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Operators Induced by Toeplitz and Hankel Operators

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Abstract. In this paper, we introduce a new class of operators called *Toep-Hank* operators on the space $H^2(\mathbb{T})$ with the help of already existing class of operators namely Hankel and Toeplitz operators and discuss its structural properties. A matrix characterization for an operator to be *Toep-Hank* operator is obtained. Commutators of *Toep-Hank* operators and Toeplitz operators are computed.

Keywords: Toeplitz operators; Hankel operators; Hilbert-Schmidt operator; Hyponormal operator.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{C}$ denote the set of integers and the set of complex numbers respectively. Let $\mathbb{D}$ be the open unit disk in the set of complex numbers and its boundary is the unit circle $\mathbb{T}$. The classical Hilbert space $L^2(\mathbb{T})$ is the space of functions $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ with $\|f\| = \left(\sum_{n=-\infty}^{\infty} |a_n|^2\right)^{1/2} < \infty$. The norm $\|\cdot\|$ of $L^2(\mathbb{T})$ is induced by the inner product $\langle f, g\rangle = \sum_{n=-\infty}^{\infty} a_n \overline{b}_n$ for $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and $g(z) = \sum_{n=-\infty}^{\infty} b_n z^n$ in $L^2(\mathbb{T})$. The collection

$\{e_n : n \in \mathbb{Z}\}$, where $e_n(z) = z^n$ for each $z \in \mathbb{T}$, is the standard orthonormal basis for $L^2(\mathbb{T})$. The symbol $H^2(\mathbb{T})$ denotes the space generated by $\{e_n : n \geq 0\}$ and is subspace of $L^2(\mathbb{T})$. The symbol $L^\infty(\mathbb{T})$ is used to denote the space of all essentially bounded complex valued measurable functions on $\mathbb{T}$.

The Hardy space H^2 of analytic functions in the open unit disk $\mathbb{D}$ is defined as $H^2 = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$. It is customary to identify the functions of H^2 with the space of their boundary functions (see [10, 19]). The boundary functions correspond to those functions in L^2 whose negative Fourier coefficients vanish and hence are in $H^2(\mathbb{T})$. With this identification, the space $H^2(\mathbb{T})$ is called the space of all analytic functions having power series representations with square summable complex coefficients. We refer to [10, 18, 19] as well as the references therein, for the details of the spaces $L^2(\mathbb{T})$, $H^2(\mathbb{T})$ and $L^\infty(\mathbb{T})$. If $\phi \in L^\infty(\mathbb{T})$ is given by $\phi(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ then the use of the elements $\overline{\phi}$, ϕ^* and $\widetilde{\phi}$ of $L^\infty(\mathbb{T})$ is very much visible on these spaces, particularly when the study involves multiplication operators over these spaces, where $\overline{\phi}(z) = \sum_{k=-\infty}^{\infty} \bar{a}_k z^{-k}$, $\phi^*(z) = \sum_{k=-\infty}^{\infty} \bar{a}_k z^k$ and $\widetilde{\phi}(z) = \sum_{k=-\infty}^{\infty} a_{-k} z^k$.

Let P denote the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ and M_ϕ denote the multiplication operator on $L^2(\mathbb{T})$ defined as $M_\phi f = \phi f$ for each f in $L^2(\mathbb{T})$. The name Hankel matrix, appeared first in the dissertation of Hankel [14] in 1861, is a square complex matrix (finite or infinite) that is constant on each diagonal orthogonal to the main diagonal. A Toeplitz matrix, on the other hand, is the one whose entries are constant along each diagonal. Hankel and Toeplitz operators on $H^2(\mathbb{T})$ induced by a symbol $\phi \in L^\infty(\mathbb{T})$ are respectively defined as $H_\phi = PJM_\phi\big|_{H^2}$ and $T_\phi = PM_\phi\big|_{H^2}$, where J (flip operator) is an operator from $L^2(\mathbb{T})$ to $L^2(\mathbb{T})$ defined as $J(e_n) = e_{-n}$ for all $n \in \mathbb{Z}$. From the work of Nehari [17] and Power [19], various equivalent forms can be seen for an operator to be Hankel and Toeplitz and it is found that an operator A on $H^2(\mathbb{T})$ is Hankel (Toeplitz) if its matrix representation with respect to standard orthonormal basis $\{e_n : n \geq 0\}$ of $H^2(\mathbb{T})$ is a Hankel (Toeplitz) matrix equivalently it satisfies the equation $U^*A = AU$ ($U^*AU = A$), where U is the unilateral shift operator on $H^2(\mathbb{T})$.

