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On Projection Algebras

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Abstract. Projection algebras (spaces) can be interpreted as an algebraic version of ultrametric algebras. Computer scientists use these algebras as a convenient means for algebraic specification of process algebras. A topos-theoretic notion related to the topos **PRO** of projection algebras is the ideal topology which has been investigated by the authors. In this manuscript, first, we show that for any element m of the monoid $\mathbb{N}^{\infty}$ of extended natural numbers with the binary operation $m \cdot n = \min\{m, n\}$ the notion of 'j^m-separated' is the same as 'm-separated' in **PRO**. Then using this fact the associated sheaf functor with respect to j^m is explicitly described. Furthermore, for any $m \in \mathbb{N}^{\infty}$, we provide another form of j^m in terms of the implication map $\downarrow (m+1) \Rightarrow (-)$ on the Heyting algebra of (right) ideals of $\mathbb{N}^{\infty}$. Finally, we establish a strict chain of De Morgan (sheaf) topoi in **PRO**.

Keywords: Projection algebra; (Lawvere-Tierney) topology; Associated sheaf functor; Topos.

1. Introduction

The notion of projection algebra was first introduced by Ehrig et al. in [6]. Computer scientists use these algebras for the specification of infinite objects (process) which can not be denoted by finite terms ([7, 8]). Projection algebras have been studied by Giuli, Ebrahimi and Mahmoudi in extensive form; for example we refer the reader to [2, 3, 4, 5, 9]. A projection algebra is in fact a (right) M-act for the monoid $M = \mathbb{N}^{\infty}(= \mathbb{N} \cup \{\infty\})$ of extended natural numbers with the operation $m \cdot n = \min\{m, n\}$, for any $m, n \in \mathbb{N}^{\infty}$. A projection morphism between projection algebras, which is also called an equivariant map, is a function $f : A \to B$ such that $f(an) = f(a)n$, for every $n \in \mathbb{N}^\infty$ and $a \in A$. We denote **PRO** for the category of all projection algebras and equivariant maps. Notice that the category **PRO** is a kind of presheaf categories and so is a topos. In fact, $PRO \cong Sets^{\mathbb{N}^{\infty\,\mathrm{op}}}$ where $\mathbb{N}^\infty$ is considered as a category whose only object is $\mathbb{N}^\infty$ and morphisms are members of $\mathbb{N}^\infty$. For more detail in general case we refer the reader to [12, Proposition I.7.6].

In this paper we consider one type of (Lawvere-Tierney) topology on projection algebras which is so-called the ideal topology. We describe some topos-theoretic concepts for these kind of algebras.

In this manuscript, we use the following notations and notions:

The set of all ideals of the monoid $\mathbb{N}^\infty$ is

$$\mathrm{Idl}(\mathbb{N}^\infty) = \{\downarrow k |\ k \in \mathbb{N}^\infty\} \cup \{\mathbb{N}^\infty\}$$

in which, for $k \in \mathbb{N}^\infty$, $\downarrow k = \{x \in \mathbb{N}^\infty |\ x < k\}$. Also notice that $\mathbb{N} = \downarrow \infty$ is an ideal of $\mathbb{N}^\infty$. Then the set $\mathrm{Idl}(\mathbb{N}^\infty)$ endowed with the action

$$(\downarrow k) \cdot s = \{t \in \mathbb{N}^\infty |\ st \in \downarrow k\}$$

is a projection algebra which stands for the subobject classifier of **PRO** with the monomorphism

$$\mathrm{true} : \Theta \rightarrowtail \mathrm{Idl}(\mathbb{N}^\infty)$$

given by $\mathrm{true}(\theta) = \mathbb{N}^\infty$ in which $\Theta = \{\theta\}$ is the singleton set endowed with the trivial action.

