Southeast Asian Bulletin of Mathematics (2019) 43: 165–181

Southeast Asian
Bulletin of
Mathematics

Differential Identities and Generalized Derivations in Prime Rings with Involution

Abdelkarim Boua
Sidi Mohammed Ben Abdellah University, Polydisciplinary Faculty, LSI, Taza, Morocco
Email: abdelkarimboua@yahoo.fr; karimoun2006@yahoo.fr

Mohammad Ashraf
Department of Mathematics, Aligarh Muslim University, Aligarh-202002 India
Email: mashraf80@hotmail.com

Received 3 January 2017
Accepted 8 September 2017

Communicated by Nguyen Van Sanh

AMS Mathematics Subject Classification(2000): 16U80, 16N60, 16W25

Abstract. Let $\mathcal{R}$ be a $*$-ring with the center $Z(\mathcal{R})$. In the present paper we study commutativity of a prime ring which admits derivations d, g satisfying any one of the properties:

(i) $d([x, x^*]) \pm [x, x^*] \in Z(\mathcal{R})$,
(ii) $d(x \circ x^*) \pm x \circ x^* \in Z(\mathcal{R})$,
(iii) $d([x, x^*]) \pm x \circ x^* \in Z(\mathcal{R})$,
(iv) $d(x \circ x^*) \pm [x, x^*] \in Z(\mathcal{R})$,
(v) $d(x) \circ d(x^*) \pm x \circ x^* \in Z(\mathcal{R})$,
(vi) $d(x)g(x^*) \pm [x, x^*] \in Z(\mathcal{R})$ and
(vii) $d(x)g(x^*) \pm x \circ x^* \in Z(\mathcal{R})$,

for all $x \in \mathcal{R}$. Further, some related results are also obtained if $\mathcal{R}$ admits a generalized derivation F satisfying either of the properties

$$F(xx^*) \pm F(x)F(x^*) \in Z(\mathcal{R})$$

for all $x \in \mathcal{R}$ or

$$F(xx^*) \pm F(x^*)F(x) \in Z(\mathcal{R})$$

for all $x \in \mathcal{R}$. Finally, an example has been provided to justify the hypotheses in various results.

Keywords: Prime rings; Involution; Derivation; Generalized derivations.

1. Introduction

Throughout this paper $\mathcal{R}$ denotes an associative ring with the center $Z(\mathcal{R})$. For any $x, y \in \mathcal{R}$, $[x, y]$ will denote the Lie product $xy - yx$ while $x \circ y$ will represent the Jordan product $xy + yx$. However, given two subsets A and B of $\mathcal{R}$, then $[A, B]$ will denote the additive subgroup of $\mathcal{R}$ generated by all elements of the form $[a, b]$ where $a \in A$ and $b \in B$ and $A \circ B$ is defined similarly. A ring $\mathcal{R}$ is said to be 2-torsion free if $2a = 0$ (where $a \in \mathcal{R}$) implies $a = 0$. Recall that $\mathcal{R}$ is prime if $a\mathcal{R}b = \{0\}$ implies that $a = 0$ or $b = 0$ and $\mathcal{R}$ is semiprime if $a\mathcal{R}a = \{0\}$ implies $a = 0$. It is straightforward to see that a prime ring with characteristic different from two is 2-torsion free. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a derivation of $\mathcal{R}$ if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{R}$. A derivation d is said to be inner if there exists $a \in \mathcal{R}$ such that $d(x) = ax - xa$ for all $x \in \mathcal{R}$. Following Bresar [16], an additive mapping $F : \mathcal{R} \to \mathcal{R}$ is called a generalized derivation if there exists a derivation $d : \mathcal{R} \to \mathcal{R}$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in \mathcal{R}$. The concept of generalized derivation includes both the concept of derivation and the concept of left multiplier (i.e., an additive mapping $F : \mathcal{R} \to \mathcal{R}$ satisfying $F(xy) = F(x)y$ for all $x, y \in \mathcal{R}$). An additive map $x \mapsto x^*$ of $\mathcal{R}$ into itself is called an involution if

(i) $(xy)^* = y^*x^*$ and
(ii) $(x^*)^* = x$ holds

for all $x, y \in \mathcal{R}$. A ring equipped with an involution is known as a ring with involution or $*-$ring. An element x in a ring with involution $*$ is said to be Hermitian if $x^* = x$ and skew-Hermitian if $x^* = -x$. The sets of all Hermitian and skew-Hermitian elements of $\mathcal{R}$ will be denoted by $H(\mathcal{R})$ and $S(\mathcal{R})$ respectively. If $\mathcal{R}$ is 2-torsion free then every $x \in \mathcal{R}$ can be uniquely represented in the form $2x = h + k$ where $h \in H(\mathcal{R})$ and $k \in S(\mathcal{R})$. The involution is said to be of the first kind if $Z(\mathcal{R}) \subseteq H(\mathcal{R})$; otherwise, it is said to be of the second kind. In the latter case $S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$.

Over the last two decades several authors have investigated the relationship between the commutativity of the ring $\mathcal{R}$ and certain special types of maps on $\mathcal{R}$. In the second section of this paper, we discuss the commutativity of a prime ring $\mathcal{R}$ with involution of the second kind involving a nonzero derivation d satisfying any one of the following properties:

(i) $d([x, x^*]) \pm [x, x^*] \in Z(\mathcal{R})$,
(ii) $d(x \circ x^*) \pm (x \circ x^*) \in Z(\mathcal{R})$,
(iii) $d([x, x^*]) \pm (x \circ x^*) \in Z(\mathcal{R})$,
(iv) $d(x \circ x^*) \pm [x, x^*] \in Z(\mathcal{R})$,
(v) $d(x) \circ d(x^*) \pm x \circ x^* \in Z(\mathcal{R})$,
(vi) $d_1(x)d_2(x^*) \pm [x, x^*] \in Z(\mathcal{R})$ and
(vii) $d_1(x)d_2(x^*) \pm x \circ x^* \in Z(\mathcal{R})$,

for all $x \in \mathcal{R}$.

In the third section, we investigate the commutativity of $*$-ring which admits a generalized derivation F satisfying either of the following conditions:

(i) $F(xx^*) \pm F(x)F(x^*) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ or (ii) $F(xx^*) \pm F(x^*)F(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. In the end of the each section example has been provided, to show that the conditions imposed in the hypotheses of various theorems are not altogether superfluous.