Hankel operators are an important tool in function theory on the unit circle. Together with Toeplitz operators they constitute two most important classes of operators on Hardy spaces. Barría and Halmos in 1982 added a new dimension to the study of Hankel and Toeplitz operators by focussing the attention of mathematicians towards a new direction by proposing the operator equation $U^*XU = \lambda X$ for an arbitrary complex number λ. This direction got a great momentum with the work of Sun [22], Avendano [3] and Datt and Aggarwal [8], when many new operator equations were discussed and the notions like $\lambda-$Toeplitz operators, $\lambda-$Hankel operators and $(\lambda, \mu)-$Hankel operators came into existence. For many decades, a lot of applications of Hankel and Toeplitz operators in many directions have been explored such as interpolation problems [1], rational approximation and stationary processes [18].

Another dimension came in the study of Hankel and Toeplitz operators with the work of Ho [15] in the year 1995 when he introduced the notion of slant

Toeplitz operators. Spectral properties of slant Toeplitz operators are used to discuss the smoothness of wavelets and the spectral radius of slant Toeplitz operators is associated to the Bessov regularity of the solutions of the refinement equation (see [21, 23])). Goodman, Micchelli and Ward [11] showed the connection between the spectral radii and conditions for the solutions of certain differential equations being in Lipschitz classes.

In this paper, we are interested to thought of operators having matrix representation formed by taking the columns of a Hankel matrix and a Toeplitz matrix alternatively or operators whose matrix representation provides a Hankel matrix if only even columns are considered and a Toeplitz matrix if only odd columns are considered. A structural formula for such operators is derived and these operators are named as *Toep-Hank* operators. We discuss some structural properties of this class of operators and also compute the commutator of these operators with Toeplitz operators. Like the Toeplitz and Hankel operators, there is a dearth of compact *Toep-Hank* operators and the only compact *Toep-Hank* operator is the zero operator.

2. Toep-Hank: Operator and Matrix

Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\mathbb{T})$. Consider the following Hankel and Toeplitz matrices formed from the coefficients of ϕ

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_{-2} & a_{-3} & a_{-4} & a_{-5} & \cdots \\ a_{-3} & a_{-4} & a_{-5} & a_{-6} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ a_{-1} & a_0 & a_1 & a_2 & \cdots \\ a_{-2} & a_{-1} & a_0 & a_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

We observed that the corresponding Hankel and Toeplitz operators on $H^2(\mathbb{T})$ for these matrices are defined as $H_\phi (= PJM_\phi\big|_{H^2})$ and $T_{z\widetilde{\phi}} (= PM_{z\widetilde{\phi}}\big|_{H^2})$. We can also express the Toeplitz operator $T_{z\widetilde{\phi}}$ as $T_{z\widetilde{\phi}} = PJM_\phi J_0$ or $T_{z\widetilde{\phi}} = UT_{\widetilde{\phi}} + B_\phi$, where J_0 is a bounded linear operator from $H^2(\mathbb{T})$ to $H^2(\mathbb{T})^\perp$ defined as $J_0(e_n) = e_{-n-1}$ for all $n \geq 0$, U is the unilateral shift operator on $H^2(\mathbb{T})$ given by $U(e_n) = e_{n+1}$ for all $n \geq 0$ and B_ϕ on $H^2(\mathbb{T})$ as $B_\phi e_n = a_{n+1} e_0 \ \forall\, n \geq 0$. Clearly B_ϕ is a rank one operator on $H^2(\mathbb{T})$ satisfying $\|B_\phi\| \leq \|\phi\|_\infty$. Now define the operators V and Λ on $H^2(\mathbb{T})$ as

$$V(e_n) = \begin{cases} e_{\frac{n-1}{2}} & \text{if n is odd,} \\ 0 & \text{if n is even} \end{cases} \qquad \text{and} \qquad \Lambda(e_n) = \begin{cases} e_{\frac{n}{2}} & \text{if n is even,} \\ 0 & \text{if n is odd} \end{cases}$$

for $n \geq 0$. The operators V and Λ satisfy $\|V\| = \|\Lambda\| = 1$. Now we want to know whether there is an operator on $H^2(\mathbb{T})$ whose matrix consists of all even columns from the given Hankel matrix and all odd columns from the given

