Southeast Asian Bulletin of Mathematics (2017) 41: 807–814

Southeast Asian
Bulletin of
Mathematics

A Characterization of Noetherian Modules by the Class of One-Sided Strongly Prime Submodules

Nguyen Trong Bac
Dept. of Math., Faculty of Science, Mahidol University, Bangkok, Thailand, and
Dept. of Basic Sicenes, Thai Nguyen Univ. of Eco. and Business Administration, Thai Nguyen, Vietnam
E-mail: bacnt2008@gmail.com

Nguyen Van Sanh*
Department of Mathematics, Faculty of Science, Mahidol University, Bangkok 10400, Thailand
Dept. of Math., Faculty of Science, Center of Excellence in Math., Mahidol University, Bangkok 10400, Thailand
Email: nguyen.san@mahidol.ac.th

Received 22 September 2015
Accepted 28 March 2016

Communicated by K. P. Shum

AMS Mathematics Subject Classification (2000): 16D50, 16D70, 16D80

Abstract. In this paper, we introduce the classes of strongly prime and one-sided strongly prime submodules and use these classes to characterize Noetherian modules. A finitely generated right R-module M is Noetherian if and only if every one-sided strongly prime submodule is finitely generated. This result can be considered as a generalization of Cohen's Theorem in 1950.

Keywords: Strongly prime submodules; One-sided strongly prime submodules; One-sided strongly prime ideals; Cohen's theorem.

1. Introduction and Preliminaries

Throughout this paper, all rings are associative rings with identity and all modules are unitary right R- modules. Let R be a ring and M, a right R-module.

*Corresponding author.

Denote $S = End_R(M)$, the endomorphism ring of the module M. A submodule X of M is called a *fully invariant* submodule if $f(X) \subset X$, for any $f \in S$. Especially, a right ideal of R is a fully invariant submodule of R_R if it is a two-sided ideal of R. The class of all fully invariant submodules of M is non-empty and closed under intersections and sums. A right R-module M is called a *self-generator* if it generates all its submodules. Following Sanh et al. [14], a fully invariant proper submodule X of M is called a *prime submodule* of M if for any ideal I of $S = \mathrm{End}_R(M)$, and any fully invariant submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$. A fully invariant submodule X of M is called a *strongly prime submodule* of M if for any $\varphi \in S = \mathrm{End}_R(M)$, any $m \in M$, if $\varphi(m) \in X$, then either $\varphi(M) \subset X$ or $m \in X$. The basic Theorem 2.1 in [14] shows that the class of prime submodules of a given module has some properties similar to that of prime ideals in an associative ring. Following that theorem, a fully invariant proper submodule X of M is prime if and only if for any $\varphi \in S$, any $m \in M$, $\varphi Sm \subset X$ implies that $\varphi(M) \subset X$ or $m \in X$. Using this property one can see that every strongly prime submodule is prime. It is natural to ask a question that when a prime submodule is strongly prime and we will answer it in section 2. For a commutative ring, the two notions of prime and strongly prime ideals are coincided.

In this paper, we investigate the classes of strongly prime and one-sided strongly prime submodules and use them to characterize Noetherian modules.

2. On Strongly Prime and One-Sided Strongly Prime Submodules

Definition 2.1. *A proper fully invariant submodule U of M is called strongly prime if for any $f \in S$, any $m \in M$, $f(m) \in U$, then either $f(M) \subset U$ or $m \in U$. Especially, an ideal I of a ring R is strongly prime if for any $a, b \in R, ab \in I$, then either $a \in I$ or $b \in I$.*

Definition 2.2. *A proper submodule X of M is called one-sided strongly prime if for any $f \in S$ and $m \in M$ such that $f(X) \subset X$ and $f(m) \in X$, then either $f(M) \subset X$ or $m \in X$. In particular, a right ideal $P \subsetneq R$ is an one-sided strongly prime right ideal if for any $a, b \in R$ such that $aP \subset P, ab \in P$, then either $a \in P$ or $b \in P$.*

The following proposition is clear by the remark above.