Recall [13, p. 219] that an equivariant map $j : \mathrm{Idl}(\mathbb{N}^\infty) \to \mathrm{Idl}(\mathbb{N}^\infty)$ is said to be a *Lawvere-Tierney topology* or briefly a *topology* on **PRO** whenever the following hold:

(i) $j(\mathbb{N}^\infty) = \mathbb{N}^\infty$;

(ii) $j \circ j = j$;

(iii) $j(\downarrow k_1 \cap \downarrow k_2) = j(\downarrow k_1) \cap j(\downarrow k_2)$, for any $k_1, k_2 \in \mathbb{N}^\infty$.

Furthermore, on the topos **PRO**, topologies j are in one-to-one correspondence with *universal closure operators* $\overline{(\cdot)}$ (i.e., for all sub projection algebras B, B_1 and B_2 of a projection algebra A it has the properties $B \subseteq \overline{B}$, $\overline{\overline{B}} = B$, $\overline{B_1 \cap B_2} = \overline{B_1} \cap \overline{B_2}$, and for each equivariant map $f : E \to F$, we have $\overline{f^{-1}(U)} = f^{-1}(\overline{U})$ for each sub projection algebra U of F.).

For any $m \in \mathbb{N}^\infty$, the topology on **PRO**, corresponding to the m-closure operator C^m which is a universal closure operator [11], is called the *ideal topology* on **PRO** and is denoted by $j^m : \mathrm{Idl}(\mathbb{N}^\infty) \to \mathrm{Idl}(\mathbb{N}^\infty)$ in which the m-closure operator is defined [5] by

$$C^m_A(B) = \{a \in A |\ \forall t \leq m, at \in B\} \tag{1}$$

for any sub projection algebra $B \subseteq A$ (see also [1]). In fact, the equivariant map j^m is defined by

$$j^m(\downarrow k) = \{s \in \mathbb{N}^\infty |\ \forall t \leq m, st \in \downarrow k\}, \tag{2}$$

for any $k \in \mathbb{N}^\infty$. One can easily check that $j^m(\downarrow k) = \{s \in \mathbb{N}^\infty | \, sm \in \downarrow k\}$ for any $k \in \mathbb{N}^\infty$. Notice that if $m = \infty$, then C^∞ is called *sequentially closure operator* (or *s-closure operator* for short) and is denoted by C^s. (see also [9]). We write j^s for the ideal topology j^∞ associated to the s-closure operator.

Let j be a topology on **PRO** and $\overline{(\cdot)}$ the universal closure operator associated to j. Also, consider a sub projection algebra $B \subseteq A$. Then

(i) B is said to be *j-closed* in A if $\overline{B} = B$.
(ii) B is said to be *j-dense* in A if $\overline{B} = A$.

We may assume that any monomorphism in **PRO** (up to isomorphism) is an inclusion map. In this route, an S-act C is called a *j-sheaf* whenever for any j-dense sub projection algebra $B \subseteq A$, one can uniquely extend any equivariant map $h : B \to C$ to a map $g : A \to C$, i.e. $g|_B = h$. We say that C is *j-separated* if the arrow g exists, it is unique.

It is straightforward to see that in **PRO** for any $m \in \mathbb{N}^\infty$, a sub projection algebra $B \subseteq A$ is j^m-dense iff $At \subseteq B$ for all $t \leq m$, or equivalently $Am \subseteq B$ (see also [5]).

Let us now briefly describe the contents of other sections. We start, in section 2, to show that the two notions m-separated and j^m-separated coincide in **PRO**, for any $m \in \mathbb{N}^\infty$. Afterwards, an explicit expression of the associated sheaf functor with respect to the ideal topology j^m on **PRO** is given. In section 3, we show that, for any $m \in \mathbb{N}^\infty$, j^m coincides with the implication map $\downarrow (m+1) \Rightarrow (-)$ on the Heyting algebra $\mathrm{Idl}(\mathbb{N}^\infty)$. Finally, we establish a strict chain of De Morgan (sheaf) topoi in **PRO**.

2. The Associated Sheaf Functor with Respect to the Ideal Topology

In the present section, we restrict our attention to give an explicit expression of the associated sheaf functor with respect to the ideal topology j^m on **PRO**.