Toeplitz matrix. At this stage of time, one can easily see that one such operator is given by $H_\phi\Lambda + T_{z\tilde{\phi}}V$ and its matrix with respect to the standard basis of $H^2(\mathbb{T})$ is nothing but

$$\begin{bmatrix} a_0 & a_1 & a_{-1} & a_2 & a_{-2} & a_3 & a_{-3} & a_4 & a_{-4} & \cdot\cdot \\ a_{-1} & a_0 & a_{-2} & a_1 & a_{-3} & a_2 & a_{-4} & a_3 & a_{-5} & \cdot\cdot \\ a_{-2} & a_{-1} & a_{-3} & a_0 & a_{-4} & a_1 & a_{-5} & a_2 & a_{-6} & \cdot\cdot \\ a_{-3} & a_{-2} & a_{-4} & a_{-1} & a_{-5} & a_0 & a_{-6} & a_1 & a_{-7} & \cdot\cdot \\ a_{-4} & a_{-3} & a_{-5} & a_{-2} & a_{-6} & a_{-1} & a_{-7} & a_0 & a_{-8} & \cdot\cdot \\ a_{-5} & a_{-4} & a_{-6} & a_{-3} & a_{-7} & a_{-2} & a_{-8} & a_{-1} & a_{-9} & \cdot\cdot \\ a_{-6} & a_{-5} & a_{-7} & a_{-4} & a_{-8} & a_{-3} & a_{-9} & a_{-2} & a_{-10} & \cdot\cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot \end{bmatrix}.$$

In order to get a general structure for all such operators, we first consider an operator K from $H^2(\mathbb{T})$ to $L^2(\mathbb{T})$ defined as $K(e_{2n}) = e_n$, $K(e_{2n+1}) = e_{-n-1}$ for all $n \geq 0$. Clearly $\|K\| = 1$ and the adjoint K^* of K from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$, is indeed, $K^*(e_n) = e_{2n}$, $K^*(e_{-n-1}) = e_{2n+1}$ for all $n \geq 0$. Further, one can observe that $K^*K = I$ on $H^2(\mathbb{T})$ and $KK^* = I$ on $L^2(\mathbb{T})$. Also, we see that $(H_\phi\Lambda + T_{z\tilde{\phi}}V)e_n = PJM_\phi K e_n$ for each $n \geq 0$. This motivates us to define the following.

Definition 2.1. *Let $\phi \in L^\infty(\mathbb{T})$. A Toep-Hank operator G_ϕ on $H^2(\mathbb{T})$ induced by ϕ is given by $G_\phi = PJM_\phi K$.*

Clearly, $\|G_\phi\| = \|PJM_\phi K\| \leq \|M_\phi\| = \|\phi\|_\infty$. Also, if $\phi = 0$ then from the fact that $G_\phi = H_\phi\Lambda + T_{z\tilde{\phi}}V$, it is trivial to conclude that $G_\phi = 0$.

The class $\{G_\phi | \phi \in L^\infty(\mathbb{T})\}$ is a subspace of $\mathfrak{B}(H^2(\mathbb{T}))$, the space of all bounded operators on $H^2(\mathbb{T})$. Also, the correspondence $\phi \to G_\phi$ is an injective linear mapping from $L^\infty(\mathbb{T})$ into $\mathfrak{B}(H^2(\mathbb{T}))$.

The study of Toeplitz and Hankel operators is connected to the matrix theory with their characterization in terms of matrices (see [9, 17, 19]). Now we proceed ahead to obtain a matrix characterization for *Toep-Hank* operators. If $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ is the Fourier expansion of ϕ and $\{\alpha_{i,j}\}_{i,j\geq 0}$ denotes the matrix of the operator G_ϕ, then it is evident (from the matrix structure of G_ϕ seen earlier) that $\langle \alpha_{i,j} \rangle = \langle a_{-i-n} \rangle$, if $j = 2n$ and $\langle \alpha_{i,j} \rangle = \langle a_{-i+n+1} \rangle$, if $j = 2n+1$, $n \geq 0$. Further, the matrix satisfies the following relations:

$$\begin{cases} \alpha_{k+j,2j-1} = \alpha_{k,0} \quad \text{for } k \geq 0,\ j \geq 1, \\ \alpha_{k-j,2j} = \alpha_{k,0} \quad \text{for } 0 \leq j \leq k, \\ \alpha_{k,2k+2j+1} = \alpha_{0,2j+1} \quad \text{for } k \geq 0,\ j \geq 0. \end{cases} \tag{1}$$

We now define a *Toep-Hank* matrix as follows:

Definition 2.2. *A Toep-Hank matrix is defined as a one way infinite matrix* $\{\alpha_{i,j}\}_{i,j\geq 0}$ *satisfying the relation* (1).

It is well known that an operator on $H^2(\mathbb{T})$ is Hankel (Toeplitz) if and only if its matrix with respect to the orthonormal basis of $H^2(\mathbb{T})$ is Hankel (Toeplitz). Ho in [16] has shown the applications of composition operator C_{z^n} in the study of sampling operators and average operators. For the study of composition operators induced by self mapping on a measurable space, we refer [6] and [20]. We find the appearance of these operators in the next result as follows.