2. Derivations in Rings with Involution

Rings satisfying differential identities occupy an important place in modern noncommutative ring theory and recently many authors obtained several interesting results concerning structure and commutativity of rings satisfying certain differential identities (see [1, 9, 8, 7, 4, 5, 6, 14, 10, 11, 12, 18, 21, 24] etc. for partial reference). In 1992 Daif and Bell [17] proved that if $\mathcal{R}$ is a prime ring which admits a derivation d and a nonzero ideal I of $\mathcal{R}$ such that $d([x, y]) - [x, y] = 0$ for all $x, y \in I$ or $d([x, y]) + [x, y] = 0$ for all $x, y \in I$, then $\mathcal{R}$ is commutative. This result was further generalized by Hongan [21] who proved that if $\mathcal{R}$ is a semiprime ring which admits a derivation d such that $d([x, y]) \pm [x, y] \in Z(\mathcal{R})$ for all $x, y \in I$, a nonzero ideal of $\mathcal{R}$, then I is a central ideal of $\mathcal{R}$. In particular if $I = \mathcal{R}$, then $\mathcal{R}$ is commutative. In a similar direction, Oukhtite et al. [24] obtained commutativity of $\mathcal{R}$ if $\mathcal{R}$ is 2-torsion free prime ring which admits a derivation d satisfying $d(x \circ y) \pm (x \circ y) \in \mathcal{R}$, for all x, y in a nonzero Jordan ideal of $\mathcal{R}$. Very recently Dar and Ali [18] proved that if a prime ring $\mathcal{R}$ with involution $'*'$ of the second kind and with characteristic different from two admits a nonzero derivation d such that $d([x, x^*]) \pm [x, x^*] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative. In the present section of this paper, we generalize the above-mentioned result in the setting of $*$-ring and obtain the commutativity of a prime ring satisfying rather weaker identities. We begin our investigation with the following:

Theorem 2.1. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind with the char$(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a derivation d, then the following conditions are equivalent:*

(i) $d([x, x^*]) \pm [x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$.
(ii) $d(x \circ x^*) \pm x \circ x^* \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$.
(iii) $\mathcal{R}$ *is commutative.*

In order to develop the proof of the above theorem, we begin with some well known results for rings which will be used frequently in the subsequent discussion. The proof of the first lemma is straightforward in the setting of prime rings.

Lemma 2.2. *Let $\mathcal{R}$ be a prime ring. If z is a nonzero central element such that $xz \in Z(\mathcal{R})$, then $x \in Z(\mathcal{R})$.*

The following two lemmas are easy consequences of [11, Theorems 2.9 and 2.10] respectively, while the proof of Lemma 2.5 can be seen in [24].

Lemma 2.3. *If $\mathcal{R}$ is a prime ring in which $[x,y] \in Z(\mathcal{R})$ for all $x,y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Lemma 2.4. *If $\mathcal{R}$ is a prime ring of characteristic different from two in which $x \circ y \in Z(\mathcal{R})$ for all $x,y \in R$, then $\mathcal{R}$ is commutative.*

Lemma 2.5. *Let $\mathcal{R}$ be a prime ring of characteristic different from two and J be a nonzero Jordan ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a derivation d such that either $d(x \circ y) - x \circ y \in Z(\mathcal{R})$ for all $x,y \in J$, or $d(x \circ y) + x \circ y \in Z(\mathcal{R})$ for all $x,y \in J$, then $\mathcal{R}$ is commutative.*

We are now well-equipped to prove our theorems.

Proof of Theorem 2.1. It is clear that (iii) implies (i) and (ii). So we need to prove that (i)⇒(iii) and (ii)⇒(iii).

(i)⇒(iii) Suppose that $\mathcal{R}$ satisfies $d([x,x^*]) - [x,x^*] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. If $d = 0$, then $-[x,x^*] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. Replacing x by $x + y^*$, we find that $-[x,y] - [y^*,x^*] \in Z(\mathcal{R})$ for all $x,y \in \mathcal{R}$. Replace y by ys again where $s \in Z(\mathcal{R}) \cap S(\mathcal{R})$ and use the primeness of $\mathcal{R}$, to obtain $-[x,y] + [y^*,x^*] \in Z(\mathcal{R})$ for all $x,y \in \mathcal{R}$. Combining the latter two expressions and using the fact that $char(\mathcal{R}) \neq 2$, we arrive at $[x,y] \in Z(\mathcal{R})$ for all $x,y \in \mathcal{R}$ and Lemma 2.3, forces that $\mathcal{R}$ is commutative.

If $Z(\mathcal{R}) = \{0\}$, then $d([x,x^*]) = \pm[x,x^*]$, and by Theorem 3.4 of [18], we conclude that $\mathcal{R}$ is commutative.

Hence onward we assume that $d \neq 0$ and $Z(\mathcal{R}) \neq \{0\}$. We are given that

$$d([x,x^*]) - [x,x^*] \in Z(\mathcal{R}) \quad \text{for all } x \in \mathcal{R}. \tag{1}$$

A linearization of (1) yields that

$$d([x,y^*]) + d([y,x^*]) - [x,y^*] - [y,x^*] \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}. \tag{2}$$

writing y^* instead of y we get

$$d([x,y]) + d([y^*,x^*]) - [x,y] - [y^*,x^*] \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}. \tag{3}$$

Taking yh instead of y where $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$ and using (3), we obtain

$$\big([x,y] + [y^*,x^*]\big)d(h) \in Z(\mathcal{R}) \quad \text{for all } x,y \in \mathcal{R}. \tag{4}$$

Since $d(h) \in Z(\mathcal{R})$, by Lemma 2.2, we find that either $d(h) = 0$ or $[x,y] + [y^*,x^*] \in Z(\mathcal{R})$ for all $x,y \in \mathcal{R}$. If $d(h) = 0$ for all $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$, then putting k^2 instead of h, where $k \in Z(\mathcal{R}) \cap S(\mathcal{R})$, we arrive at $d(k)k = 0$ for all $k \in Z(\mathcal{R}) \cap S(\mathcal{R})$. Now since the center of a prime ring is free from zero divisors, we find that $d(k) = 0$ or $k = 0$. Since $k = 0$ also implies $d(k) = 0$, we may write $d(k) = 0$ for all $k \in Z(\mathcal{R}) \cap S(\mathcal{R})$.

