Southeast Asian Bulletin of Mathematics (2019) 43: 193–206

Southeast Asian
Bulletin of
Mathematics


# Simplicity of Special Algebras over Laurent Polynomial Algebra

Woo Jeon
Department of Natural & Applied Sciences, University of Wisconsin-Green Bay, USA
Email: jeonw@uwgb.edu

Ki-Bong Nam
Department of Mathematics, University of Wisconsin-Whitewater, USA
Email: namk@uww.edu

Received 20 October 2016
Accepted 22 June 2017

Communicated by A.C. Kim

**AMS Mathematics Subject Classification(2000):** 17B65, 17B66, 17B70

**Abstract.** Special algebras of Cartan type have been generalized in several directions. In this paper, we define them based on the Laurent polynomial algebra over a field of characteristic 0. These algebras are infinite dimensional Lie algebras and spanned by $D_{ij}^h(f) = h\{\partial_j(f)\partial_i - \partial_i(f)\partial_j\}$ where $f$ and $h$ are polynomials. First we extend polynomials to Laurent polynomials and to rational functions later. We derive necessary and sufficient conditions for simple Lie algebras over Laurent polynomials, and we investigate the unique minimal ideal for the case of a field of rational function.

**Keywords:** Special Lie algebras; Generalized Cartan type; Lie algebras of Cartan type $S$ or $S^*$; Infinite dimensional simple Lie algebras.

## 1. Introduction

Let $F$ be a field of characteristic 0 and $B_n = F[x_1, \ldots, x_n]$, the polynomial algebra over $F$ with $n$ indeterminates. If we let

$$W_n = Span_F\{f\partial_i : f \in B_n, i = 1, \ldots, n\},$$

where $\partial_i = \frac{\partial}{\partial x_i}$, the partial derivative with respect to $x_j$ acting on $B_n$, then $W_n$ becomes an infinite dimensional vector space over $F$. This $W_n$ becomes a Lie algebra under the product

$$[f\partial_i, g\partial_j] = f\partial_i(g)\partial_j - g\partial_j(f)\partial_i, \text{ for } f, g \in B_n \text{ and } i, j = 1, \ldots, n.$$

This $W_n$ is a subspace of the space of linear operators on $B_n$ and it is the classical Witt algebra. Note that each element of $W_n$ is $F$-bilinear and $W_n$ itself is obviously $\mathbb{Z}^n$-graded as $W_n = \bigoplus_{(m_1,\ldots,m_n)\in\mathbb{Z}^n} W_{(m_1,\ldots,m_n)}$ where $W_{(m_1,\ldots,m_n)}$ is a subspace of $W_n$ and spanned by $\{x_1^{m_1}\cdots x_n^{m_n}\partial_i \mid 1 \le i \le n\}$. Here, $B_n$ acts on $W_n$ as a left module and $W_n$ acts on $B_n$ as algebra derivations, i.e., the action $B_n \times W_n \longrightarrow W_n$ is given by $g(f\partial_i) = (gf)\partial_i$ and the action $W_n \times B_n \longrightarrow B_n$ is given by $(f\partial_i)g = f\partial_i(g)$. We call $B_n$ *the base algebra* for $W_n$.

The algebras of this type of vector fields are called *Lie algebras of Cartan type* $W$ and known to be simple Lie algebras. These are also subalgebras of generalized Witt algebras defined by N. Kawamoto (see [4]).

In 1996, M. Osborn generalized $W_n$ to $W_n^*$ through extensions of $B_n$. These extensions include $B_n^* = F[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$, the Laurent polynomial algebra, and $W_n^*$, the corresponding Lie algebra, is still simple (type $W^*$) (see [6]).

Later in 2000, $W_n^*$ was generalized further using extensions of $B_n$ (or $B_n^*$) to $\widetilde{B_n} = F(x_1, \ldots, x_n)$, the field of rational functions with $n$ indeterminates, and it was proved that the simplicity is still kept (type $\widetilde{W}$) (see [2]).

In this paper, we consider a class of subalgebras (denoted by $L_n^*(h)$) of $W_n^*$ with the base algebra $B_n^*$. This subalgebra is called *a Lie algebra of generalized Cartan type* $S^*$ or *a special algebras of generalized Cartan type*. There have been various generalizations of $L_n^*(h)$ by several authors (see [1, 3, 5, 6]).

In the next section, we review the definition and some basic facts from [6] and [3] about $L_n^*(h)$. For a special case, our main results on simplicity of these algebras are found in Section 3 and 4, and the general case is investigated in Section 5. In Section 6, we review a further generalization and re-investigate the size of the unique minimal ideal (see [3]) in this case.

## 2. Lie Algebras of Cartan Type $S^*$

We mainly follow the notation in [2] and [3]. For a fixed nonzero Laurent polynomial $h \in B_n^*$, we define the following linear transformation from $B_n^*$ to $W_n^*$:

$$D_{ij}^h(f) = h\{\partial_j(f)\partial_i - \partial_i(f)\partial_j\}, \tag{1}$$

for all $i, j = 1, \ldots, n$. Note that $D_{ij}^h(f) = 0$ if and only if $i = j$ or $f$ does not depend on $x_i$ and $x_j$. In particular, $D_{ij}^h(f) = 0$ if and only if $f \in F$ when $n = 2$ and $i \neq j$.