Lemma 2.3. *If matrix of any bounded linear operator A defined on* $H^2(\mathbb{T})$ *is a Toep-Hank matrix, then* AC_{z^2} *is a Hankel operator and* $AM_zC_{z^2}$ *is a Toeplitz operator.*

Proof. Let $\{\alpha_{i,j}\}_{i,j\geq 0}$ denote matrix of A with respect to the basis $\{e_n\}_{n\geq 0}$ satisfying (1). Let $\{\lambda_{i,j}\}_{i,j\geq 0}$ and $\{\beta_{i,j}\}_{i,j\geq 0}$ be the matrices of AC_{z^2} and $AM_zC_{z^2}$ respectively with respect to the usual basis of $H^2(\mathbb{T})$. Then for $i \geq 1, j \geq 0$, $\lambda_{i-1,j+1} = \left\langle AC_{z^2}z^{j+1}, z^{i-1}\right\rangle = \left\langle Az^{2j+2}, z^{i-1}\right\rangle = \alpha_{i-1,2j+2} = \alpha_{i,2j} = \left\langle Az^{2j}, z^i\right\rangle = \left\langle AC_{z^2}z^j, z^i\right\rangle = \lambda_{i,j}$ and $\beta_{i+1,j+1} = \left\langle AM_zC_{z^2}z^{j+1}, z^{i+1}\right\rangle = \beta_{i,j}$ for $i \geq 0, j \geq 0$. This completes the proof. ■

Now we obtain a necessary and sufficient condition for any operator to be a *Toep-Hank* operator as follows.

Theorem 2.4. *A necessary and sufficient condition for an operator on* $H^2(\mathbb{T})$ *to be a Toep-Hank operator is that its matrix with respect to the orthonormal basis* $\{e_n\}_{n\geq 0}$ *is a Toep-Hank matrix.*

Proof. We only need to prove the sufficient part. Assume that A is an operator on $H^2(\mathbb{T})$ whose matrix $\{\alpha_{i,j}\}_{i,j\geq 0}$ is a *Toep-Hank* matrix. Using Lemma 2.3, AC_{z^2} is a Hankel operator and $AM_zC_{z^2}$ is a Toeplitz operator. Let $AM_zC_{z^2} = T_\psi$ and $AC_{z^2} = H_\zeta$ for some $\psi(z) = \sum_{n=-\infty}^{\infty} b_nz^n, \zeta(z) = \sum_{n=-\infty}^{\infty} c_nz^n \in L^\infty(\mathbb{T})$. Let $\{\beta_{i,j}\}_{i,j\geq 0}$ and $\{\lambda_{i,j}\}_{i,j\geq 0}$ be the matrices of $AM_zC_{z^2}$ and AC_{z^2} respectively. Then using the definition of Toeplitz operator, we have $\beta_{i,j} = b_{i-j}$ for $i, j \geq 0$. Thus, $< \psi, e_k >= b_k = \beta_{k,0} = \left\langle AM_zC_{z^2}z^0, z^k\right\rangle = \left\langle Az, z^k\right\rangle = \alpha_{k,1} = \alpha_{k-1,0}$ for $k \geq 1$ and $< \psi, e_{-k} >= b_{-k} = \beta_{0,k} = \left\langle AM_zC_{z^2}z^k, z^0\right\rangle = \left\langle Az^{2k+1}, z^0\right\rangle = \alpha_{0,2k+1}$ for $k \geq 0$.

We now define a complex valued function $\phi(z) = \sum_{n=-\infty}^{\infty} a_nz^n$ such that $b_n = a_{-n+1}$ for $n \in \mathbb{Z}$ so that $\psi = z\widetilde{\phi}$. This gives that $\phi \in L^\infty(\mathbb{T})$. Similarly, by the definition of Hankel operator, $\lambda_{i,j} = c_{-i-j}$ for $i, j \geq 0$ which implies $< \zeta, e_{-k} >= c_{-k} = \lambda_{k,0} = \left\langle AC_{z^2}z^0, z^k\right\rangle = \left\langle Az^0, z^k\right\rangle = \alpha_{k,0} = a_{-k}$ for $k \geq 0$. Thus, we have $\zeta - \phi \in zH^2$ which yields that $H_\zeta = H_\phi$ (Using [4, Theorem 4.1.4]). Hence, AC_{z^2} is a Hankel operator H_ϕ and $AM_zC_{z^2}$ is a Toeplitz operator $T_{z\widetilde{\phi}}$ with ϕ defined as above.