Proposition 2.3. *Every strongly prime submodule is prime.*

Proposition 2.4. *Every maximal submodule is an one-sided strongly prime submodule. In particular, every maximal right ideal of a ring R is an one-sided strongly prime right ideal.*

Proof. Let U be a maximal submodule of M and $\varphi \in S, m \in M$ such that $\varphi(U) \subset U$ and $\varphi(m) \in U$. Suppose that $m \notin U$. Then $U + mR = M$ and hence

$\varphi(M) = \varphi(U) + \varphi(m)R \subset U$, proving that U is an one-sided strongly prime submodule. ■

Next, we will present some examples of strongly prime and one-sided strongly prime submodules:

(1) Every prime ideal in a right duo ring is a strongly prime ideal. Indeed, suppose that P is a prime ideal and $ab \in P$. Put $C = \{c \in R \mid ac \in P\}$. We can verify that C is a right ideal. Since R is a right duo ring, C is a two-sided ideal. Note that from $ab \in P$, we see that $b \in C$. Since C is a two-sided ideal of R, we can see that $Rb \subset C$. This shows that $aRb \subset P$, proving that P is a strongly prime ideal.

(2) Every prime submodule in a duo module is a strongly prime submodule. In fact, suppose that U is a prime submodule and M, a duo module. Let $\varphi(m) \in U$ for any $\varphi \in S$ and $m \in M$. Then we have $U \supset \varphi(m)R = \varphi(mR)$. Since M is a duo module, we see that mR is a fully invariant submodule of M. This implies that $S(mR) = mR$. Hence $\varphi(mR) = \varphi S(mR) \subset U$. By the primeness of U, either $\varphi(M) \subset U$ or $m \in U$, showing that U is a strongly prime submodule of M.

(3) Let $\mathbb{M}_3(k)$ be a matrix ring and k be a division ring. Let R be the following subring of $\mathbb{M}_3(k)$:

$$R := \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} .$$

Let $P \subset R$ be the right ideal of R of the form $P := \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$.

It is easy to verify that, if $xP \subset P$, then $x_{12} = 0; x_{13} = 0; x_{23} = 0$ and $x_{32} = 0$. Suppose that $xy \in P$. Then $x_{11}y_{11} = 0; x_{11}y_{12} = 0; x_{11}y_{13} = 0; x_{21}y_{11} + x_{22}y_{21} = 0; x_{21}y_{13} + x_{22}y_{23} = 0; x_{31}y_{11} + x_{33}y_{31} = 0$ and $x_{31}y_{12} + x_{33}y_{32} = 0$. From $x_{12} = 0, x_{13} = 0, x_{23} = 0$ and $x_{32} = 0$, we can see that either $x \in P$ or $y \in P$, proving that P is an one-sided strongly prime right ideal of R.

Definition 2.5. *A right R-module M is called strongly prime if 0 is a strongly prime submodule of M. A ring R is called a strongly prime ring if 0 is a strongly prime ideal of R.*

We have the following proposition.

Proposition 2.6. *Let M be a quasi-projective right R-module. The following statements are equivalent:*

(1) *X is a strongly prime submodule of M,*

(2) *M/X is a strongly prime module.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\bar{\varphi}(\bar{m}) = \bar{0}$, where $\bar{\varphi} \in End(M/X)$. This implies that $\bar{\varphi}\nu(m) = \bar{0}$. Since M is a quasi-projective module, we can find $f \in S$ such that $\nu f = \bar{\varphi}\nu$, where ν is the natural epimorphism from M to $\bar{M} = M/X$. By (1), either $f(M) \subset X$ or $m \in X$. If $f(M) \subset X$, then $\bar{\varphi}(M/X) = \bar{\varphi}\nu(M) = \bar{0}$. If $m \in X$, then we have $\nu(m) = \bar{m} = \bar{0}$. Hence $\bar{0}$ is a strongly prime submodule of M/X, showing that M/X is a strongly prime module.

(2) $\Rightarrow$ (1). Let $\varphi(m) \in X$, for some $\varphi \in S$ and $m \in M$. Then $\nu\varphi(m) = \bar{0}$. Since X is a fully invariant submodule of M, we can find an endomorphism $f \in \bar{S} = End(M/X)$ such that $\nu\varphi = f\nu$. It follows that $f(\bar{m}) = \bar{0}$, which is a strongly prime submodule. Hence either $f(\bar{M}) = \bar{0}$ or $\bar{m} = \bar{0}$. If $f(\bar{M}) = \bar{0}$, then $f\nu(M) = \bar{0}$. This shows that $\nu\varphi(M) = \bar{0}$. Hence $\varphi(M) \subset X$. If $\bar{m} = \bar{0}$, then $m \in X$. This proves that X is a strongly prime submodule. ■

Note that in the proof (2) $\Rightarrow$ (1), we do not need the quasi-projectivity of M. The following corollary is a direct consequence of proposition above.