Let $L_n^*(h)$ be the vector space spanned by (1) where $n$ indicates the number of indeterminates. Then $L_n^*(h)$ becomes a Lie subalgebra of algebras of Cartan type $W^*$ (see [2], [3]) from the following calculation:

$$\begin{aligned}[D_{ij}^h(f), D_{kl}^h(g)] = {} & D_{il}^h(h\partial_k(g)\partial_j(f)) - D_{ik}^h(h\partial_l(g)\partial_j(f)) \\ & + D_{jk}^h(h\partial_l(g)\partial_i(f)) - D_{jl}^h(h\partial_k(g)\partial_i(f)).\end{aligned} \tag{2}$$

Since $B_n^*$ acts on $W_n^*$ as a left module and $W_n^*$ acts on $B_n^*$ as algebra derivations, this product can be written as

$$[D_{ij}^h(f), D_{kl}^h(g)] = D_{ij}^h(f)D_{kl}^h(g) - D_{kl}^h(g)D_{ij}^h(f).$$

This $L_n^*(h)$ is called *a Lie algebra of generalized Cartan type* $S^*$, *a special Lie algebra of generalized Cartan type*, or simply a type $S^*$ algebra.

For type $S^*$ algebras, it is known that if $h$ is invertible, the derived algebra $L_n^*(h)^{(1)} = [L_n^*(h), L_n^*(h)]$ is simple (see [6]), and we want to derive a case that $L_n^*(h)$ is simple. We assume that $h$ is a nonzero monomial in $B_n^*$ during this and next sections. We state the following simple lemmas derived from the calculation (2) by setting $k = i$ and $l = j$.

**Lemma 2.1.** $[D_{ij}^h(f), D_{ij}^h(g)] = D_{ij}^h(h\{\partial_i(g)\partial_j(f) - \partial_i(f)\partial_j(g)\})$.

First, we consider $L_2^*(h)$ with 2 indeterminates. Let $h = x_1^r x_2^s$. Then an immediate consequence of Lemma 2.1 is

**Lemma 2.2.** *If* $h = x_1^r x_2^s$, *then*

$$[D_{12}^h(x_1^m x_2^n), D_{12}^h(x_1^p x_2^q)] = (pn - mq)D_{12}^h(x_1^{m+p+r-1} x_2^{n+q+s-1}).$$

*Proof.*

$$\begin{aligned} & [D_{12}^h(x_1^m x_2^n), D_{12}^h(x_1^p x_2^q)] \\ = \ & D_{12}^h(h\{\partial_1(x_1^p x_2^q)\partial_2(x_1^m x_2^n) - \partial_1(x_1^m x_2^n)\partial_2(x_1^p x_2^q)\}) \\ = \ & D_{12}^h(h\{p x_1^{p-1} x_2^q n x_1^m x_2^{n-1} - m x_1^{m-1} x_2^n q x_1^p x_2^{q-1}\}). \end{aligned}$$ ■

This lemma leads the following useful lemma.

**Lemma 2.3.** *If* $h = x_1^r x_2^s$, *then* $[D_{12}^h(x_1^m x_2^n), D_{12}^h(x_1^p x_2^q)] = 0$ *if and only if* $x_1^m x_2^n$ *and* $x_1^p x_2^q$ *are powers of a single monomial, or* $m + p + r - 1 = 0 = n + q + s - 1$.

*Proof.* If $x_1^m x_2^n = (x_1^a x_2^b)^c$ and $x_1^p x_2^q = (x_1^a x_2^b)^d$, $pn - mq = (ad)(bc) - (ac)(bd) = 0$, and the rest of the proof is obvious by Lemma 2.2. ■

We can also state Lemma 2.3 as $x_1^m x_2^n$ *and* $x_1^p x_2^q$ *are powers of a single monomial if and only if* $pn - mp = 0$. Also, a special case of $h = x_1^r x_2^s$ for Lemma 2.3 is $[D_{12}^h(x_1^m x_2^n), D_{12}^h(x_1^p x_2^q)] = 0$ if and only if $pn - mp = 0$ or $m + p = 0 = n + q$.

## 3. Simple Lie Algebras of Cartan Type $S^*$ with 2 Indeterminates

In this section, $n$ is a assumed to be 2 for $L_2^*(h)$ and we prove that $L_2^*(h)$ is simple if $h = x_1 x_2$. We call $D_{12}^h(f)$ *a monomial element* whenever $f$ is a monomial.

Let $P$ be a non-zero ideal of $L_2^*(h)$ and let $z$ be a nonzero element of $P$, i.e.

$$0 \neq z = D_{12}^h(f) \in P \triangleleft L_2^*(h),$$

for some $f \in B_2^*$. Note that we can write $z$ as

$$\sum_{i=1}^{l_1}\sum_{j=1}^{l_2} c_{ij} D_{12}^h(x_1^{m_i} x_2^{n_j}),$$

or

$$\sum_{i=1}^{l} c_i D_{12}^h(x_1^{m_i} x_2^{n_i}), \tag{3}$$

for some positive integer $l_1$, $l_2$, $l$, and $c_{ij}, c_i \in F$, and $m_i, n_j, n_i \in \mathbb{Z}$ for each $i, j = 1, \ldots, l$. We use the latter notation to define the following.

Let *the length of* $z$ be the smallest number $l$ such that $z$ can be expressed as a linear combination of $l$ of monomial elements of $L_2^*(h)$. If the length of $z$ is $l$, there must be a unique expression for $z$ in (3) satisfying

(i) $c_i \neq 0$ for all $i = 1, \ldots, l$.
(ii) $(m_i, n_i) \neq (m_j, n_j)$ when $i \neq j$.
(iii) $(m_i, n_i) \neq (0, 0)$ for all $i = 1, \ldots, l$.

We call it *the simplified form of* $z$. For example, the length of $D_{12}^h(2x_1^3x_2^{-1} - x_1^{-1}x_2^3 + 5)$ is 2 and its simplified form is $2D_{12}^h(x_1^3x_2^{-1}) - D_{12}^h(x_1^{-1}x_2^3)$. Also, note that $z$ is a non-zero monomial element if and only if the length is 1. Next lemma proves the existence of a non-zero monomial element in any ideal.