Recall [5, Lemma 3.15] that for any $m \in \mathbb{N}^\infty$, a projection algebra A is said to be *m-separated* whenever for all $a, b \in A$,

$$am = bm \Longrightarrow a = b. \tag{3}$$

The following shows that two notions of m-separated and j^m-separated projection algebras coincide.

Lemma 2.1. *For any $m \in \mathbb{N}^\infty$, a projection algebra A is m-separated iff it is j^m-separated.*

Proof. (Necessity). Let $B \subseteq C$ be a j^m-dense sub projection algebra and $f, g : C \to A$ two equivariant maps such that $f|_B = g|_B$. Since $B \subseteq C$ is j^m-dense we have $Cm \subseteq B$. Then, for any $c \in C$ we get $f(cm) = g(cm)$ and so, $f(c)m = g(c)m$. Since $f(c)$ and $g(c)$ are in A, we get $f(c) = g(c)$ by assumption, and hence, $f = g$.

(Sufficiency). We prove that A satisfies (3). To do so, let $a, b \in A$ such that $am = bm$. Consider the j^m-dense sub projection algebra $m\mathbb{N}^\infty \subseteq \mathbb{N}^\infty$. Now take the equivariant map $h : m\mathbb{N}^\infty \to A$ given by $h(mn) = amn$. Moreover, consider two equivariant maps $f, g : \mathbb{N}^\infty \to A$ defined by $f(\infty) = a$ and $g(\infty) = b$. Then, for any $n \in \mathbb{N}^\infty$ we get

$$f(mn) = f(\infty)mn = amn = bmn = g(mn).$$

Therefore, we found two equivariant maps $f, g : \mathbb{N}^\infty \to A$ for which $f|_{m\mathbb{N}^\infty} = h = g|_{m\mathbb{N}^\infty}$. That A is j^m-separated implies that $f = g$ and then, $a = b$. ■

From now on, we use the notion of m-separated instead of j^m-separated.

Take $m \in \mathbb{N}^\infty$ and a projection algebra A. We write $\mathrm{Equiv}_m(A)$ for the projection algebra of all equivariant maps $f : \downarrow(m+1) \to A$ endowed with the action

$$(f \cdot n)(t) = f(nt) \tag{4}$$

for any $n \in \mathbb{N}^\infty$ and $t < m + 1$. We point out that the $\mathrm{Equiv}_\infty(A)$ is just the projection algebra of all Cauchy sequences over A (see also [14]).

The following provides the main property of the projection algebra $\mathrm{Equiv}_m(A)$ which we are interested in.

Lemma 2.2. *Let $m \in \mathbb{N}^\infty$ and A be a projection algebra. Then, the projection algebra $\mathrm{Equiv}_m(A)$ is m-separated. Also, it is j^m-injective and so, j^m-sheaf if A is m-separated.*

Proof. Take a projection algebra A and $f, g \in \mathrm{Equiv}_m(A)$ such that $f \cdot m = g \cdot m$. Under these conditions, for any $n \in \downarrow(m+1)$ we get $(f \cdot m)(n) = (g \cdot m)(n)$, and in particular, one has $(f \cdot m)(m) = (g \cdot m)(m)$. By the action of $\mathrm{Equiv}_m(A)$, defined as in (4), we have $f(m^2) = g(m^2)$, or equivalently, $f(m) = g(m)$. Then, one has

$$f(n) = f(m \cdot n) = f(m)n = g(m)n = g(m \cdot n) = g(n)$$

and so, $f = g$.

To investigate the second part of the Lemma, suppose further that A to be an m-separated. Let $B \subseteq C$ be a j^m-dense sub projection algebra and $f : B \to \mathrm{Equiv}_m(A)$ an arbitrary equivariant map. Then we define a map $g : C \to \mathrm{Equiv}_m(A)$ by

$$g(c)(t) = f(ct)(t) \text{ for each } t \in \downarrow(m+1). \tag{5}$$

For simplicity, for any $c \in C$ we denote the map $g(c) : \downarrow(m+1) \to A$ by g_c. To prove that g is well defined, first for any $c \in C$ we show that g_c is equivariant.

