

$$\gamma(k) = \tfrac{\ell^{\frac{3}{2}}}{k}\Big(\tfrac{\ell^2\sqrt{k+\ell}}{2k^2} + \tfrac{2\alpha^2}{\sqrt{k}}(\sqrt{k+\ell}-\sqrt{k})(\sqrt{k-\ell}-\sqrt{k})$$
$$+(\alpha-1)^2\sqrt{k+\ell} + 2(\alpha-1)\Delta_{\alpha(\ell)}\sqrt{k} + \Delta^2_{\alpha(\ell)}\sqrt{k-\ell}\Big)v(k) \tag{31}$$

and

$$\beta(k) = \frac{2(k-\ell)}{k\sqrt{k}}\Delta_{\ell}\sqrt{k} - \frac{\ell}{k}. \tag{32}$$

Since $\frac{2(k-\ell)}{k\sqrt{k}}\Delta_\ell\sqrt{k} > 0$, from $\left(1+\frac{\ell}{k}\right)^{\frac{1}{2}} < 1 + \frac{1}{2}\frac{\ell}{k}$, we obtain

$$-\frac{\ell}{k} < \beta(k) < -\frac{\ell^2}{k^2},\ \ k \in [a,\infty). \tag{33}$$

Further, since $(\sqrt{k+\ell}-\sqrt{k})(\sqrt{k-\ell}-\sqrt{k}) = -\frac{\ell^2}{(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k-\ell}+\sqrt{k})}$ and

$$\begin{aligned}&(\alpha-1)^2\sqrt{k+\ell} + 2(\alpha-1)\Delta_{\alpha(\ell)}\sqrt{k} + \Delta^2_{\alpha(\ell)}\sqrt{k-\ell}\\ =&\alpha^2(\sqrt{k+\ell}-\sqrt{k}+\sqrt{k-\ell}-\sqrt{k})\\ =&\alpha^2\ell\frac{\sqrt{k-\ell}-\sqrt{k+\ell}}{(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k-\ell}+\sqrt{k})},\end{aligned}$$

we get

$$\gamma(k) = \frac{\ell^{\frac{3}{2}}}{k\sqrt{k}}\Big(\frac{\ell^2\sqrt{k+\ell}}{2k\sqrt{k}} + \frac{-2\alpha^2\ell^2 + \alpha^2\ell\sqrt{k}(\sqrt{k-\ell}-\sqrt{k+\ell})}{(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k}+\sqrt{k-\ell})}\Big)v(k).$$

From Lemmas 3.4 and 3.5,

$$\gamma(k) < \tfrac{\ell^{\frac{3}{2}}}{k\sqrt{k}}\Big(\tfrac{4\alpha^2\ell^2\sqrt{k+\ell}}{2\sqrt{k}(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k}+\sqrt{k-\ell})} + \tfrac{-2\alpha^2\ell^2+\alpha^2\ell\sqrt{k}(\sqrt{k-\ell}-\sqrt{k+\ell})}{(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k}+\sqrt{k-\ell})}\Big)v(k)$$
$$= \frac{2\alpha^2\ell^{\frac{7}{2}}}{k\sqrt{k}(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k}+\sqrt{k-\ell})}\left(\frac{\sqrt{k+\ell}}{\sqrt{k}} - \frac{\sqrt{k}}{\sqrt{k+\ell}+\sqrt{k-\ell}} - 1\right)v(k). \tag{34}$$

By Lemma 3.5, we find $\gamma(k) < 0$, for all $k \in [a,\infty)$. Thus from Lemma 3.6 and $\gamma(k) < 0$,

$$\Delta_{\alpha(\ell)}\left(z(k)\prod_{r=0}^{\left[\frac{k-a}{\ell}\right]-1}(1+\beta(j+a+r\ell))^{-1}\right)<0,\ \text{for all } k\in[a+\ell,\infty),$$

which is same as

$$\alpha^{\left\lceil\frac{k+\ell}{\ell}\right\rceil}\Delta_\ell\left(\frac{z(k)\prod_{r=0}^{\left[\frac{k-a}{\ell}\right]-1}(1+\beta(j+a+r\ell))^{-1}}{\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}}\right)<0,\quad \text{for all } k\in[a+\ell,\infty),$$

i.e. $\left(\frac{z(k)\prod_{r=0}^{\left[\frac{k-a}{\ell}\right]-1}(1+\beta(j+a+r\ell))^{-1}}{\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}}\right)$ is decreasing by ℓ steps.