**Lemma 3.1.** *If $h = x_1x_2$, then any non-zero ideal $P$ of $L_2^*(h)$ contains a non-zero monomial element.*

*Proof.* Suppose $z = \sum_{i=1}^{l} c_i D_{12}^h(x_1^{m_i} x_2^{n_i})$ is a non-zero element in $P$ with the shortest length $l$ among all non-zero elements of $P$. Also suppose that it is the simplified form of $z$. We want to prove that $l = 1$. Assume that $l > 1$.

*Claim:* The element $z$ can be chosen such that not all $x_1^{m_i}x_2^{n_i}$'s are powers of a single monomial.

Assume that all $x_1^{m_i}x_2^{n_i}$'s are powers of a single monomial, i.e.

$$x_1^{m_i} x_2^{n_i} = (x_1^a x_2^b)^{t_i}, \tag{4}$$

for some non-zero integers $t_i$, $i = 1, \ldots, l$. Then (3) becomes $\sum_{i=1}^{l} c_i D_{12}^h(x_1^a x_2^b)^{t_i}$.

Note that all $t_i$'s are distinct and $(a, b) \neq (0, 0)$. Then, by Lemma 2.2 and the linearity of $D_{12}^h$, the following element is in $P$ for any $p, q \in \mathbb{Z}$:

$$[z, D_{12}^h(x_1^p x_2^q)] = \sum_{i=1}^{l} c_i t_i (pb - aq) D_{12}^h(x_1^{at_i+p} x_2^{bt_i+q}) \in P, \tag{5}$$

In particular, when $p = d$ and $q = 0$, (5) becomes

$$[z, D^h_{12}(x_1^d)] = \sum_{i=1}^{l} c_i t_i db D^h_{12}((x_1^a x_2^b)^{t_i} x_1^d) \in P. \tag{6}$$

Since each integer has only finitely many distinct integer factors, there are infinitely many integers $d$ such that not all $(x_1^a x_2^b)^{t_i} x_1^d$'s are powers of a single monomial. Therefore, choosing a proper $d$, (6) gives a new element with the same length $l$, with one exception: $b = 0$. Then, we apply $p = 0$ and $q = d$ to (5), and using a similar argument we can make an element in $P$ with the same length $l$, with one exception: $a = 0$, which leads a contradiction and therefore without loss of generality, we assume that not all $x_1^{m_i} x_2^{n_i}$'s of $z$ are powers of a single monomial. Let's say that $x_1^{m_1} x_2^{n_1}$ and $x_1^{m_2} x_2^{n_2}$ are not powers of a single monomial. Then by Lemma 2.3, $m_2 n_1 - m_1 n_2 \neq 0$ and the following product produces an element with a length less than $l$:

$$\begin{aligned} &[z, D^h_{12}(x_1^{m_1} x_2^{n_1})] \qquad (7)\\ =& \sum_{i=1}^{l} c_i (m_1 n_i - m_i n_1) D^h_{12}(x_1^{m_i+m_1} x_2^{n_i+n_1}) \\ =& \ c_1(m_1 n_1 - m_1 n_1) D^h_{12}(x_1^{m_1+m_1} x_2^{n_1+n_1}) \\ &+ c_2(m_1 n_2 - m_2 n_1) D^h_{12}(x_1^{m_2+m_1} x_2^{n_2+n_1}) + \cdots \end{aligned}$$

with one exception: $m_2 + m_1 = 0 = n_2 + n_1$. But, in this case $m_2 n_1 - m_1 n_2 = -m_1 n_1 - m_1(\text{-}n_1) = 0$, a contradiction. This implies that $l$ is not the shortest if $l > 1$. Therefore $l = 1$. ■

Now we prove the main theorem of this section.

**Theorem 3.2.** *If $h = x_1 x_2$, $L_2^*(h)$ is simple.*

*Proof.* Suppose $P$ is a non-zero ideal of $L_2^*(h)$ when $h = x_1 x_2$. Then by Lemma 3.1, there exists a non-zero monomial element $z = D^h_{12}(x_1^m x_2^n)$ in $P$. Let $z' = D^h_{12}(x_1^p x_2^q)$ be an arbitrary non-zero monomial element in $L_2^*(h)$. Obviously $(p, q) \neq (0, 0)$. By taking the Lie brackets of $z \in P$ and some monomial element(s) in $L_2^*(h)$, we want to make $z'$, and this will show that $L_2^*(h)$ is simple. Consider

$$[z, D^h_{12}(x_1^{p-m} x_2^{q-n})] = \{n(p - m) - (q - n)m\} D^h_{12}(x_1^p x_2^q) \in P. \tag{8}$$

We can see that any monomial element in $L_2^*(h)$ is, with one exception, contained in $P$ from (8). The exception occurs when

$$n(p - m) - (q - n)m = np - qm = 0, \tag{9}$$

i.e. $(p, q)$ and $(m, n)$ are linearly dependent. For this case, we assume that $(p, q) \neq (m, n)$ since $D^h_{12}(x_1^m x_2^n)$ is already contained in $P$. We consider three different cases for this exception of linearly dependent $(p, q)$ and $(m, n)$.

*Case 1.* $z = D^h_{12}(x_1^m x_2^n)$ with $m \neq 0$ and $n \neq 0$.