Since $char(\mathcal{R}) \neq 2$, every $z \in Z(\mathcal{R})$ can be represented as $2z = h + k$, where $h \in H(R) \cap Z(\mathcal{R})$ and $k \in Z(\mathcal{R}) \cap S(\mathcal{R})$. This implies that $2d(z) = d(2z) = d(h + k) = d(h) + d(k) = 0$, which forces that $d(z) = 0$ for all $z \in Z(\mathcal{R})$.

Replacing y by ys in (3) where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$ we get

$$\big(d([x, y]) - d([y^*, x^*]) - [x, y] + [y^*, x^*]\big)s \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}.$$

This yields that

$$d([x, y]) - d([y^*, x^*]) - [x, y] + [y^*, x^*] \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{5}$$

Combining (2) with (5) and using the fact $char(\mathcal{R}) \neq 2$, we find that $d([x, y]) - [x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. By Theorem 1 of [21], we conclude that $\mathcal{R}$ is commutative.

In the second case suppose that $[x, y] + [y^*, x^*] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, replacing y by ys, where $s \in Z(R) \cap S(R) \setminus \{0\}$, we get $([x, y] - [y^*, x^*])s \in Z(\mathcal{R})$ so that $[x, y] - [y^*, x^*] \in Z(\mathcal{R})$. Combining the last equation with our assumption we obtain $[x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, proving that $\mathcal{R}$ is commutative by Lemma 2.3.

Further using the similar techniques as used after (1), we obtain the same conclusion in the case if $d([x, x^*]) + [x, x^*] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$.

(ii)$\Rightarrow$(iii) Now assume that

$$d(x \circ x^*) - x \circ x^* \in Z(\mathcal{R}) \quad \text{for all } x \in \mathcal{R}. \tag{6}$$

A linearization of (6), yields that

$$d(x \circ y^*) + d(y \circ x^*) - x \circ y^* - y \circ x^* \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R},$$

and hence

$$d(x \circ y) + d(y^* \circ x^*) - x \circ y - y^* \circ x^* \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{7}$$

Substituting yh for y in (7), where $h \in Z(\mathcal{R}) \cap H(\mathcal{R}) \setminus \{0\}$, and using it again, we obtain

$$\big(x \circ y + y^* \circ x^*\big)d(h) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{8}$$

By primeness of $\mathcal{R}$, we find that either $d(h) = 0$ or $x \circ y + y^* \circ x^* \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$.

If $x \circ y + y^* \circ x^* \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, then in particular, for $y \in Z(\mathcal{R})$, it is easy to see that $xy + x^*y^* \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. Taking $y \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$, we get $x + x^* \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. On the other hand for $y \in Z(R) \cap H(R)$, we obtain $x - x^* \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, and we conclude that $x \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, which forces that $\mathcal{R}$ is commutative.

Suppose that $d(h) = 0$ for all $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$. This implies that $d(z) = 0$ for all $z \in Z(\mathcal{R})$. Now replacing y by ys in (7), where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$, we arrive at $\big(d(x \circ y) - d(y^* \circ x^*) - x \circ y + y^* \circ x^*\big)s \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. and hence

$$d(x \circ y) - d(y^* \circ x^*) - x \circ y + y^* \circ x^* \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{9}$$

Combining (9) and (7), it follows that $d(x \circ y) - x \circ y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. By Lemma 2.5, we conclude that $\mathcal{R}$ is commutative.

By using the similar arguments, we obtain the same conclusion in the case if $d(x \circ x^*) + x \circ x^* \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. This completes the proof. ■

Following is an immediate consequence of the above theorem:

Corollary 2.6. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind such that the char$(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a derivation d such that either $d(xx^*) - xx^* \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ or $d(xx^*) + xx^* \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

In [18], Dar and Ali showed that if a prime ring $\mathcal{R}$ with involution $*$ of the second kind with $char(R) \neq 2$ and $\mathcal{R}$ admits a derivation d such that either (i) $d([x, x^*]) \pm (x \circ x^*) = 0$ for all $x \in \mathcal{R}$ or (ii) $d(x \circ x^*) \pm [x, x^*] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative. In view of the above mentioned result, it is natural to ask what we can say about the commutativity of a prime ring $\mathcal{R}$ if the underlying properties (i) and (ii) are replaced by weaker assumptions viz. $d([x, x^*]) \pm (x \circ x^*) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ or $d(x \circ x^*) \pm [x, x^*] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ respectively. In this direction, we have succeeded in establishing the following result:

Theorem 2.7. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind with the char$(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a derivation d, then the following conditions are equivalent:*

(i) $d([x, x^*]) \pm x \circ x^* \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$.
(ii) $d(x \circ x^*) \pm [x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$.
(iii) $\mathcal{R}$ *is commutative.*

Proof. It is clear that (iii) implies (i) and (ii). We need to prove that (i)⇒(iii) and (ii)⇒(iii).

(i)⇒(iii) We are given that

$$d([x, x^*]) - x \circ x^* \in Z(\mathcal{R}) \quad \text{for all } x \in \mathcal{R}. \tag{10}$$

Linearizing (10), we get $d([x, y^*]) + d([y, x^*]) - x \circ y^* - y \circ x^* \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, i.e.;

$$d([x, y]) + d([y^*, x^*]) - x \circ y - y^* \circ x^* \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{11}$$

Putting yh instead of y in (11), where $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$ and using (11), we obtain

$$\big([x,y] + [y^*, x^*]\big)d(h) \in Z(\mathcal{R}) \quad \text{for all } \; x, y \in \mathcal{R}, \tag{12}$$

which is the same identity as (4). Thus by the same argument as we have used to get (5) from (4), we find that

$$d([x,y]) - x \circ y \in Z(\mathcal{R}) \quad \text{for all } \; x, y \in \mathcal{R}. \tag{13}$$

In particular, for $y = x$, we obtain $x^2 \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. Replacing x by x^2 in (13) we arrive at $x^2 y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Using Lemma 2.2, we get either $x^2 = 0$ for all $x \in \mathcal{R}$ or $y \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$. It is clear that the first case gives a contradiction and second forces that $\mathcal{R}$ is commutative.

If $d([x, x^*]) + x \circ x^* \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, then using the similar arguments as above, we can prove that $\mathcal{R}$ is commutative.