So, we have $AC_{z^2}f_1(z) = H_\phi f_1(z) = PJM_\phi f_1(z) = PJM_\phi K f_1(z^2) = G_\phi f_1(z^2)$ for each $f_1(z) \in H^2(\mathbb{T})$. Also $AM_zC_{z^2}f_2(z) = T_{z\widetilde{\phi}}f_2(z) = PJM_\phi J_0 f_2(z) = PJM_\phi(z^{-1}f_2(z^{-1})) = PJM_\phi K(zf_2(z^2)) = G_\phi(zf_2(z^2))$ for each $f_2(z) \in H^2(\mathbb{T})$. Let $h(z) \in H^2(\mathbb{T})$ be arbitrary. Then we can write $h(z) = h_1(z^2) + zh_2(z^2)$ with $h_1, h_2 \in H^2(\mathbb{T})$. Hence, $A(h(z)) = A(h_1(z^2) + zh_2(z^2)) = A(h_1(z^2)) + A(zh_2(z^2)) = AC_{z^2}h_1(z) + AM_zC_{z^2}h_2(z) = G_\phi(h(z))$. Thus A is a *Toep-Hank* operator with symbol ϕ. ■

The adjoint ${G_\phi}^*$ of a *Toep-Hank* operator G_ϕ is an operator on $H^2(\mathbb{T})$ to $H^2(\mathbb{T})$ satisfying ${G_\phi}^* = K^* M_{\overline{\phi}} J\big|_{H^2}$ and we are interested to study the conditions under which it is an isometry.

We note that for $\phi \neq 0$, any *Toep-Hank* operator is not self adjoint. Also $G_\phi {G_\phi}^* = (PJM_\phi K)(K^* M_{\overline{\phi}} J\big|_{H^2}) = PJM_\phi M_{\overline{\phi}} J\big|_{H^2} = PM_{\widetilde{\phi}\phi^*}\big|_{H^2} = T_{\widetilde{\phi}\phi^*}$. Hence $G_\phi {G_\phi}^*$ is a Toeplitz operator with symbol $\widetilde{\phi}\phi^*$. Thus we observe that for $\widetilde{\phi}\phi^* = 1$, $G_\phi {G_\phi}^* = P = I$ on $H^2(\mathbb{T})$ which implies G_ϕ is a coisometry on $H^2(\mathbb{T})$.

In [5] and [10], it has been observed that the only isometric Toeplitz operators are those of the form T_ϕ, where ϕ is an inner function on the unit disk whereas the only Toeplitz operators that are partial isometries are of the form T_ϕ and T_ϕ^* (ϕ being an inner function). As far as Hankel operators are concerned, any partially isometric Hankel operator must be of the form $H_{\overline{\phi}}$ for an inner function ϕ (see [18]). However, we now provide the conditions for the operators ${G_\phi}^*$ and G_ϕ to be an isometry and partial isometry respectively.

Theorem 2.5. *${G_\phi}^*$ is an isometry on $H^2(\mathbb{T})$ if and only if $\widetilde{\phi}\phi^* = 1$.*

Proof. Let ${G_\phi}^*$ be an isometry. Then $G_\phi {G_\phi}^* = I$. But $G_\phi {G_\phi}^* = T_{\widetilde{\phi}\phi^*}$ which gives $T_{\widetilde{\phi}\phi^*} = I = P = PM_1$ on $H^2(\mathbb{T})$. Thus $PM_{\widetilde{\phi}\phi^* - 1} = 0$ implying $\widetilde{\phi}\phi^* = 1$. Converse follows evidently. ■

Theorem 2.6. *G_ϕ is a partial isometry on $H^2(\mathbb{T})$ if and only if $\widetilde{\phi}\phi^* = 1$.*
Proof. If $\widetilde{\phi}\phi^* = 1$, then G_ϕ is a co-isometry and hence partial isometry. For the converse, suppose G_ϕ is a partial isometry on $H^2(\mathbb{T})$ then $G_\phi {G_\phi}^* G_\phi = G_\phi$ which gives $G_\phi = T_{\widetilde{\phi}\phi^*} G_\phi$. Thus $(I - T_{\widetilde{\phi}\phi^*})G_\phi = 0$ which implies $(I - T_{\widetilde{\phi}\phi^*})G_\phi f = 0$ for each $f \in H^2(\mathbb{T})$.