First consider the following product:

$$\begin{aligned}[[z, D^h_{12}(x_1^{p-m})], D^h_{12}(x_2^{q-n})] &= [n(p-m)D^h_{12}(x_1^p x_2^n), D^h_{12}(x_2^{q-n})] \\ &= n(p-m)p(n-q)D^h_{12}(x_1^p x_2^q) \in P. \quad (10)\end{aligned}$$

Note that $(p, q) \neq (m, n)$ is equivalent to $m \neq p$ and $n \neq q$ in this case since $m = p$ if and only if $n = q$ by (9). So, from (10), we can see that any monomial element $D^h_{12}(x_1^p x_2^q)$ in $L_2^*(h)$ is contained in $P$ if $p \neq 0$. To get $D^h_{12}(x_2^q) \in P$, consider the following product:

$$\begin{aligned}[[z, D^h_{12}(x_2^{q-n})], D^h_{12}(x_1^{-m})] &= [m(n-q)D^h_{12}(x_1^m x_2^q), D^h_{12}(x_1^{-m})] \\ &= m(q-n)mqD^h_{12}(x_2^q) \in P.\end{aligned}$$

Therefore, every monomial element $D^h_{12}(x_1^p x_2^q)$ in $L_2^*(h)$ is contained in $P$ in this case.

*Case 2.* $z = D^h_{12}(x_1^m x_2^n)$ with $m \neq 0$ and $n = 0$.

This case means that $z = D^h_{12}(x_1^m) \in P$ and we consider the following product:

$$[z, D^h_{12}(x_1^{p-m} x_2^q)] = -mqD^h_{12}(x_1^p x_2^q) \in P.$$

This implies that any monomial element $D^h_{12}(x_1^p x_2^q)$ in $L_2^*(h)$ is contained in $P$ if $q \neq 0$ in this case. To get $D^h_{12}(x_1^p) \in P$, consider the following product:

$$\begin{aligned}[[[z, D^h_{12}(x_2)], D^h_{12}(x_1^{p-m})], D^h_{12}(x_2^{-1})] &= [[-mD^h_{12}(x_1^m x_2), D^h_{12}(x_1^{p-m})], D^h_{12}(x_2^{-1})] \\ &= [m(m-p)D^h_{12}(x_1^p x_2), D^h_{12}(x_2^{-1})] \\ &= m(m-p)pD^h_{12}(x_1^p) \in P.\end{aligned}$$

Therefore, every monomial element $D^h_{12}(x_1^p x_2^q)$ in $L_2^*(h)$ is contained in $P$ in this case too.

*Case 3.* $z = D^h_{12}(x_1^m x_2^n)$ with $m = 0$ and $n \neq 0$.

This case is similar to Case 2. Here $z = D^h_{12}(x_2^n) \in P$ and the following products prove the desired result:

$$\begin{aligned}[z, D^h_{12}(x_1^p x_2^{q-n})] &= npD^h_{12}(x_1^p x_2^q) \in P, \\ [[[z, D^h_{12}(x_1)], D^h_{12}(x_2^{q-n})], D^h_{12}(x_1^{-1})] &= [[nD^h_{12}(x_1 x_2^n), D^h_{12}(x_2^{q-n})], D^h_{12}(x_1^{-1})] \\ &= [n(n-q)D^h_{12}(x_1 x_2^q), D^h_{12}(x_1^{-1})] \\ &= n(q-n)qD^h_{12}(x_2^q) \in P.\end{aligned}$$

Since $(m, n) = (0, 0)$ implies that $z$ is zero, these are only cases we need to consider. Therefore, since every monomial element in $L_2^*(h)$ is in every non-zero ideal, $L_2^*(h)$ is a simple Lie algebra. ■

*Remark 3.3.* Let's call $(m, n)$ the root of $D^h_{12}(x_1^m x_2^n)$. Then, we can say that any root can be produced by (8) with one exception (9). This exception occurs if we want to produce any root on the line $y = \frac{n}{m}x$. (8) can be explained by the following diagram:

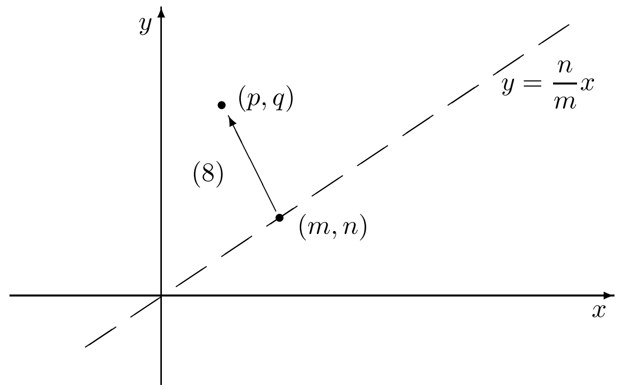



**Fig. 1**

And the 3 cases for the exception can be explained by Fig. 2:

*Remark 3.4.* These 3 exceptions might be combined as one linearly dependent case to show that $D^h_{12}(x_1^p x_2^q) \in P$. We don't present this approach in this paper.