If $z(k)\prod_{r=0}^{\left[\frac{k-a}{\ell}\right]-1}(1+\beta(j+a+r\ell))^{-1}>0$, for all $k\in[a+\ell,\infty)$, then $z(k)>0$, for all $k\in[a+\ell,\infty)$, from (30), we find $\Delta_{\alpha(\ell)}w(k)>0$ and hence $w(k+\ell)>\alpha w(k)$, for all $k\in[a+\ell,\infty)$, but this contradicts (26).
If there exists a real $K\geq a+\ell$ such that

$z(K+j)\prod_{r=0}^{\left[\frac{K-a}{\ell}\right]-1}(1+\beta(j+a+r\ell))^{-1}=p_j<0$ for all $0\leq j<\ell$, then

$z(k)\prod_{r=0}^{\left[\frac{k-a}{\ell}\right]-1}(1+\beta(j+a+r\ell))^{-1}<p_j$ for all $k\in[K,\infty)$,

i.e. $z(k)<p_j\prod_{r=0}^{\left[\frac{k-a}{\ell}\right]-1}(1+\beta(j+a+r\ell))$.

However from (33), $1+\beta(k)>(k-\ell)/k>0$ and $j=k-a-\left[\frac{k-a}{\ell}\right]\ell$, it follows that $z(k)<p_j(j+a-\ell)/(k-\ell)$, and hence from (30), we find $\Delta_{\alpha(\ell)}w(k)<p_j(j+a-\ell)$. Since $w(k)\to 0$, $k\geq K+2\ell\Rightarrow \frac{1}{\ell}(k-K-\ell)\geq 1$, we get $w(k+\ell)<\alpha w(k)+p_j(j+a-\ell)$ which yields $w(k)<\alpha w(k-\ell)+p_j(j+a-\ell)$ and hence for all $k\in[K+2\ell,\infty)$, $w(k)<\alpha w(K+\ell)+\frac{1}{\ell}p_j(j+a-\ell)(k-K-\ell)$. Since $k\geq K+2\ell\Rightarrow k-K\geq 2\ell$, $\frac{1}{\ell}(k-K-\ell)\geq 1$. But this implies that $w(k)\to-\infty$, and again we get a contradiction to (26). Combining the above arguments, we find that our assumption $v(k)>0$ for all $k\in[a,\infty)$ is not correct, and this completes the proof. ■

Example 3.8. For the generalized difference equation $\Delta^2_{\alpha(\ell)}u(k)=k_\ell^{(n-2)}\Big((k+\ell)\big(k(1-2\alpha)+2\ell(1-(n-2)\alpha)\big)+\alpha(k-(n-2)\ell)(k-(n-1)\ell)\Big)$ (17) is not satisfied. Hence $u(k)\neq 0$ for all $k\in(2\ell,\infty)$. Infact $u(k)=k_\ell^{(3)}\in\ell_{2(\alpha(\ell))}$ is a solution.

Theorem 3.9. *Let for all $(k,u)\in[0,\infty)\times\mathbb{R}$ and $\alpha>1$ the function $f(k,u)$ be defined and*

$$|f(k,u)|\leq \ell^q k^{-q}|u|,\ q>\frac{5}{2}. \tag{35}$$

Then, if $u(k)\in c_{0(\alpha(\ell))}$ is a solution of (1), there exists a positive $k_1\geq a$ $(a\geq 4\ell)$ such that $u(k)=0$ for all $k\in[k_1,\infty)$.

Proof. Let $u(k)$ be a solution of (1) such that $\lim_{r\to\infty}\frac{|u(a+j+r\ell)|}{\alpha^{(r+1)}}=0$. Then, $\lim_{k\to\infty}\Delta_{\alpha(\ell)}\frac{u(k)}{\alpha(\lceil\frac{k+\ell}{\ell}\rceil)}=\lim_{k\to\infty}\Delta^2_{\alpha(\ell)}\frac{u(k)}{\alpha(\lceil\frac{k+\ell}{\ell}\rceil)}=0$ for all $\ell>0$. Thus, for this solution also the relation (20) holds. Further, since there exists a constant $c_j>0$ such that $\frac{|u(k)|}{\alpha^{(r+1)}}\le c_j$ for all $k\in[k_1,\infty)$, where $0\le j=k-\left[\frac{k}{\ell}\right]\ell<\ell$, we find that