Now any $f \in H^2(\mathbb{T})$ can be written as $f(z) = f_1(z^2) + zf_2(z^2)$ for some $f_1, f_2 \in H^2(\mathbb{T})$, thus we have, $(PJM_\phi - PM_{\widetilde{\phi}\phi^*}PJM_\phi)K(f_1(z^2) + zf_2(z^2)) = 0$. This implies $(PJM_\phi - PM_{\widetilde{\phi}\phi^*}PJM_\phi)(f_1(z) + z^{-1}f_2(z^{-1})) = 0$ for all $f_1, f_2 \in H^2(\mathbb{T})$. Thus $(PJM_\phi - PM_{\widetilde{\phi}\phi^*}PJM_\phi) = 0$ (Since K is onto),i.e., $(PJM_\phi - PM_{\widetilde{\phi}\phi^*}PJM_\phi)J_0 = 0$ which gives $(I - T_{\widetilde{\phi}\phi^*})T_{z\widetilde{\phi}} = 0$ hence $T_{1-\widetilde{\phi}\phi^*}T_{z\widetilde{\phi}} = 0$. Using ([5, Theorem8 (Corollary1)]) we have the result. ■

3. Properties and Commutators

In this section we study compactness and hyponormality of G_ϕ alongwith the commutators of G_ϕ and ${G_\phi}^*$ with $T_{\psi(z^2)}$. It is known that there are no non zero compact Toeplitz operators and the set $\{H_\phi : \phi \in H^\infty + C\}$ defines the set of all compact Hankel operators (Hartman's Theorem), C being the set of all continuous complex-valued functions in $\mathbb{T}$ (see [18]). Further in case of Hankel operators, H_ϕ is Hilbert-Schmidt if and only if $\widetilde{\phi}_+$ has finite Dirichlet integral, where $\widetilde{\phi}_+(z) = \sum_{n=0}^{\infty} a_{-n} z^n$. We see in the next result that the only condition for a *Toep-Hank* operator G_ϕ to be Hilbert-Schmidt is that $\phi = 0$.

Theorem 3.1. *G_ϕ is a Hilbert-Schmidt operator if and only if $\phi = 0$.*

Proof. Let G_ϕ be a Hilbert-Schmidt operator, where $\phi = \sum_{i=-\infty}^{\infty} a_i e_i \in L^\infty(\mathbb{T})$. Then

$$\begin{aligned}
&\sum_{n=0}^{\infty} \big\langle G_\phi e_n, G_\phi e_n \big\rangle \\
&= \sum_{n=0}^{\infty} \big\langle G_\phi e_{2n}, G_\phi e_{2n} \big\rangle + \sum_{n=0}^{\infty} \big\langle G_\phi e_{2n+1}, G_\phi e_{2n+1} \big\rangle \\
&= \sum_{n=0}^{\infty} \big\langle PJM_\phi e_n, PJM_\phi e_n \big\rangle + \sum_{n=0}^{\infty} \big\langle PJM_\phi e_{-n-1}, PJM_\phi e_{-n-1} \big\rangle \\
&= \sum_{n=0}^{\infty} \big\langle \sum_{i=0}^{\infty} a_{-i-n} e_i, \sum_{j=0}^{\infty} a_{-j-n} e_j \big\rangle + \sum_{n=0}^{\infty} \big\langle \sum_{i=0}^{\infty} a_{-i+n+1} e_i, \sum_{j=0}^{\infty} a_{-j+n+1} e_j \big\rangle \\
&= \sum_{n=0}^{\infty} \big(\sum_{i=0}^{\infty} |a_{-i-n}|^2 \big) + \sum_{n=0}^{\infty} \big(\sum_{i=0}^{\infty} |a_{-i+n+1}|^2 \big).
\end{aligned}$$

This gives that $\sum_{n=0}^{\infty} \|G_\phi e_n\|^2$ is finite only if $|a_i| = 0$ for each i. Hence $\phi = 0$. The converse is trivial. Hence the result. ∎

We see the dearth of compact *Toep-Hank* operators and prove the following.

Theorem 3.2. *The only compact Toep-Hank operator is zero operator.*

Proof. Let G_ϕ be compact with $\phi = \sum_{i=-\infty}^{\infty} a_i e_i \in L^\infty(\mathbb{T})$. Since $e_n \to 0$ weakly, therefore by compactness criterion, $G_\phi e_n \to 0$ strongly, i.e., $\|G_\phi e_{2n+1}\| \to 0$ as $n \to \infty$. Now $\|G_\phi e_{2n+1}\|^2 = \|PJM_\phi K e_{2n+1}\|^2 = \|\sum_{i=0}^{\infty} a_{-i+n+1} e_i\|^2 = \sum_{i=0}^{\infty} |a_{-i+n+1}|^2$. This implies $a_i = 0$ for each i. Hence $\phi = 0$. ∎