(ii)⇒(iii) Assume that

$$d(x \circ x^*) - [x, x^*] \in Z(\mathcal{R}) \quad \text{for all } \; x \in \mathcal{R}. \tag{14}$$

Linearizing (14), we get

$$d(x \circ y^*) + d(y \circ x^*) - [x, y^*] - [y, x^*] \in Z(\mathcal{R}) \quad \text{for all } \; x, y \in \mathcal{R}.$$

This can be further written as

$$d(x \circ y) + d(y^* \circ x^*) - [x, y] - [y^*, x^*] \in Z(\mathcal{R}) \quad \text{for all } \; x, y \in \mathcal{R}. \tag{15}$$

In (15), replacing yh in place of y, where $h \in Z(\mathcal{R}) \cap H(\mathcal{R}) \setminus \{0\}$, and making use of (15), we get

$$\big(x \circ y + y^* \circ x^*\big)d(h) \in Z(\mathcal{R}) \quad \text{for all } \; x, y \in \mathcal{R}. \tag{16}$$

Since (16) and (8) are identical, by the same techniques as we have used in the proof of Theorem 2.1, we arrive at $\mathcal{R}$ is commutative or $d(x \circ y) - d(y^* \circ x^*) - [x, y] + [y^*, x^*] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Combining the above expression with (15), we find that

$$d(x \circ y) - [x, y] \in Z(\mathcal{R}) \quad \text{for all } \; x, y \in \mathcal{R}. \tag{17}$$

In particular, for $y = x$, we obtain $d(x^2) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ and by Theorem 4.14 of [23], we conclude that $\mathcal{R}$ is commutative.

Further, if $d(x \circ x^*) + [x, x^*] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, then using the similar arguments as above, we can prove that $\mathcal{R}$ is commutative. Thereby completing the proof of the theorem. ■

In [7] Ashraf et al. proved that if $\mathcal{R}$ is a 2-torsion free semiprime ring which admits a nonzero derivation d such that $d(x) \circ d(y) = \pm(x \circ y)$, for all x, y in a nonzero ideal of $\mathcal{R}$, then $\mathcal{R}$ is commutative. Recently, in [2] Ali et al. generalized the mentioned identity by replacing y by x^*. In this line of investigation, we consider rather a weaker situation when $d(x) \circ d(x^*) \pm x \circ x^* \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ and prove the following theorem:

Theorem 2.8. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind with the char$(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a derivation d such that $d(x) \circ d(x^*) \pm x \circ x^* \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. Suppose that

$$d(x) \circ d(x^*) - x \circ x^* \in Z(\mathcal{R}) \quad \text{for all } x \in \mathcal{R}. \tag{18}$$

Replacing x by $x + y$ in (18) and using it again, we find that

$$\big(d(x) \circ d(y^*) - x \circ y^*\big) + \big(d(y) \circ d(x^*) - y \circ x^*\big) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R},$$

and hence from the above equation, we arrived at

$$\big(d(x) \circ d(y) - x \circ y\big) + \big(d(y^*) \circ d(x^*) - y^* \circ x^*\big) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{19}$$

Replacing y by yh, where $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$, the above identity reduces to

$$\big(d(x) \circ y + y^* \circ d(x^*)\big)d(h) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{20}$$

By Lemma 2.2, we find that $d(h) = 0$ for all $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$ or $d(x) \circ y + y^* \circ d(x^*)$ for all $x, y \in \mathcal{R}$.

If $d(h) = 0$ for all $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$, then $d(z) = 0$ for all $z \in Z(\mathcal{R})$. Replacing y by ys in (19), where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$, we have

$$\Big(\big(d(x) \circ d(y) - x \circ y\big) - \big(d(y^*) \circ d(x^*) - y^* \circ x^*\big)\Big)s \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R},$$

so that

$$\big(d(x) \circ d(y) - x \circ y\big) - \big(d(y^*) \circ d(x^*) - y^* \circ x^*\big) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{21}$$

On comparing (19) with (21), we find that

$$d(x) \circ d(y) - x \circ y \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}.$$

If $Z(\mathcal{R}) = \{0\}$, then $d(x) \circ d(y) = x \circ y$ for all $x, y \in \mathcal{R}$ in view of Theorem 4.4 of [7], we conclude that $\mathcal{R}$ is commutative. Therefore by the above argument, we obtain $d(x)d(y) = xy$ for all $x, y \in \mathcal{R}$ and replacing y by ry in the above relation and using it again, yields that $d(x)rd(y) = 0$ for all $x, y, r \in \mathcal{R}$ so that $d(x)\mathcal{R}d(x) = \{0\}$ for all $x \in \mathcal{R}$. By primeness of $\mathcal{R}$, we conclude that $d = 0$ which implies that $x\mathcal{R}x = \{0\}$ for all $x \in \mathcal{R}$ by our hypothesis. Using primeness of $\mathcal{R}$ again, we obtain $\mathcal{R} = \{0\}$, a contradiction.

Now assume that $Z(\mathcal{R}) \neq \{0\}$ and $d(x) \circ d(y) - x \circ y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Since $d(Z(\mathcal{R})) \subseteq Z(\mathcal{R})$, for $y \in Z(\mathcal{R})$, the above relation yields that

$$d(x)d(y) - xy \in Z(\mathcal{R}) \quad \text{for all} \quad x \in \mathcal{R}. \tag{22}$$

Replacing x by xt where $t \in Z(\mathcal{R})$ in (22) and using the definition of d, we find that

$$t(d(x)d(y) - xy) + d(t)xd(y) \in Z(\mathcal{R}) \quad \text{for all} \quad x \in \mathcal{R}, y, t \in Z(\mathcal{R}). \tag{23}$$

This reduces to

$$d(t)xd(y) \in Z(\mathcal{R}) \quad \text{for all} \quad x \in \mathcal{R}, y, t \in Z(\mathcal{R}). \tag{24}$$

Using Lemma 2.2, we obtain that $d(Z(\mathcal{R})) = \{0\}$ or $\mathcal{R}$ is commutative.

If $d(Z(\mathcal{R})) = \{0\}$, we arrive at $xz \in Z(\mathcal{R})$ for $x \in \mathcal{R}$, $z \in Z(\mathcal{R})$ and so again using Lemma 2.2 with $Z(\mathcal{R}) \neq \{0\}$, we conclude that $\mathcal{R}$ is commutative.

Suppose that $d(x) \circ y + y^* \circ d(x^*) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Replacing y by ys where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$, we find that $\big(d(x) \circ y - y^* \circ d(x^*)\big)s \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, so that $d(x) \circ y - y^* \circ d(x^*) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Combining the previous equation with our assumption we obtain $d(x) \circ y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. In particular, for $y \in Z(\mathcal{R}) \setminus \{0\}$, we obtain $d(x)y \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$ and Lemma 2.2 with Theorem 4 of [13], assures that $\mathcal{R}$ is commutative.