## 4. A Necessary Condition for Simple Lie Algebras of Cartan Type $S^*$

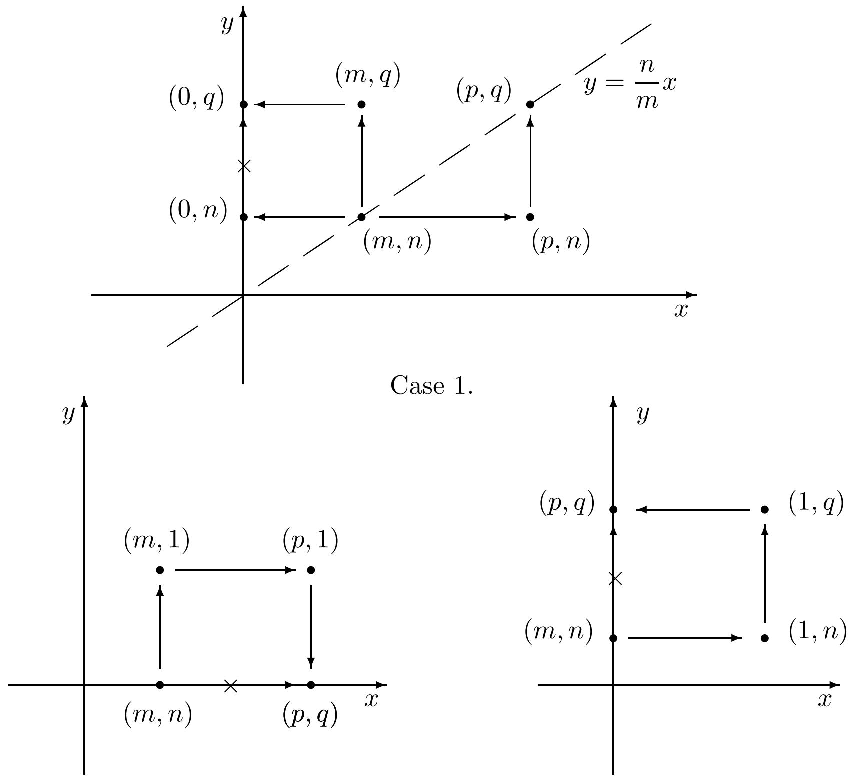
We want to show that Section 3 is the only case for $L_2^*(h)$ to be a simple Lie algebra. The following is the main theorem of this section and it is the general case: $L_n^*(h)$ for any integer $n \geq 2$ ($L_1(h)$ is a zero algebra). We need this result for the next section and the case of $n = 2$ applies to this section.

**Theorem 4.1.** *Let* $h = x_1^{r_1} \cdots x_n^{r_n} \in B_n^*$. *If* $(r_1, \cdots, r_n) \neq (1, \cdots, 1)$, *then* $L_n^*(h)$ *is not a simple Lie algebra.*

*Proof.* Suppose that $(r_1, \cdots, r_n) \neq (1, \cdots, 1)$. Let us assume that $L_n^*(h)$ is simple. Since $p = D^h_{12}(x_1^{r_1-1} \cdots x_n^{r_n-1})$ is a nonzero element of $L_n^*(h)$, we let $< p >$ be the Lie ideal generated by $p$, where

$$< p >= \bigcap \{P \subseteq L_n^*(h) \quad | \quad P \text{ is Lie ideal containing } p\}$$

Then, $p \in L_n^*(h) =< p >$ since $L_n^*(h)$ is simple so $p$ is a Lie product of two elements in $L_n^*(h)$, one from $< p >$ and one from $L_n^*(h)$. Suppose $[z, z'] =$




Case 1.

Case 2. Case 3.

**Fig. 2**

$D^h_{12}(x_1^{r_1-1}\cdots x_n^{r_n-1})$. Note that we can write $z$ and $z'$ as following

$$z = \sum_{s=1}^{w} \sum_{1\le i<j\le n} c_{ij} D^h_{ij}(x_1^{m_{s1}}\cdots x_n^{m_{sn}}),$$

$$z' = \sum_{t=1}^{w'} \sum_{1\le k<l\le n} d_{kl} D^h_{kl}(x_1^{n_{t1}}\cdots x_n^{n_{tn}}),$$

where $c_{ij}, d_{kl} \in F$, the exponents of each monomial elements are integers, and $w, w'$ are some positive integers. Then by the bilinearity of the Lie bracket, we have

$$\begin{aligned} p &= [z, z'] \\ &= \sum_{s=1}^{w}\sum_{t=1}^{w'} \sum_{1\le i<j\le n}\sum_{1\le k<l\le n} c_{ij} d_{kl} [D^h_{ij}(x_1^{m_{s1}}\cdots x_n^{m_{sn}}), D^h_{kl}(x_1^{n_{t1}}\cdots x_n^{n_{tn}})] \end{aligned}$$

$$= D^h_{12}(x_1^{r_1-1}\cdots x_n^{r_n-1}). \tag{11}$$

Let's denote $x^a = x_1^{a_1}\cdots x_n^{a_n}$ and $(x^a)_{ij} = x_a x_i^{-1}x_j^{-1}$. Note that $x^a x^b = x^{a+b}$ makes sense using this notation. Then by (2), the product in (11) becomes

$$\begin{aligned}
&[D^h_{ij}(x_1^{m_{s1}}\cdots x_n^{m_{sn}}), D^h_{kl}(x_1^{n_{t1}}\cdots x_n^{n_{tn}})] = [D^h_{ij}(x^{m_s}), D^h_{kl}(x^{n_t})]\\
=\ & D^h_{il}(h\partial_k(x^{n_t})\partial_j(x^{m_s}) - D^h_{ik}(h\partial_l(x^{n_t})\partial_j(x^{m_s})\\
&+D^h_{jk}(h\partial_l(x^{n_t})\partial_i(x^{m_s})) - D^h_{jl}(h\partial_k(x^{n_t})\partial_i(x^{m_s}))\\
=\ & D^h_{il}(m_{sj}n_{tk}(x^{r+m_s+n_t})_{jk}) - D^h_{ik}(m_{sj}n_{tl}(x^{r+m_s+n_t})_{jl})\\
&+D^h_{jk}(m_{si}n_{tl}(x^{r+m_s+n_t})_{il}) - D^h_{jl}(m_{si}n_{tk}(x^{r+m_s+n_t})_{ik}).
\end{aligned} \tag{12}$$

Note that we can rewrite the summation notation in (12) as following because we only need to produce $D^h_{12}$

$$\begin{aligned}
&\sum_{s=1}^{w}\sum_{t=1}^{w'}\sum_{1\le i<j\le n}\sum_{1\le k<l\le n}\\
&= \sum_{s=1}^{w}\sum_{t=1}^{w'}\Bigg\{\sum_{\substack{(i,j)=(1,2)\\(k,l)=(1,2)}} + \sum_{\substack{(i,j)=(1,2)\\k=1\\3\le l\le n}} + \sum_{\substack{(i,j)=(1,2)\\k=2\\3\le l\le n}} \qquad (13)\\
&\quad + \sum_{\substack{i=1\\3\le j\le n\\(k,l)=(1,2)}} + \sum_{\substack{i=1\\3\le j\le n\\k=2\\3\le l\le n}} + \sum_{\substack{i=2\\3\le j\le n\\k=1\\2\le l\le n}}\Bigg\} \qquad (14)
\end{aligned}$$

Now, we compute each of 6 summations of (13) using (12) for the case of (11).