To achieve this, for any $t \in\downarrow (m+1)$ and $n \in \mathbb{N}^\infty$, one has

$$\begin{aligned} g_c(tn) &= f(ctn)(tn) \\ &= (f(ct)\cdot n)(tn) \quad (ct \in B \text{ and } f \text{ is equivarinat}) \\ &= f(ct)(tn^2) \quad (\text{by } (4)) \\ &= f(ct)((tn)^2) \quad (t = t^2) \\ &= f(ct)(tn) \\ &= [f(ct)(t)]n \quad (f(ct) \text{ is equivarinat}) \\ &= g_c(t)n. \end{aligned}$$

Next we prove that g as in (5) is equivariant as well. To do this, take elements $c \in C$, $n \in \mathbb{N}^\infty$. Then, for every $t \in\downarrow (m+1)$ one has

$$\begin{aligned} g(cn)(t) &= f(cnt)(t) \\ &= f(cn^2t)(t) \\ &= f(cnt)(nt) \\ &= g_c(nt) \\ &= (g_c \cdot n)(t) \\ &= (g(c)\cdot n)(t). \end{aligned}$$

Therefore, $g(cn) = g(c)\cdot n$. Also, $g|_B = f$. Because, for any $b \in B$ and $t \in\downarrow (m+1)$, we have

$$\begin{aligned} g(b)(t) &= f(bt)(t) \\ &= (f(b)t)(t) \quad (f \text{ is equivariant}) \\ &= f(b)(t^2) \quad (\text{by } (4)) \\ &= f(b)(t) \end{aligned}$$

and then, $g|_B = f$. ∎

An examination of [13, Corollary V.3.6] shows that, the inclusion functor $Sep_{j^m}(PRO) \hookrightarrow PRO$ has a left adjoint $L : PRO \longrightarrow Sep_{j^m}(PRO)$ defined by $L(A) = A/\sigma_A$ in which σ_A stands for the following projection algebra congruence on A

$$\sigma_A = \{(a,b) \in A \times A |\ am = bm\}. \tag{6}$$

The next result provides the second part of the structure of the associated sheaf functor regarded to the ideal topology j^m on **PRO**.

Theorem 2.3. *For any $m \in \mathbb{N}^\infty$, the inclusion functor $\iota : Sh_{j^m}(PRO) \hookrightarrow Sep_{j^m}(PRO)$ has a left adjoint*

$$\mathrm{Equiv}_m : Sep_{j^m}(PRO) \longrightarrow Sh_{j^m}(PRO)$$

which assigns to any m-separated projection algebra A the j^m-sheaf $\mathrm{Equiv}_m(A)$ *and to any equivariant map $f : A \to B$ the equivariant map* $\mathrm{Equiv}_m(f) : \mathrm{Equiv}_m(A) \to \mathrm{Equiv}_m(B)$ *given by the composition to f.*

Proof. Take an m-separated projection algebra. One has a monomorphism $\upsilon_A : A \rightarrowtail \mathrm{Equiv}_m(A)$ given by $\upsilon_A(a)(n) = an$, for any $a \in A$ and $n \in \downarrow (m+1)$. The monomorphism υ_A is j^m-dense because for any $f \in \mathrm{Equiv}_m(A)$ the equivariant map $f \cdot m : \downarrow (m+1) \to A$ is equal to the element $\upsilon_A(f(m))$ of $\upsilon_A(A)$.