Brown and Halmos [5, page 98] proved that Toeplitz operator T_ϕ is normal if and only if $\phi = \alpha + \beta\rho$, where α and β are complex numbers and ρ is a real valued function in $L^\infty(\mathbb{T})$. There are many results concerning hyponormality of

Toeplitz operators, although the characterization was given by C.C. Cowen [7], which states that the Toeplitz operator T_ϕ ($\phi = f + \overline{g}$ for f and g in $H^2(\mathbb{T})$) is hyponormal if and only if $g = c + T_{\overline{h}} f$ for some constant c and some function h in $H^\infty(\mathbb{T})$ with $\|h\|_\infty \leq 1$. In the similar fashion, we now try to investigate the existence of normal and hyponormal *Toep-Hank* operators. We observe that $G_\phi {G_\phi}^* \neq {G_\phi}^* G_\phi$ as $G_\phi {G_\phi}^* = T_{\widetilde{\phi}\phi^*}$ whereas ${G_\phi}^* G_\phi = (K^* M_{\overline{\widetilde{\phi}}} J)(PJM_\phi K) = K^* M_{z\overline{\widetilde{\phi}}}(I-P)M_{\phi z^{-1}} K$ (Using $JPJ = M_z(I-P)M_{z^{-1}}$). Hence G_ϕ is a non normal operator provided ϕ is a non zero function. We prove in the next result that like the normal *Toep-Hank* operators, the only hyponormal *Toep-Hank* operator is the zero operator.

Theorem 3.3. *G_ϕ is hyponormal if and only if $\phi = 0$.*

Proof. Let G_ϕ be hyponormal. Let $\phi = \sum_{i=-\infty}^{\infty} a_i e_i \in L^\infty(\mathbb{T})$. Therefore, we have, $\|G_\phi f\|^2 \geq \|G_\phi^* f\|^2$ for each $f \in H^2(\mathbb{T})$. In particular, taking $f(z) = e_0$, we get $|a_i| = 0$ for $i \geq 1$,i.e., $a_i = 0$ for $i \geq 1$. Further on putting $f(z) = e_2, e_4, e_6, e_8, \cdots$, we get $a_i = 0$ for each i. Hence $\phi = 0$. For $\phi = 0$, G_ϕ is trivially hyponormal. ■

It can be observed that for the operator $G_\phi = H_\phi \Lambda + T_{z\widetilde{\phi}} V$, the adjoint $G_\phi^* = \Lambda^* H_\phi^* + V^* T_{z\widetilde{\phi}}^*$, where Λ^* and V^* on $H^2(\mathbb{T})$ are defined as $\Lambda^*(e_n) = e_{2n}$ and $V^*(e_n) = e_{2n+1}$ for $n \geq 0$. Λ and V also satisfies the relations $\Lambda\Lambda^* = I$, $VV^* = I$, $V\Lambda^* = 0$ and $\Lambda V^* = 0$. Using these properties, one can easily prove that G_ϕ and G_ϕ^* satisfies $G_\phi \Lambda^* = H_\phi$, $G_\phi V^* = T_{z\widetilde{\phi}}$, $\Lambda G_\phi^* = H_\phi^*$ and $VG_\phi^* = T_{z\widetilde{\phi}}^*$. In the next results we will compute the commutators $\left[G_\phi, T_{\psi(z^2)}\right]$ and $\left[G_\phi^*, T_{\psi(z^2)}\right]$ for some specific symbols ϕ and ψ in $L^\infty(\mathbb{T})$ and try to figure out under which conditions these commutators become zero. For basic definition and properties of commutators one can refer ([12],[13]).

Theorem 3.4. *Let $\phi = \sum_{i=-\infty}^{1} a_i z^i \in L^\infty(\mathbb{T})$ and $\psi \in H^\infty(\mathbb{T})$. Then the commutator $\left[G_\phi, T_{\psi(z^2)}\right]$ of operators G_ϕ and $T_{\psi(z^2)}$ is $H_{\phi\psi - \psi(z^2)\phi}\Lambda + T_{z\widetilde{\phi}\psi - z\widetilde{\phi}\psi(z^2)} V$. Further, it is zero if ψ is constant.*

Proof. Using [4, Theorem 3.2.11] and the facts that $\Lambda T_{\psi(z^2)} = T_\psi \Lambda$ and $VT_{\psi(z^2)} = T_\psi V$, we find that $G_\phi T_{\psi(z^2)} = H_\phi \Lambda T_{\psi(z^2)} + T_{z\widetilde{\phi}} V T_{\psi(z^2)} = H_\phi T_\psi \Lambda + T_{z\widetilde{\phi}} T_\psi V = (PJM_\phi)(PM_\psi)\Lambda + T_{z\widetilde{\phi}} T_\psi V = H_{\phi\psi}\Lambda + T_{z\widetilde{\phi}\psi} V$.