*Summation 1:* $(12) = D^h_{12}(m_{s2}n_{t1}(x^{r+m_s+n_t})_{21}) + D^h_{21}(m_{s1}n_{t2}(x^{r+m_s+n_t})_{12})$. Then, since $(x^{r+m_s+n_t})_{21} = (x^{r+m_s+n_t})_{12}$, we have

$$(12) = (m_{s2}n_{t1} - m_{s1}n_{t2})D^h_{12}((x^{r+m_s+n_t})_{12}).$$

Note that $(x^{r+m_s+n_t})_{12} = x_1^{r_1-1}\cdots x_n^{r_n-1}$ by (11). This implies that $m_{s1} + n_{t1} = 0 = m_{s2} + n_{t2}$. Therefore, the first summation in (13) is zero.

*Summation 2:* $(12) = D^h_{21}(m_{s1}n_{tl}(x^{r+m_s+n_t})_{1l})$, $3 \le l \le n$. Then, since $(x^{r+m_s+n_t})_{1l} = (x^{r+m_s+n_t})_{12}$ by (11), we derive that $m_{s1} + n_{t1} = 0 = m_{sl} + n_{tl}$ for $3 \le l \le n$. Recall that $D^h_{21} = -D^h_{12}$. Therefore, after changing $D^h_{21}$ to $D^h_{12}$ and putting the negative sign inside, the coefficient of the monomial element in this case becomes

$$-m_{s1}n_{tl} = n_{t1}n_{tl}, \qquad 3 \le l \le n. \tag{15}$$

*Summation 3:* (12) $=$ $-D^h_{12}(m_{s2}n_{tl}(x^{r+m_s+n_t})_{sl})$, $3 \leq l \leq n$. Then, since $(x^{r+m_s+n_t})_{2l} = (x^{r+m_s+n_t})_{12}$ by (11), we derive that $m_{s2} + n_{t2} = 0 = m_{sl} + n_{tl}$ for $3 \leq l \leq n$. Therefore, as above, the coefficient of the monomial element in this case becomes

$$-m_{s2}n_{tl} = n_{t2}n_{tl}, \qquad 3 \leq l \leq n. \tag{16}$$

*Summation 4:* (12) $=$ $D^h_{12}(m_{sj}n_{t1}(x^{r+m_s+n_t})_{j1})$, $3 \leq j \leq n$. Then, by a similar argument to above, the coefficient of the monomial element in this case becomes

$$m_{sj}n_{t1} = -n_{tj}n_{t1}, \qquad 3 \leq j \leq n,$$

which is cancelled with (15) of the summation 2.

*Summation 5:* (12) $=$ $-D^h_{12}(m_{sj}n_{tl}(x^{r+m_s+n_t})_{jl})$, $3 \leq l \leq n$. Then, the coefficient of the monomial element in this case becomes

$$-m_{sj}n_{tl} = n_{tj}n_{tl}, \qquad 3 \leq j \leq n, \quad 3 \leq l \leq n \tag{17}$$

*Summation 6:* (12) $= -D^h_{21}(m_{sj}n_{tl}(x^{r+m_s+n_t})_{jl})$, $3 \leq j \leq n$ and $2 \leq j \leq n$. Then, the coefficient of the monomial element in this case becomes

$$m_{sj}n_{tl} = -n_{tj}n_{tl}, \qquad 3 \leq j \leq n, \quad 2 \leq l \leq n$$

which is cancelled with (16) of the summation 3 and (17) of the summation 5.

Therefore, $p = [z, z'] = 0$ but $p = D^h_{12}(x_1^{r_1-1} \cdots x_n^{r_n-1}) \neq 0$ since $(r_1, \cdots, r_n) \neq (1, \cdots, 1)$, a contradiction. Therefore, $L^*_n(h)$ is not simple. ∎

The following is a combination of Theorem 3.2 and a special case of Theorem 4.1 for $n = 2$.

**Corollary 4.2.** *Let $h = x_1^r x_2^s \in B^*_2$. Then, $L^*_2(h)$ is a simple Lie algebra if and only if $(r, s) = (1, 1)$.*

## 5. A Sufficient Condition

In this section, we consider the general case of Theorem 3.2 with more than 2 indeterminates, i.e. $n \geq 3$. Let $h = x_1^{r_1} \cdots x_n^{r_n} \in B^*_n$ and assume that $n \geq 3$. First, we re-state and prove Lemmas 5 and 6 in [3] with more details for our case. Once again, all the exponents are integers.