Next, we constitute an assignment $\beta : \mathrm{id}_{Sep_{j^m}(PRO)} \stackrel{\cdot}{\longrightarrow} \mathrm{Equiv}_m \circ \iota$, by the rule $\beta_A = \upsilon_A : A \rightarrowtail \mathrm{Equiv}_m(A)$, in which $\mathrm{id}_{Sep_{j^m}(PRO)}$ stands for the identity functor on $Sep_{j^m}(PRO)$, and we denote any $\upsilon_A(a) : \downarrow (m+1) \to A$ by υ_a. We prove that β is a natural transformation. To verify the aim, let $f : A \to B$ be an equivariant map. We have to show that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\beta_A} & \mathrm{Equiv}_m(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{Equiv}_m(f)} \\ B & \xrightarrow{\beta_B} & \mathrm{Equiv}_m(B). \end{array}$$

For any $a \in A$, we have $(\mathrm{Equiv}_m(f) \circ \beta_A)(a) = \mathrm{Equiv}_m(f)(\upsilon_a) = f\upsilon_a$ and $(\beta_B \circ f)(a) = \upsilon_{f(a)}$. But, for any $t \in \downarrow (m+1)$ we have

$$f\upsilon_a(t) = f(at) = f(a)t = \upsilon_{f(a)}(t).$$

Therefore, $\mathrm{Equiv}_m(f) \circ \beta_A = \beta_B \circ f$.

The natural transformation β will be the unit of the desired adjoint $\mathrm{Equiv}_m \dashv \iota$. To achieve this, let A be an arbitrary m-separated projection algebra, B a j^m-sheaf, and $f : A \to B$ an equivariant map. Since B is j^m-sheaf, in the following diagram, there exists a unique equivariant map g such that $g\upsilon_A = f$,

$$\begin{array}{ccc} A & \stackrel{\upsilon_A}{\rightarrowtail} & \mathrm{Equiv}_m(A) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\ B. & & \end{array}$$

This completes the proof. ∎

Roughly speaking, the inclusion functor

$$Sh_{j^m}(PRO) \hookrightarrow PRO$$

has a left adjoint, called sheafification functor,

$$a : PRO \longrightarrow Sh_{j^m}(PRO)$$

defined by $a(A) = \mathrm{Equiv}_m(A/\sigma_A)$ where σ_A is the projection algebra congruence defined as in (6).

3. A Chain of De Morgan Topoi in PRO

Recall that [13, p. 56] there is a binary operation called 'implication' on $\mathrm{Idl}(\mathbb{N}^\infty)$, the projection algebra of ideals of $\mathbb{N}^\infty$, defined by

$$(\downarrow k_1) \Rightarrow (\downarrow k_2) = \{s \in \mathbb{N}^\infty |\ \text{for all } t \in \mathbb{N}^\infty;\ \text{if } st \in \downarrow k_1 \text{ then } st \in \downarrow k_2\} \tag{7}$$

for any $k_1, k_2 \in \mathbb{N}^\infty$. It is straightforward to check that, for any $k \in \mathbb{N}^\infty$, the implication map $(\downarrow k) \Rightarrow (-) : \mathrm{Idl}(\mathbb{N}^\infty) \to \mathrm{Idl}(\mathbb{N}^\infty)$ defined by $[(\downarrow k) \Rightarrow (-)](\downarrow n) = (\downarrow k) \Rightarrow (\downarrow n)$ for any $n \in \mathbb{N}^\infty$ is a topology on **PRO**.

The following gives us another structure of the ideal topology j^m on **PRO** using this implication.

Proposition 3.1. *For any $m \in \mathbb{N}^\infty$, the ideal topology j^m coincides with the implication topology $\downarrow (m+1) \Rightarrow (-)$ on* **PRO**.

Proof. Let $\downarrow k$ be an arbitrary principal ideal of $\mathbb{N}^\infty$. Let $n \in [\downarrow (m+1) \Rightarrow (\downarrow k)]$. For any $t \leq m$, $nt \leq m$ and by (7), $nt \in \downarrow k$. Then, by (2), one has $n \in j^m(\downarrow k)$. On the other direction, let $s \in j^m(\downarrow k)$. For any $t \in \mathbb{N}^\infty$ such that $st \leq m$, by (2), we get $s^2 t \in \downarrow k$ and hence, $st \in \downarrow k$. By (7), $s \in [\downarrow (m+1) \Rightarrow (\downarrow k)]$. ∎

In what follows, we establish a De Morgan topos by means of the ideal topology j^m on projection algebras.