$$\begin{aligned}\sum_{r=0}^{\infty}(r+1)\frac{|f((k+r\ell),u(k+r\ell))|}{\alpha^{(r+1)}}&\le\sum_{r=0}^{\infty}\Big(r+\frac{k}{\ell}\ell^q(k+r\ell)^{-q}\frac{|u(k+r\ell)|}{\alpha^{(r+1)}}\Big)\\&=\sum_{r=0}^{\infty}(k+r\ell)^{1-q}\ell^{q-1}\frac{|u(k+r\ell)|}{\alpha^{(r+1)}}\\&\le c_j\ell^{q-1}\sum_{r=0}^{\infty}(k+r\ell)^{1-q}\text{ where }j=k-\left[\frac{k}{\ell}\right]\ell\\&=c_j\ell^{q-1}\left[k^{1-q}+\sum_{r=1}^{\infty}(k+r\ell)^{1-q}\right]\\&=c_j\ell^{q-1}\left[k^{1-q}+\ell^{1-q}\sum_{r=1}^{\infty}(\frac{k}{\ell}+r)^{1-q}\right]\\&=c_j\ell^{q-1}\left[k^{1-q}+\ell^{1-q}\left[\frac{\left(\frac{k}{\ell}\right)^{2-q}}{2-q}+r\right]_{\frac{k}{\ell}}^{\infty}\right]\\&=c_j\ell^{q-1}\left[k^{1-q}+\frac{k^{2-q}}{\ell(q-2)}\right]<\infty,\text{for all }k\in[k_1,\infty).\end{aligned}$$

Therefore, this solution also has the representation (20). Now as in Theorem 3.7. we define
$\bar{v}(k)=\sum_{r=0}^{\infty}(r+1)(k+r\ell)^{-q}\frac{|u(k+r\ell)|}{\alpha^{(r+2)}}=\sum_{r=0}^{\infty}\ell^{-q}(r+1)(\frac{k}{\ell}+r)^{-q}\frac{|u(k+r\ell)|}{\alpha^{(r+2)}}$.
Since $q>\frac{5}{2}$ we find
$\bar{v}(k)\le\ell^{-q}\sum_{r=0}^{\infty}(r+1)(\frac{k}{\ell}+r)^{-2}\frac{|u(k+r\ell)|}{\alpha^{(r+2)}}=\ell^{2-q}\sum_{r=0}^{\infty}(r+1)(k+r)^{-2}\frac{|u(k+r\ell)|}{\alpha^{(r+2)}}$
$\bar{v}(k)\le\ell^{2-q}\frac{\ell^{-\frac{3}{2}}}{\sqrt{k-\ell}}\left\{\sum_{r=0}^{\infty}\frac{|u(k+r\ell)|^2}{\alpha^{(r+1)^2}}\right\}^{\frac{1}{2}}$.
Hence, we define
$\bar{w}(k)=\ell^{q-\frac{1}{2}}\sqrt{k-\ell}\bar{v}(k),\ \bar{z}(k)=\frac{1}{k-\ell}\Delta_{\alpha(\ell)}\bar{w}(k)$,
$\bar{\gamma}(k)=\frac{\ell^{q-\frac{1}{2}}}{k}\Big(\ell^q\frac{\sqrt{k+\ell}}{2k^q}+\frac{2\alpha^2}{\sqrt{k}}(\sqrt{k+\ell}-\sqrt{k})(\sqrt{k-\ell}-\sqrt{k})$
$+(\alpha-1)^2\sqrt{k+\ell}+2(\alpha-1)\Delta_{\alpha(\ell)}\sqrt{k}+\Delta^2_{\alpha(\ell)}\sqrt{k-\ell}\Big)\bar{v}(k)$,
$\bar{\beta}(k)=\frac{2(k-\ell)}{k\sqrt{k}}\Delta_{\alpha(\ell)}\sqrt{k}-\frac{\ell}{k}$,
and apply similar analysis to see that there exists a positive integer k_1 such that $u(k)=0$ for all $k\in[k_1,\infty)$. ∎

Example 3.10. For the generalized difference equation $\Delta^2_{\alpha(\ell)}u(k)=k^2(1-\alpha)^2+2\ell(1-\alpha)(2k+\ell)+2\ell^2$ (35) is not satisfied and hence $u(k)\ne0$ for all $k\in(0,\infty)$. Infact $u(k)=k^2$ is a solution which belongs to $c_{0(\alpha(\ell))}$.

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