**Lemma 5.1.** *Any non-zero ideal $P$ of $L^*_n(h)$ contains a nonzero element $z = \sum_{i=1}^n f_i\partial_i$ such that for some $k$, $f_k \notin F[x_k^{\pm 1}]$. We will call $f_i$ the ith coefficient part of $z$.*

*Proof.* Suppose $z$ is a non-zero element in $P$ having the form

$$0 \neq z = \sum_{i=1}^{n} f_i \partial_i \in P,$$

where $f_i \in F[x_i^{\pm 1}]$, for all $i = 1, 2, \ldots, n$. Since $z$ is not zero, assume that $f_l \neq 0$ for some $l \in \{1, \cdots, n\}$. Let us choose $k \neq l$ and $q \neq k, l$ $(n \geq 2)$. Then,

$$[z, D_{kq}^h(x_l^m x_q)] = \sum_{i=1}^{n} [f_i \partial_i, x_l^m h \partial_k] = \sum_{i=1}^{n} \{f_i \partial_i (x_l^m h) \partial_k - x_l^m h \partial_k (f_i) \partial_i\} \in P,$$

since $P$ is an ideal and the $k$th coefficient part is

$$\begin{aligned}
&\sum_{i=1}^{n} f_i \partial_i (x_l^m h) - x_l^m h \partial_k (f_k) \\
&= \sum_{i \neq l} f_i x_l^m \partial_i (h) + f_l \{m x_l^{m-1} h + x_l^m \partial_l (h)\} - x_l^m h \partial_k (f_k) \\
&= x_l^{m-1} \Big\{ \sum_{i \neq l} f_i x_l \partial_i (h) + f_l \{m h + x_l \partial_l (h)\} - x_l h \partial_k (f_k) \Big\}, \qquad (18)
\end{aligned}$$

which is not zero for some $m$ since $f_l = 0$ if (18) is zero for all $m$. Then since the $k$th coefficient part of this new element in $P$ is not contained in $F[x_k^{\pm 1}]$ for some $m$, therefore this satisfies the condition of the lemma. ■

Now, we establish the following lemma.

**Lemma 5.2.** *Suppose $P$ is a non-zero ideal of $L_n^*(h)$. Then, $L_n^*(h)^{(1)} \cap P \neq \{0\}$ when $n \geq 3$.*

*Proof.* By Lemma , we know that there is a nonzero $z = \sum_{i=1}^{n} f_i \partial_i \in P$ with $f_k \notin F[x_k^{\pm 1}]$ for some $k$. Let $\gamma = x_1^{m_1} \cdots x_n^{m_n}$ and consider

$$\begin{aligned}
&[z, D_{rs}^h(\gamma)] = \sum_{i=1}^{n} [f_i \partial_i, h \partial_s (\gamma) \partial_r - h \partial_r (\gamma) \partial_s] \\
&= \sum_{i=1}^{n} \Big\{ f_i \partial_i (h \partial_s (\gamma)) \partial_r - h \partial_s (\gamma) \partial_r (f_i) \partial_i - f_i \partial_i (h \partial_r (\gamma)) \partial_s + h \partial_r (\gamma) \partial_s (f_i) \partial_i \Big\} \\
&= \sum_{i=1}^{n} \Big\{ h \{\partial_r (\gamma) \partial_s (f_i) - \partial_s (\gamma) \partial_r (f_i)\} \partial_i + f_i \partial_i (h \partial_s (\gamma)) \partial_r - f_i \partial_i (h \partial_r (\gamma)) \partial_s \Big\}. \qquad (19)
\end{aligned}$$

Note that $[z, D_{rs}^h(\gamma)] \in L_n^*(h)^{(1)} \cap P$. Assume that $[z, D_{rs}^h(\gamma)]$ is zero for all $m_i \in \mathbb{Z}, i = 1, \ldots, n$. If we choose $r, s \neq k$ $(n \geq 3)$, then by (19), the $k$th coefficient part of $[z, D_{rs}^h(\gamma)]$ is

$$h\{\partial_r (\gamma) \partial_s (f_k) - \partial_s (\gamma) \partial_r (f_k)\} = 0.$$

This means that if $r \neq k, \partial_r(f_k) = 0$ and it's true for any $r$ and $k$ as long as $r \neq k$, which implies that for any $k$, $f_k$, the $k$th coefficient part of $z$ is in $F[x_k^{\pm 1}]$, a contradiction. Therefore, $[z, D_{rs}^h(\gamma)]$ is a non-zero element in the intersection and this proves the lemma. ■

Recall that Theorem 3.9 in [6] says $L(h)_n^{(1)}$ is simple if $h$ is invertible and the proof is under the assumption of $n \geq 3$. Therefore, we assume the same and note that under our assumption, every $h$ is invertible. Now, we establish the following lemma.

**Lemma 5.3.** *If $h = x_1 \cdots x_n$, any monomial element in $L_n^*(h)$ is a product of two monomial elements.*

*Proof.* Suppose that $z = D_{ij}^h(x_1^{p_1} \cdots x_n^{p_n})$ is a monomial element in $L_n^*(h)$ for $i, j = 1, \cdots, n$ and $p_1, \cdots, p_n \in \mathbb{Z}$. Since the proposition is obvious if $z$ is zero, we assume that $z$ is not zero. This implies that $(p_i, p_j) \neq (0, 0)$ by the definition of $D_{ij}^h$. Then, we consider the following product:

$$\begin{aligned}
&[D_{ij}^h(x_1^{m_1} \cdots x_n^{m_n}), D_{ij}^h(x_1^{p_1 - m_1 - 1} \cdots x_n^{p_n - m_n - 1} x_i x_j)] \\
&= \{(p_i - m_i)m_j - m_i(p_j - m_j)\} D_{ij}^h(x_1^{p_1} \cdots x_n^{p_n}) \\
&= (p_i m_j - m_i p_j) D_{ij}^h(x_1^{p_1} \cdots x_n^{p_n}) \qquad (20)
\end{aligned}$$

Assume that $p_j = 0$, i.e. we want to make $D_{ij}^h(x_1^{p_1} \cdots x_{j-1}^{p_{j-1}} x_{j+1}^{p_{j+1}} \cdots x_n^{p_n})$. Then, since $p_i \neq 0$, (20) is not zero if we use a non-zero $m_j$. If $p_j \neq 0$, there exists $k \in \mathbb{Z}$ such that $p_j k + p_i \neq 0$. Let $p_j k + p_i = l$. Then, if we set $m_i = -k = \frac{p_i - l}{p_j}$ and $m_j = 1$, (20) becomes $l D_{ij}^h(x^{p_1} \cdots x^{p_n})$, which is not zero. Therefore, we can say that any monomial element in $L_n^*(h)$ is a product of two monomial elements. ■

Now, we prove the main theorem.