On using [4, Corollary 4.5.5], a simple computation shows that $T_{\psi(z^2)} G_\phi = T_{\psi(z^2)} H_\phi \Lambda + T_{\psi(z^2)} T_{z\widetilde{\phi}} V = H_{\psi(z^2)\phi}\Lambda + T_{z\widetilde{\phi}\psi(z^2)} V$. Therefore $\left[G_\phi, T_{\psi(z^2)}\right] = G_\phi T_{\psi(z^2)} - T_{\psi(z^2)} G_\phi = H_{\phi\psi - \psi(z^2)\phi}\Lambda + T_{z\widetilde{\phi}\psi - z\widetilde{\phi}\psi(z^2)} V$. Moreover if ψ is constant then $\psi(z) = \psi(z^2)$, which gives $\left[G_\phi, T_{\psi(z^2)}\right] = 0$. ■

Theorem 3.5. *Let $\phi \in L^\infty(\mathbb{T})$ and $\psi \in H^\infty(\mathbb{T})$. Then $\left[G_\phi^*, T_{\psi(z^2)}\right] = G_{\phi\psi^*(z^2)}^* -$*

$(\Lambda^* H^*_{\overline{\psi}\phi} + V^* T^*_{z\overline{\psi}\widetilde{\phi}})$. *Further, it is zero if* ψ *is constant.*

Proof. We have

$$G^*_\phi T_{\psi(z^2)} = \Lambda^* H^*_\phi T_{\psi(z^2)} + V^* T^*_{z\widetilde{\phi}} T_{\psi(z^2)} = \Lambda^* H^*_{\phi\psi^*(z^2)} + V^* T^*_{z\widetilde{\phi}\overline{\psi}(z^2)} = G^*_{\phi\psi^*(z^2)}.$$

Similarly,

$$T_{\psi(z^2)} G^*_\phi = T_{\psi(z^2)} \Lambda^* H^*_\phi + T_{\psi(z^2)} V^* T^*_{z\widetilde{\phi}} = \Lambda^* H^*_{\overline{\psi}\phi} + V^* T^*_{z\overline{\psi}\widetilde{\phi}}$$

(using the facts that $\Lambda^* T_\psi = T_{\psi(z^2)}\Lambda^*$ and $V^* T_\psi = T_{\psi(z^2)} V^*$). Therefore $\left[G^*_\phi, T_{\psi(z^2)}\right]$ has the desired form. Also it can be easily verified that if ψ is constant, then $\left[G^*_\phi, T_{\psi(z^2)}\right] = 0$. ■

The operator G_ϕ, $\phi \in L^\infty(\mathbb{T})$ considered in the paper gives a Hankel(Toeplitz) matrix when its even (odd) columns are taken. If we talk of the operators of the form $T_\phi\Lambda + H_\phi V (= D_\phi)$, $\phi \in L^\infty(\mathbb{T})$ then it has a matrix representation which gives a Toeplitz (Hankel) matrix when its even (odd) columns are considered. As we have seen that G_ϕ is expressed in the form $PJM_\phi K$, but this is not the case with D_ϕ which restricts us to obtain the following results for the operator $T_\phi\Lambda + H_\phi V (= D_\phi)$, the proof of which follow along the lines of proof for the corresponding results done for G_ϕ.

Proposition 3.6. *Let* $\phi \in L^\infty(\mathbb{T})$, $\psi \in L^\infty(\mathbb{T})$.

(i) *The class* $\{D_\phi | \phi \in L^\infty(\mathbb{T})\}$ *is a subspace of the class of bounded operators on* $H^2(\mathbb{T})$.

(ii) D_ϕ *is a Hilbert-Schmidt operator if and only if* $\phi = 0$.

(iii) *The only compact operator* D_ϕ *is zero operator.*

(iv) *A bounded linear operator* A *on* $H^2(\mathbb{T})$ *belongs to the class* $\{D_\phi | \phi \in L^\infty(\mathbb{T})\}$ *if and only if its matrix* $\{\alpha_{i,j}\}$ *satisfies the following:*

  (a) $\alpha_{k+j,2j} = \alpha_{k,0}$ *for* $k \geq 0$, $j \geq 0$.

  (b) $\alpha_{k-j,2j+1} = \alpha_{k,1}$ *for* $0 \leq j \leq k$.

  (c) $\alpha_{k,2j+2k} = \alpha_{0,2j}$ *for* $k \geq 0$, $j \geq 1$.

  (d) $\alpha_{0,2j+1} = \alpha_{0,2j}$ *for* $j \geq 0$.

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