Using the similar arguments one can prove that (ii)$\Rightarrow$(iii). ■

Theorem 2.9. *Let $\mathcal{R}$ be a prime ring with involution '$*$' of the second kind with the char$(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits derivations d_1, d_2, then the following conditions are equivalent:*

(i) $d_1(x)d_2(x^*) \pm [x, x^*] \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$.
(ii) $d_1(x)d_2(x^*) \pm x \circ x^* \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$.
(iii) $\mathcal{R}$ *is commutative.*

Proof. It is clear that (iii) implies (i) and (ii). Therefore, we need to prove that (i)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iii).

(i)$\Rightarrow$(iii) Suppose that

$$d_1(x)d_2(x^*) - [x, x^*] \in Z(\mathcal{R}) \quad \text{for all} \quad x \in \mathcal{R}. \tag{25}$$

Replacing x by $x + y^*$ in (25) and using it again, we arrive at

$$d_1(x)d_2(y) - [x, y] + d_1(y^*)d_2(x^*) - [x^*, y^*] \in Z(\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{26}$$

Replacing y by yh, where $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$ and using (26), we obtain

$$d_1(x)yd_2(h) + d_1(h)y^*d_2(x^*) \in Z(\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{27}$$

Taking ys where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$ instead of y in (27) and using Lemma 2.2, we get

$$d_1(x)yd_2(h) - d_1(h)y^*d_2(x^*) \in Z(\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{28}$$

Combining (27) with (28) we find that

$$d_1(x)yd_2(h) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{29}$$

Since $d_2(h) \in Z(\mathcal{R})$, $d_1(x)y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$ or $d_2(h) = 0$ by Lemma 2.2.

If $d_1(x)y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, then $d_1(x)ys = sd_1(x)y$ for all $x, y, s \in \mathcal{R}$. Replacing y by yt in latter expression and using it again, we arrive at $d_1(x)\mathcal{R}[s,t] = \{0\}$ for all $x, s, t \in \mathcal{R}$. Since $\mathcal{R}$ is prime, we have $d_1 = 0$ or $\mathcal{R}$ is commutative.

If $d_1 = 0$, then from the equation (26), we get $-[x, y] - [x^*, y^*] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$ and replacing y by ys where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$, we find that $-[x, y] + [x^*, y^*] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Combining the two latter relations, we arrive at $-[x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$ and by Lemma 2.3, we conclude that $\mathcal{R}$ is commutative.

If $d_2(h) = 0$, then $d_2(z) = 0$ for all $z \in \mathcal{R}$. Replacing y by ys, where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$ in (26), then for all $x, y \in \mathcal{R}$,

$$\big(d_1(x)d_2(y) - [x, y]\big)s - \big(d_1(y^*)d_2(x^*) - [x^*, y^*]\big)s - y^*d_1(s)d_2(x^*) \in Z(\mathcal{R}). \tag{30}$$

Using again (26), we arrive at

$$\big(d_1(x)d_2(y) - [x, y]\big)s + \big(d_1(y^*)d_2(x^*) - [x^*, y^*]\big)s \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{31}$$

Combining (30) with (31) we find that

$$2\big(d_1(x)d_2(y) - [x, y]\big)s - y^*d_1(s)d_2(x^*) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{32}$$

$$2(d_1(x)d_2(y) - [x, y])s^2 + 2xd_2(y)d_1(s)s + y^*d_2(x^*)d_1(s)s \in Z(\mathcal{R}).$$

From the above, we also have

$$2(d_1(x)d_2(y) - [x, y])s^2 - y^*d_2(x^*)d_1(s)s \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}.$$

Combining the last two equations and using the fact $char(\mathcal{R}) \neq 2$, we obtain

$$(xd_2(y) + y^*d_2(x^*))d_1(s)s \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}.$$

By Lemma 2.1, we get

$$xd_2(y) + y^*d_2(x^*) \in Z(\mathcal{R}) \ \text{ or } \ d_1(s) = 0 \quad \text{for all } x, y \in \mathcal{R}. \tag{33}$$

Suppose that $xd_2(y) + y^*d_2(x^*) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Replacing y by ys, where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) - \{0\}$ and using Lemma 2.2, we get $xd_2(y) - y^*d_2(x^*) \in Z(\mathcal{R})$. Combining the above expression with our assumption, we conclude that $xd_2(y) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Putting rx in place of x and using primeness of $\mathcal{R}$ twice, we can easily conclude that $d_2 = 0$ or $\mathcal{R}$ is commutative. According to the above, the last two conditions give that $\mathcal{R}$ is commutative.

Assume that $d_1(s) = 0$.

Replacing x by xs, where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$ in (32), we conclude that

$$2\big(d_1(x)d_2(y) - [x, y]\big)s^2 \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{34}$$

Since $s^2 \in Z(\mathcal{R}) \setminus \{0\}$, by Lemma 2.2, we obtain that

$$d_1(x)d_2(y) - [x, y] \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{35}$$

Now, if $Z(\mathcal{R}) = \{0\}$, then $d_1(x)d_2(y) = [x, y]$ for all $x, y \in \mathcal{R}$. Substituting yx for x in the preceding equation, we get

$$\begin{aligned} d_1(yx)d_2(y) &= [yx, y] \\ &= y[x, y] \\ &= yd_1(x)d_2(y) \quad \text{for all } \ x, y \in \mathcal{R}. \end{aligned}$$

We also observe that

$$\begin{aligned} d_1(yx)d_2(y) &= (d_1(y)x + yd_1(x))d_2(y) \\ &= d_1(y)xd_2(y) + yd_1(x)d_2(y) \quad \text{for all } \ x, y \in \mathcal{R}. \end{aligned}$$

Comparing the last two expressions for $d_1(yx)d_2(y)$, we find that $d_1(y)xd_2(y) + yd_1(x)d_2(y) = yd_1(x)d_2(y)$ for all $x, y \in \mathcal{R}$. This gives $d_1(y)\mathcal{R}d_2(y) = \{0\}$ for all $y \in \mathcal{R}$. By primeness of $\mathcal{R}$, we must have either $d_1(y) = 0$ or $d_2(y) = \{0\}$ for all $y \in \mathcal{R}$. In this case by our hypothesis, we obtain $[x, y] = 0$ for all $x, y \in \mathcal{R}$ and Lemma 2.3 forces that $\mathcal{R}$ is commutative.