Corollary 2.7. *Let I be an ideal of the ring R. Then I is a strongly prime ideal if and only if R/I is a strongly prime ring.*

Lemma 2.8. *Let M, N be right R-modules and $f : M \longrightarrow N$ be an epimorphism. Suppose that $Kerf$ is a fully invariant submodule of M. Then,*

(1) *For any $\varphi \in S$, there exists $\phi \in \bar{S} = End(N)$ such that $\phi f = f\varphi$.*
(2) *If V is a fully invariant submodule of N, then $U = f^{-1}(V)$ is a fully invariant submodule of M.*

Proof. (1) Let $y \in N$. Then $y = f(m)$ for some $m \in M$. Put $\psi(y) = f\varphi(m)$. If $y = f(m) = f(m')$, then $m - m' \in Kerf$. Since $Kerf$ is a fully invariant submodule of $M, \varphi(m - m') \in Kerf$. Thus $f\varphi(m - m') = 0$, proving that ψ is well-defined and moreover it is an R-homomorphism with $f\varphi = \psi f$.

(2) Suppose that V is a fully invariant submodule of N and $U := f^{-1}(V)$. Then by homomorphism theorem, for each $\varphi \in S$, there exists $\alpha \in S$ such that $f\varphi = \alpha f$. Since $Kerf$ is fully invariant, $f\varphi(U) = \alpha f(U) = \alpha(V) \subset V$. This shows that $\varphi(U) \subset f^{-1}(V) = U$, i.e., U is a fully invariant submodule of M. ■

Lemma 2.9. *Let M be a quasi-projective module and P, a strongly prime submodule of M. If $A \subset P$ is a fully invariant submodule of M, then P/A is a strongly prime submodule of M/A.*

Proof. Let $\bar{S} = End_R(M/A), \varphi \in \bar{S}$ and $m + A \in M/A$ with $\varphi(m + A) \subset P/A$. By the quasi-projectivity of M, we can find an endomorphism $f \in S$ such that $\varphi\nu = \nu f$ where $\nu : M \longrightarrow M/A$ is the natural epimorphism. From $f(m) + A = \nu f(m) = \varphi\nu(m) = \varphi(m + A) \in P/A$, we see that $f(m) \in P$. By hypothesis, either $m \in P$ or $f(M) \subset P$. This implies that either $m + A \in P/A$ or $\varphi(M/A) = (f(M) + A)/A \subset P/A$, showing that P/A is strongly prime. ■

Proposition 2.10. *Let M be a quasi-projective module and $f : M \longrightarrow N$ be an epimorphism such that $Kerf$ is a fully invariant submodule of M. Then,*

(1) *If Y is a strongly prime submodule of N, then $X = f^{-1}(Y)$ is a strongly prime submodule of M.*
(2) *If X is a strongly prime submodule of M, then $f(X)$ is a strongly prime submodule of N.*

Proof. (1) By Lemma 2.8, $X = f^{-1}(Y)$ is a fully invariant submodule of M. It is easy to see that X is different M. Suppose that $\varphi \in S$ and $\varphi(m) \in X$. We will show that either $\varphi(M) \subset X$ or $m \in X$. From Lemma 2.8 again, there exists $\gamma \in S^{'} = End(N)$ such that $\gamma f = f\varphi$. From $\varphi(m) \in X$, we can see that $f\varphi(m) \in f(X) = Y$. Since $\gamma f = f\varphi$, we have $\gamma f(m) \in Y$. By assumption, we must have either $f(m) \in Y$ or $\gamma(N) \subset Y$. If $\gamma(N) \subset Y$, then $\gamma f(M) \subset Y$. It follows that $f\varphi(M) \subset Y$. Hence $\varphi(M) \subset f^{-1}(Y) = X$. If $f(m) \in Y$, then $m \in f^{-1}(Y) = X$. Therefore X is a strongly prime submodule.

(2) Note that $f(X)$ is a fully invariant submodule of N. Suppose that $f(X) = N = f(M)$. Then we have $M \subset X + Kerf = X$, a contradiction. This implies that $f(X)$ is different N. Let $\gamma(n) \in f(X)$, where $\gamma \in S' = End(N)$. We will show that $\gamma(N) \subset f(X)$ or $n \in f(X)$. Since M is a quasi-projective module, there is $\varphi \in S$ such that $\gamma f = f\varphi$. From this, we can see that $\gamma(n) = \gamma(f(f^{-1})(n)) = f\varphi(f^{-1}(n)) \subset f(X)$. It follows that $\varphi(f^{-1}(n)) \subset X + Kerf = X$. If X is a strongly prime submodule, then we have either $\varphi(M) \subset X$ or $f^{-1}(n) \in X$. If $\varphi(M) \subset X$, then $f\varphi(M) \subset f(X)$. Thus $\gamma f(M) \subset f(X)$ and hence $\gamma(N) \subset f(X)$. If $f^{-1}(n) \in X$, then $n \in f(X)$. This shows that $f(X)$ is a strongly prime submodule. ∎