**Theorem 5.4.** *Let $h$ be a monomial. Then, $L_n^*(h)$ is a simple Lie algebra if and only if $(r_1, \cdots, r_n) = (1, \cdots, 1)$.*

*Proof.* Since the necessary part of the theorem is proved in Theorem 4.1 and the case when $n = 2$ is proved in Corollary 4.2, we assume that $(r_1, \cdots, r_n) = (1, \cdots, 1)$ and prove that $L_n^*(h)$ is simple when $n \geq 3$. Suppose $P$ is a non-zero ideal. Then, $L_n^*(h)^{(1)} \cap P$ is a non-zero ideal of $L_n^*(h)^{(1)}$ by Lemma 5.2. This implies that $L_n^*(h)^{(1)} \cap P = L_n^*(h)^{(1)}$ since $L_n^*(h)^{(1)}$ is simple by Theorem 3.9 in [6]. Therefore, $L_n^*(h)^{(1)} \subseteq P$. Suppose $z$ is any monomial element in $L_n^*(h)$. Then by Lemma 5.3, $z$ is a product of two monomial elements in $L_n^*(h)$, which implies $L_n^*(h) \subseteq L_n^*(h)^{(1)}$. Therefore $P = L_n^*(h)^{(1)} = L_n^*(h)$ and this proves that $L_n^*(h)$ is simple. ■

## 6. Further Generalization

In this section, we consider a further generalization of algebras of Cartan type $S^*$ as in [3]. We use the Laurent polynomial algebra $B_n^*$ over $F$ for previous sections. Now, we want to extend it to $\widetilde{B}_n = F(x_1, \cdots, x_n)$, the field of rational functions using $n$ indeterminates. We call the corresponding Lie algebra $\widetilde{W_n}$ and the special subalgebra $\widetilde{L_n(h)}$, i.e. $\widetilde{L_n(h)}$ is the set of sums of images under $D_{ij}^h$ when $D_{ij}^h$ is a linear transformation from $\widetilde{B}_n$ to $\widetilde{W_n}$. If $z \in \widetilde{L_n(h)}$. Then, we can write

$$z = \sum_{1 \leq i < j \leq n} D_{ij}^h(\frac{f}{g}),$$

where $f, g(\neq 0) \in B_n$. Suppose $P$ is a non-zero ideal of $\widetilde{B}_n$. Then, Lemma 7 in [3] says that $P$ includes $L_n^*(h)$ that is a simple Lie algebra if and only if $h = x_1 \cdots x_n$ by Theorem 5.4 in the previous section of this paper.

Using Theorem 1 in [3] and Lemma 5.3 in the previous section, we can also state the following under the same assumption, $h = x_1 \cdots x_n$.

**Theorem 6.1.** *Let $h = x_1 \cdots x_n$ and $P$ be the ideal of $\widetilde{L_n(h)}$ generated by a non-zero monomial element. Then, $P$ is the unique minimal ideal of $\widetilde{L_n(h)}$. Furthermore, $P$ includes $L_n^*(h)$.*

In Section 5 in [3], the size of this minimal ideal $P$ and the first result under our assumption becomes

**Proposition 6.2.** *Let $P$ be the unique minimal ideal of $\widetilde{L_2(h)}$ and $f = x_1^m x_2^n$. If $g = c_1 x^{u_1} x^{v_1} + c_2 x^{u_2} x^{v_2} \in B_2$ where $m \neq \frac{u_1}{v_1} n$ and $m \neq \frac{n(u-2-u_1)-(u_2 v_1 - u_1 v_2)}{v_2 - v_1}$, then $D_{12}^h(\frac{f}{g}) \in P$.*

Note that we cannot use Proposition 2 in [3] in our case since $r_l = 1 = r_k$. Therefore, if $h = x_1 \cdots x_n$ then $L_n^*(h)$ is simple but the size of the minimal ideal of $\widetilde{L_n(h)}$ can be smaller than the case of $h = x_1^{r_1} \cdots x_n^{r_n}$. However we expect to pursue the conditions for the simplicity of $\widetilde{L_n(h)}$ as well as their derivations and automorphisms in other papers.

**Acknowledgement.** The authors want to express special thanks to the referee for appropriate and helpful comments. They made this paper more complete.

## References

[1] D.Z. Djokovic and K.M. Zhao, Generalized Cartan type $S$ Lie algebras in characteristic zero, *Journal of Algebra* **193** (1997) 144–179.

[2] W. Jeon, A further generalization of algebras of Cartan type $W$, *Journal of Algebra* **223** (2) (2000) 535–555.

[3] W. Jeon, The special algebras of Cartan type in characteristic 0, *Communications in Algebra* **29** (6) (2001) 2319–2332.

[4] N. Kawamoto, Generalizations of Witt algebras over a field of characteristic zero, *Hiroshima Math. J.* **16** (1986) 417–426.

[5] K.-B. Nam, W and H type algebras using maps, *Southeast Asian Bull. Math.* **25** (2001) (1) 135–146.

[6] J.M. Osborn, New simple infinite dimensional Lie algebras of characteristic 0, *Journal of Algebra* **185** (3) (1996) 820–835.