If $d_1 = 0$ or $d_2 = 0$, then $[x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$ and Lemma 2.3 ensures that $\mathcal{R}$ is commutative.

Now suppose that $Z(\mathcal{R}) \neq \{0\}$ and

$$d_1(x)d_2(y) - [x, y] \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{36}$$

Replacing x by yx in (36), the above relation yields

$$d_1(y)xd_2(y) + y(d_1(x)d_2(y) - [x, y]) \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{37}$$

Applying the relations (36) and (37), it follows that

$$yd_1(y)xd_2(y) = d_1(y)xd_2(y)y \quad \text{for all } \ x, y \in \mathcal{R}. \tag{38}$$

Substituting $xd_1(y)$ for x in the letter equation, we find that

$$yd_1(y)xd_1(y)d_2(y) = d_1(y)xd_1(y)d_2(y)y \quad \text{for all } x, y \in \mathcal{R}.$$

Since $d_1(y)d_2(y) \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$ by (35), the above expression yields that

$$d_1(y)d_2(y)\mathcal{R}\big(yd_1(y)x - d_1(y)xy\big) = \{0\} \quad \text{for all } \ x, y \in \mathcal{R}.$$

Since $\mathcal{R}$ is prime, we arrive at

$$d_1(y)d_2(y) = 0 \text{ or } yd_1(y)x = d_1(y)xy \quad \text{for all } \ x, y \in \mathcal{R}. \tag{39}$$

If there is an element $y_0 \in \mathcal{R}$ such that $y_0 d_1(y_0)x = d_1(y_0)xy_0$ for all $x \in \mathcal{R}$. Replacing x by xt where $t \in \mathcal{R}$ in the letter equation and using it again, we get

$$\begin{aligned} d_1(y_0)xty_0 &= y_0 d_1(y_0)xt \\ &= d_1(y_0)xy_0 t \quad \text{for all} \quad x, t \in \mathcal{R}, \end{aligned}$$

which yields

$$d_1(y_0)\mathcal{R}[y_0, t] = \{0\} \quad \text{for all } t \in \mathcal{R}.$$

By primeness of $\mathcal{R}$, we obtain $d_1(y_0) = 0$ or $y_0 \in Z(\mathcal{R})$ which implies that $d_1(y_0)d_2(y_0) = 0$. In this case, (39) forces that

$$d_1(y)d_2(y) = 0 \quad \text{for all} \quad y \in \mathcal{R}. \tag{40}$$

Hence from Theorem 2.1 of [19], we obtain

$$d_1^2(x) = 0 \text{ or } d_2(x) \in Z(\mathcal{R}) \quad \text{for all} \quad x \in \mathcal{R}. \tag{41}$$

If there is an element x_0 of $\mathcal{R}$ which satisfy the second case, then using (40), we arrive at $d_1(x_0)\mathcal{R}d_2(x_0) = \{0\}$. By primeness of $\mathcal{R}$, we obtain $d_1(x_0) = 0$ or $d_2(x_0) = 0$, in this case (41) becomes

$$d_1^2(x) = 0 \text{ or } d_2(x) = 0 \quad \text{for all} \quad x \in \mathcal{R}.$$

But according to our hypotheses, both the cases force that $[d_1(x), y] \in Z(\mathcal{R})$ or $[x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Replacing y by xy in the first case and y by $d_1(x)y$ in the second case, we can conclude that $x \in Z(\mathcal{R})$ or $d_1(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, and hence from the above we conclude that $d_1(\mathcal{R}) \subseteq Z(\mathcal{R})$ and this forces that $\mathcal{R}$ is commutative.

Using similar arguments one can also prove the result if $\mathcal{R}$ satisfies $d_1(x)d_2(x^*) + [x, x^*] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$.

(ii)⇒(iii). Proof runs as above and hence we skip the details of the proof just to avoid repetition. ■

The following example demonstrates that the restriction of the second kind involution in various Theorems is crucial.

Example 2.10. Let $\mathcal{R} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in \mathbb{Z} \right\}$. It is obvious that R is prime ring. Next, we define the map $d : \mathcal{R} \to \mathcal{R}$ by $d\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & -\beta \\ \gamma & 0 \end{pmatrix}$, and $* : \mathcal{R} \to \mathcal{R}$ such that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$. Obviously, $Z(\mathcal{R}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z} \right\}$. Then $x^* = x$ for all $x \in Z(\mathcal{R})$, and hence $Z(\mathcal{R}) \subseteq H(\mathcal{R})$, which shows that the involution $*$ is of the first kind. Moreover, for all $A \in \mathcal{R}$, d is a derivation on $\mathcal{R}$ which satisfies the following:

(i) $d([A, A^*]) \pm [A, A^*] \in Z(\mathcal{R}$.
(ii) $d(A \circ A^*) \pm A \circ A^* \in Z(\mathcal{R})$.
(iii) $d([A, A^*]) \pm A \circ A^* \in Z(\mathcal{R})$.
(iv) $d(A \circ A^*) \pm [A, A^*] \in Z(\mathcal{R})$.
(v) $d(A)d(A^*) \pm [A, A^*] \in Z(\mathcal{R})$.
(vi) $d(A)d(A^*) \pm A \circ A^* \in Z(\mathcal{R})$.
(vii) $d(A) \circ d(A^*) \pm A \circ A^* \in Z(\mathcal{R})$.

However, $\mathcal{R}$ is not commutative.

3. Generalized Derivations in Rings with Involution

In the year 1989 Bell and Kappe [15], proved that if $\mathcal{R}$ is a semiprime ring and d is a derivation on $\mathcal{R}$ which is either an endomorphism or an anti-endomorphism on $\mathcal{R}$ then $d = 0$. Of course, derivations which are not endomorphisms or anti-endomorphisms on $\mathcal{R}$ may behave as such on certain subsets of $\mathcal{R}$ for example, any derivation d behaves as the zero endomorphism on the subring consisting of all constants (i.e., the elements x for which $d(x) = 0$). In fact, in a semiprime ring $\mathcal{R}$, d may behave as an endomorphism on a proper ideal of $\mathcal{R}$. However, as noted in [15], the behaviour of d is somewhat restricted in the case of a prime ring.