Recall from [17] that a submodule X of a right R-module M is said to have *"insertion factor property"* (briefly, an IFP-submodule) if for any endomorphism φ of M and any element $m \in M$, if $\varphi(m) \in X$, then $\varphi Sm \subset X$. A right ideal I is an IFP- right ideal if it is an IFP submodule of R_R, that is for any $a, b \in R$, if $ab \in I$, then $aRb \subset I$. A right R-module M is called an IFP-module if 0 is an IFP-submodule of M. A ring is IFP if 0 is an IFP-ideal. For more details, we refer the readers to [17]. We give the relationship between a strongly prime and prime submodule by the following theorem.

Theorem 2.11. *Let M be an R-module. A submodule X of M is a strongly prime submodule if and only if it is prime and IFP.*

Proof. Suppose that X is a strongly prime submodule of M. For any $\varphi \in S$ and for any $m \in M$, if $\varphi S(m) \subset X$, then $\varphi(m) \in X$. Since X is a strongly prime submodule, we have either $\varphi(M) \subset X$ or $m \in X$. This implies that X is a prime submodule. We assume that $\varphi(m) \in X$. We need to prove that $\varphi S(m) \subset X$. Since $\varphi(m) \in X$, we can see that either $\varphi(M) \subset X$ or $m \in X$. If $m \in X$, then we have $g(m) \in g(X) \subset X$, for all $g \in S$. This means that $S(m) \subset X$. Therefore $\varphi S(m) \subset X$. Suppose that $\varphi(M) \subset X$. We can see that $\varphi S(M) = \varphi(M) \subset X$. This follows that $\varphi S(m) \subset X$, as desired.

Suppose that X is a prime submodule and has IFP. If $\varphi(m) \in X$, then we want to show that either $\varphi(M) \subset X$ or $m \in X$. Since X has IFP, we have

$\varphi S(m) \subset X$. By the primeness of X, we can see that either $\varphi(M) \subset X$ or $m \in X$. This shows that X is a strongly prime submodule, as required. ∎

Corollary 2.12. *An ideal I of a ring R is a strongly prime ideal if and only if it is prime and IFP.*

Theorem 2.13. *Let M be a right R-module. If X is a strongly prime submodule of M, then $I_X = \{f \in S | f(M) \subset X\}$ is a strongly prime ideal of S. Conversely, if M is a self-generator and I_X is a strongly prime ideal of S, then X is a strongly prime submodule.*

Proof. Suppose that X is a strongly prime submodule. From Theorem 2.11, we see that X is prime and IFP. By [14, Theorem 1.10], I_X is a prime ideal of S. It is well known from [17, Lemma 2] that if X has IFP, then I_X is an IFP-right ideal of S. Hence I_X is a strongly prime ideal of S, by Corollary 2.12.

Conversely, suppose that M is a self-generator and I_X is a strongly prime ideal of S. Then I_X is prime and IFP. By Theorem 1.10 in [14], we see that X is prime. Similarly, from Lemma 2 in [17], X has IFP. Applying Theorem 2.11, X is a strongly prime submodule, as desired. ∎

Theorem 2.14. *Let M be a right R-module. If X is an one-sided strongly prime submodule of M, then I_X is an one-sided strongly prime right ideal of S. Conversely, if M is a self-generator and I_X is an one-sided strongly prime right ideal of S, then X is an one-sided strongly prime submodule of M.*

Proof. Suppose that X is an one-sided strongly prime submodule and $\varphi, \alpha \in S$ such that $\varphi I_X \subset I_X$ and $\varphi\alpha \in I_X$. Then $\varphi\alpha(m) \in X$ for all $m \in M$. Since M is a self-generator, we have $X = \sum_{f \in I_X} f(M)$. Hence $\varphi(X) \subset X$. We assume that $\varphi \notin I_X$. Since X is an one-sided strongly prime submodule, we must have $\alpha(m) \in X$, for all $m \in M$. This shows that $\alpha \in I_X$. Hence I_X is an one-sided strongly prime right ideal of S.

Conversely, suppose that I_X is an one-sided strongly prime right ideal of S. Since M is a self-generator, we have $I_X(M) = X$. Assume that $\varphi(X) \subset X, \varphi(m) \in X$ and $m \notin X$. We wish to prove that $\varphi(M) \subset X$. From our assumption, we can see that $\varphi I_X \subset I_X$. Put $mR = \sum_{\psi \in A} \psi(M)$, for some subset A of S. Then $X \supset \varphi(m)R = \varphi(mR) = \varphi(\sum_{\psi \in A} \psi(M)) = \sum_{\psi \in A} \varphi\psi(M)$. This implies that $\varphi\psi(M) \subset X$ for all $\psi \in A$. Since I_X is an one-sided strongly prime right ideal and $m \notin X$, we have $\varphi \in I_X$. This shows that X is an one-sided strongly prime submodule of M, as required. ∎

3. Characterizations of Noetherian Modules

Theorem 3.1. *Let M be a finitely generated right R-module. Then M is a Noetherian right R-module if and only if every one-sided strongly prime submodule of*

M is finitely generated.

Proof. One way is clear. Suppose on the contrarily that there is a submodule A of M which is not finitely generated. By Zorn's Lemma, the set $\mathcal{F} = \{X \subset M | A \subset X$ and X is not finitely generated $\}$ has a maximal element, A_0 says. Since M is finitely generated, A_0 is a proper submodule of M. We now prove that A_0 is one-sided strongly prime. Suppose that there are $\varphi \in S, m \in M$ such that $\varphi(m) \in A_0$ with $\varphi(A_0) \subset A_0$ but $\varphi(M) \not\subset A_0$ and $m \notin A_0$. Then $A_0 + \varphi(M)$ contains properly A_0, and hence it is finitely generated, that is $A_0 + \varphi(M) = x_1R + x_2R + \cdots + x_nR$ for some $x_1, x_2, \ldots, x_n \in M$. Let $K = \{a \in M | \varphi(a) \in A_0\}$. By assumption, $A_0 \subset K$ and $m \in K$. Since $m \notin A_0, K$ contains properly $A_0 + mR$ and hence it is finitely generated. Since $x_i \in A_0 + \varphi(M)$, we can write $x_i = b_i + \varphi(m_i)$, where $b_i \in A_0$ and $m_i \in M$. By definition, $\varphi(K) \subset A_0$. It follows that $b_1R + b_2R + \cdots + b_nR \subset A_0$. We now prove that $A_0 \subset b_1R + b_2R + \cdots + b_nR + \varphi(K)$. For any $w \in A_0$, we have $w \in A_0 + \varphi(M)$. We can write $w = \sum_{i=1}^n x_ir_i = \sum_{i=1}^n (b_i + \varphi(m_i))r_i = \sum_{i=1}^n b_ir_i + \sum_{i=1}^n \varphi(m_ir_i) + \varphi(\sum_{i=1}^n m_ir_i)$. Since $w \in A_0$ and $\sum_{i=1}^n b_ir_i \in A_0$, we have $\varphi(\sum_{i=1}^n m_ir_i) \in A_0$ and hence $\sum_{i=1}^n m_ir_i \in K$. This implies that $w \in b_1R + b_2R + \cdots + b_nR + \varphi(K)$. Therefore $b_1R + b_2R + \cdots + b_nR + \varphi(K) \subset A_0$. This proves that $A_0 = b_1R + b_2R + \cdots + b_nR + \varphi(K)$. Since K is finitely generated, we can see that $\varphi(K)$ is finitely generated and hence A_0 is finitely generated, which is a contradiction. Therefore, every submodule of M is finitely generated, proving that M is Noetherian. ■

Note that one-sided strongly prime right ideals are called *completely prime right ideals* in [13]. The following Corollary can be considered as an immediately consequence of our theorem.

Corollary 3.2. [13, Theorem 3.8] *A ring R is right Noetherian if and only if every one-sided strongly prime right ideal is finitely generated.*

Recall that a right R- module M is called a *duo module* if every submodule of M is a fully invariant submodule of M. A ring is called a *right duo ring* if every right ideal is a two-sided ideal. It is easy to see that a fully invariant one-sided strongly prime submodule of M is a strongly prime submodule of M. Thus, if M is a duo module, then every one-sided strongly prime submodule of M is also a strongly prime submodule of M. This leads to another corollary.

Corollary 3.3. *A finitely generated, duo right R- module is Noetherian if and only if every strongly prime submodule of M is finitely generated.*

From this corollary, putting $M = R_R$, we get:

Corollary 3.4. [3] *If R is a left (resp. right) duo ring and suppose that every prime ideal in R is finitely generated, then R is left (resp. right) Noetherian.*

Note that the definition of strongly prime ideals coincides with the usual definition of prime ideals in the commutative case. Therefore, the following Corollary is a direct consequence of Theorem 3.1.

Corollary 3.5. [4, Theorem 2] *A commutative ring R with identity is Noetherian if and only if every prime ideal of R is finitely generated.*

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