Recently, Albas [1] studied the above mentioned identities in prime rings which are central valued and proved the following theorem (for reference see Theorems 4 & 5 of [1]):

Theorem 3.1. *Let $\mathcal{R}$ is a prime ring and I be a nonzero ideal of $\mathcal{R}$, which admits a non-zero generalized derivation F with associated derivation d of R.*

(i) *If $F(xy) - F(x)F(y) \in Z(\mathcal{R})$ or $F(xy) + F(x)F(y) \in Z(\mathcal{R})$ for all $x, y \in I$, then $\mathcal{R}$ is commutative, or $F = I_{\mathcal{R}}$ or $F = -I_{\mathcal{R}}$, where $I_{\mathcal{R}}$ denotes the identity map of the ring $\mathcal{R}$.*
(ii) *If $F(xy) - F(y)F(x) \in Z(\mathcal{R})$ or $F(xy) + F(y)F(x) \in Z(\mathcal{R})$ for all $x, y \in I$, then $\mathcal{R}$ is commutative.*

In view of the above results it is natural to consider rather weaker situations namely: (i) $F(xx^*) \pm F(x)F(x^*) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, and *(ii)* $F(xx^*) \pm F(x^*)F(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. In this direction we prove the following theorem:

Theorem 3.2. *Let $\mathcal{R}$ be a prime ring with involution $*$ of the second kind such that $char(\mathcal{R}) \neq 2$. Suppose that F is a nonzero generalized derivation of $\mathcal{R}$ associated with a nonzero derivation d on $\mathcal{R}$. If $F(xx^*) \pm F(x)F(x^*) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative or $F = I_{\mathcal{R}}$ or $F = -I_{\mathcal{R}}$, where $I_{\mathcal{R}}$ denotes the identity map of the ring $\mathcal{R}$.*

Proof. By hypothesis, we have

$$F(xx^*) - F(x)F(x^*) \in Z(\mathcal{R}) \quad \text{for all } x \in \mathcal{R}. \tag{42}$$

A linearization of (42) yields that

$$F(xy^*) + F(yx^*) - F(x)F(y^*) - F(y)F(x^*) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}$$

and hence

$$F(xy) + F(y^*x^*) - F(x)F(y) - F(y^*)F(x^*) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{43}$$

Replacing y by yh, where $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$, we obtain that

$$\big(xy + y^*x^* - F(x)y - y^*F(x^*)\big)d(h) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{44}$$

By Lemma 2.2, the above relation ensures that either $d(h) = 0$ for all $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$ or $xy + y^*x^* - F(x)y - y^*F(x^*) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$.

If $d(h) = 0$ for all $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$, then $d(z) = 0$ for all $z \in Z(\mathcal{R})$. Replacing y by ys in (43), where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$, we have

$$F(xy) - F(y^*x^*) - F(x)F(y) + F(y^*)F(x^*) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{45}$$

Combining (45) with (43) we find that $F(xy) - F(x)F(y) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. By the above Theorem 3.1(i), we conclude that $\mathcal{R}$ is commutative or $F = I_{\mathcal{R}}$ or $F = -I_{\mathcal{R}}$.

Now suppose the remaining case that

$$xy + y^*x^* - F(x)y - y^*F(x^*) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{46}$$

Taking ys instead of y, where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$, we obtain

$$xy - y^*x^* - F(x)y + y^*F(x^*) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{47}$$

Using (46) together with (47), we see that

$$xy - F(x)y \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{48}$$

If $Z(\mathcal{R}) = \{0\}$, then $(F(x) - x)y = 0$ for all $x, y \in \mathcal{R}$. Therefore, $(F(x) - x)\mathcal{R}t = \{0\}$ for all $x \in \mathcal{R}$, $t \in \mathcal{R} \setminus \{0\}$ by primeness of $\mathcal{R}$, we obtain $F = I_{\mathcal{R}}$.

If $Z(\mathcal{R}) \neq \{0\}$, then for $z_0 \in Z(\mathcal{R}) \setminus \{0\}$, we have $(x - F(x))z_0 \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. By primeness of $\mathcal{R}$, we conclude that $x - F(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. Taking xy instead of x, we get $xy - F(x)y - xd(y) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$ and by (48), we obtain $xd(y) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. Replacing y by yt in the previous relation and using it again, we have $xd(y)t \in Z(\mathcal{R})$ for all $x, y, t \in \mathcal{R}$. This forces that $xd(y) = 0$ for all $x, y \in \mathcal{R}$ or $\mathcal{R}$ is commutative. Putting xr instead of x in the first case, We can write $x\mathcal{R}d(y) = \{0\}$ for all $x, y \in \mathcal{R}$ and primeness of $\mathcal{R}$, ensures that either $d = 0$ or $\mathcal{R}$ is commutative.

Using similar arguments one can prove the result if $\mathcal{R}$ satisfies the $F(xx^*) + F(x)F(x^*) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. ■

Theorem 3.3. *Let $\mathcal{R}$ be a prime ring with involution $'*'$ of the second kind with $char(\mathcal{R}) \neq 2$. Suppose that F is a nonzero generalized derivation of $\mathcal{R}$ associated with a nonzero derivation d on $\mathcal{R}$. If $F(xx^*) \pm F(x^*)F(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. Assume that

$$F(xx^*) - F(x^*)F(x) \in Z(\mathcal{R}) \quad \text{for all } \ x \in \mathcal{R}. \tag{49}$$

A linearization of (49) yields that

$$F(xy^*) + F(yx^*) - F(x^*)F(y) - F(y^*)F(x) \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}$$

and hence

$$F(xy) + F(y^*x^*) - F(x^*)F(y^*) - F(y)F(x) \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{50}$$

Replacing y by yh, where $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$, we obtain

$$\big(xy + y^*x^* - F(x^*)y^* - yF(x)\big)d(h) \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{51}$$

But by Lemma 2.2, we have either $d(h) = 0$ for all $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$ or $xy + y^*x^* - F(x^*)y^* - yF(x) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. If $d(h) = 0$ for all $h \in Z(\mathcal{R}) \cap H(\mathcal{R})$, then $d(z) = 0$ for all $z \in Z(\mathcal{R})$. Replacing y by ys in (50), where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$, we have

$$F(xy) - F(y^*x^*) + F(x^*)F(y^*) - F(y)F(x) \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{52}$$

Combining (52) with (50) we find that $F(xy) - F(y)F(x) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$ and hence by Theorem 5 of [1], we conclude that $\mathcal{R}$ is commutative.

On the other hand suppose that

$$xy + y^*x^* - F(x^*)y^* - yF(x) \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{53}$$

Taking ys instead of y where $s \in Z(\mathcal{R}) \cap S(\mathcal{R}) \setminus \{0\}$, we obtain

$$xy - y^*x^* + F(x^*)y^* - yF(x) \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{54}$$

Applying (52) together with (54), we see that

$$xy - yF(x) \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{55}$$

Replacing x by xy in (54) and using it again, we arrive at $[yxd(y), y] = 0$ for all $x, y \in \mathcal{R}$ and developing this expression, we obtain $yx[d(y), y] + y[x, y]d(y) = 0$ for all $x, y \in \mathcal{R}$. By replacing x by xt again in the last expression and simplify it, we find that $y[t, y]\mathcal{R}d(y) = \{0\}$ for all $y, t \in \mathcal{R}$. By using the primeness of $\mathcal{R}$

twice, we can easily arrive at $y \in Z(\mathcal{R})$ or $d(y) = 0$ for all $y \in \mathcal{R}$ so $d(\mathcal{R}) \subseteq Z(\mathcal{R})$ which ensures that $\mathcal{R}$ is commutative.

In the similar manner, we can prove the same conclusion for the case when $F(xx^*) + F(x^*)F(x) \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. ■

At the end, it is to remark that the restriction of the second kind involution in Theorems 3.2 & 3.3 is not superfluous.

Example 3.4. Consider the ring $\mathcal{R}$ given in the Example 2.10. Define mappings $F, d, * : \mathcal{R} \to \mathcal{R}$ by

$$F\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & -\beta \\ \gamma & 0 \end{pmatrix}, \quad F = d, \quad \text{and} \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

It is easy to see that F is a generalized derivation on $\mathcal{R}$ associated with d and $*$ is an involution of the first kind on $\mathcal{R}$ which satisfies

(i) $F(AA^*) \pm F(A)F(A^*) \in Z(\mathcal{R})$ and
(ii) $F(AA^*) \pm F(A^*)F(A) \in Z(\mathcal{R})$ for all $A \in \mathcal{R}$.

However, $\mathcal{R}$ is not commutative. Hence, in Theorems 3.2 and 3.3, the hypothesis of the second kind involution is crucial.

Remark 3.5. In retrospect, it is tempted to conjecture that the results of Sec. 2, i.e., Theorems 2.1, 2,7, 2.8, 2.9 may be extended to generalized derivation.

Acknowledgement. The authors are indebted to the referees for their useful suggestions and comments.

References

[1] E. Albas, Generalized derivations on ideals of prime rings, *Miskolc Math. Notes* **14** (1) (2013) 3–9.

[2] S. Ali, N.A. Dar, A.N. Khanon, Strong commutativity-preserving liks maps in rings with involution, *Miskolc Mathematical Notes* **16** (1) (2015) 17–24.

[3] S. Ali and S.L. Huang, On derivations in semiprime rings, *Algebr. Represent. Theor.* **15** (2012) 1023–1033.

[4] M. Ashraf, A. Ali, R. Rani, On generalized derivations of prime rings, *Southeast Asian Bull. Math.* **29** (2005) 669–675.

[5] M. Ashraf, A. Ali, S. Ali, Some commutativity theorems for rings with generalized derivations, *Southeast Asian Bull. Math.* **31** (2007) 215–421.

[6] M. Ashraf, N. Parveen, M.R. Jamal, Traces of permuting n-derivations and commutativity of rings, *Southeast Asian Bull. Math.* **38** (2014) 321–332.

[7] M. Ashraf and N. Rehman, On commutativity of rings with derivations, *Results Math.* **42** (1-2) (2002) 3–8.

[8] M. Ashraf and N. Rehman, On derivation and commutativity of rings, *East-West J. Math.* **3** (1) (2001) 87–91.

[9] M. Ashraf, N. Rehman, S. Ali, On Jordan left derivations of Lie ideals in prime rings, *Southeast Asian Bull. Math.* **25** (2001) 379–382.

[10] H.E. Bell, On derivations in near-rings II, In: *Near-rings, Nearfields and K-loops (Hamburg, 1995)*, Kulwer Academic Publishers, Netherland, 1997.

[11] H.E. Bell, A. Boua, L. Oukhtite, Semigroup ideals and commutativity in 3-prime near rings, *Comm. Alg.* **43** (2015) 1757-1770.

[12] H.E. Bell and M.N. Daif, On commutativity and strong commutativity-preserving maps, *Canad. Math. Bull.* **37** (1994) 443–447.

[13] H.E. Bell and M.N. Daif, On derivations and commutativity in prime rings, *Acta Math. Hungar* **66** (4) (1995) 337–343.

[14] H.E. Bell and G. Mason, On derivations in near-rings, *North-Holland Mathematics Studies* **137** (1987) 31–35.

[15] H.E. Bell and L.C. Kappe, Rings in which derivations satisfy certain algebraic conditions, *Acta Math. Hungar* **53** (3-4) (1989) 339–345.

[16] M. Bresar, On the distance of the composition of two derivations to the generalized derivations, *Glasg. Math. J.* **33** (1991) 89–93.

[17] M.N. Daif and H.E. Bell, Remarks on derivations on semiprime rings, *Internat. J. Math. & Math. Sci.* **15** (1992) 205–206.

[18] N.A. Dar and S. Ali, On $*$-commuting mappings and derivations in rings with involution, *Turk. J. Math.* **40** (2016) 884–894.

[19] M. Fosner and J. Vukman, Identities with generalized derivations in prime rings, *Mediterr. J. Math.* **6** (2010) 139–150.

[20] Ö. Gölbasi and K. Kaya, On Lie ideals with generalized derivations, *Sib. Math. J.* **47** (5) (2006) 862–866.

[21] M. Hongan, A note on semiprime rings with derivation, *Internat. J. Math. & Math. Sci.* **20** (2) (1997) 413–415.

[22] S.L. Huang, Generalized derivations of prime and semiprime rings, *Taiwan. J. Math.* **16** (2) (2012) 771–776.

[23] A.A.M. Kamal and K.H. Al-Shaalan, Generalization of Posner's theorems, *Bull. Malaysian Math. Soc.* **36** (4) (2013) 949–964.

[24] L. Oukhtite, A. Mamouni, M. Ashraf, Commutativity theorems for rings with differential identities on Jordan ideals, *Comment. Math. Univ. Carolin.* **54** (4) (2013) 447–457.