Theorem 3.2. *For any $m \in \mathbb{N}^\infty$, the topos $Sh_{j^m}(PRO)$ is a De Morgan topos.*

Proof. To check the claim, we show that for any j^m-sheaf A the Heyting algebra $\mathrm{Sub}_{Sh_{j^m}(PRO)}(A)$, which its structure can be found in [13, Lemma VI.1.2], satisfies De Morgan's law. Following [10], it is sufficient to show that for any j^m-sheaf A and any sub projection algebra B of A the following holds:

$$\neg_{j^m} B \vee_{j^m} \neg_{j^m} \neg_{j^m} B = A \tag{8}$$

It is known that the join of any two closed sub projection algebras is the closure of their union (see also [13]). Using [13, p. 272], we get

$$\neg B = \{a \in A |\ \forall s \in \mathbb{N}^\infty, as \notin B\},$$

in **PRO**.

First of all we prove that for any $a \in A$, the equivalence below holds:

$$(\forall s \in \mathbb{N}^\infty, \exists t \leq m, \exists k \in \mathbb{N}^\infty;\ astk \in B) \Longleftrightarrow (\exists s_0 \in \mathbb{N}^\infty;\ as_0 \in B) \tag{9}$$

By putting $s = \infty$ it is clear that the 'only if' part of (9) is true. For establishing the 'if' part, let $s \in \mathbb{N}^\infty$. Take $t = s_0 m$ and $k = \infty$. Since B is a sub projection algebra of A, the assertion follows. Next notice that, for any $a \in A$, we have $a \in \neg(\overline{\neg B})$ iff a satisfies in the left side of (9).

From (9),[13, Lemma VI.1.2] and (1), we can deduce that

$$\begin{aligned}
\neg_{j^m} B \vee_{j^m} \neg_{j^m}\neg_{j^m} B &= \overline{\overline{\neg B} \cup \neg(\overline{\neg B})} \\
&= \{a \in A |\ \forall t \leq m, at \in \overline{\neg B} \text{ or } at \in \overline{\neg(\overline{\neg B})}\ \} \\
&= \{a \in A |\ \forall t \leq m, \forall k \leq m, atk \in \neg B \\
&\quad \text{or } atk \in \neg(\overline{\neg B})\} \\
&= \{a \in A | \forall t \leq m, \forall k \leq m, (\forall s \in \mathbb{N}^{\infty}, atks \notin B) \\
&\quad \text{ or } (\exists s_0 \in \mathbb{N}^{\infty},\ atks_0 \in B)\} \\
&= A.
\end{aligned}$$

This is the required result. ■

By Lemma 2.1 and [5, Theorem 4.32], we deduce that a projection algebra A is j^s-sheaf iff it is s-separated and injective. More generally, a projection algebra A is j^m-sheaf iff it is m-separated and injective, for any $m \in \mathbb{N}^{\infty}$. Then, by [5, Lemma 3.16], we have that a chain of De Morgan topoi as $Sh_{j^1}(PRO) \subset Sh_{j^2}(PRO) \subset \ldots \subset Sh_{j^s}(PRO)$. Note that the above inclusions are strict. To show this, let $m \in \mathbb{N}^{\infty}$. Similar to the proof of [5, Lemma 3.16], consider the projection algebra $A = \{a, b\}$ given by the action $an = a$, for $n \in \mathbb{N}^{\infty}$, and $bk = a$, for $k < m$, $bk = b$, for $k \geq m$. First we prove that A is j^m-injective. To this end, let $B \subseteq C$ be a j^m-dense sub projection algebra and $f : B \to A$ an arbitrary equivariant map in **PRO**. Then we can define a map $g : C \to A$ by $g(c) = f(c)$ if $c \in B$, and $g(c) = f(c \cdot m)$ if $c \notin B$. It is straightforward to see that g is equivariant and $g|_B = f$. Next, by the proof of [5, Lemma 3.16], the projection algrbra A is a j^m-sheaf . However, since it is not $m-1$-separated, it could not be a j^{m-1}-sheaf. Finally, by Proposition 3.1, we can constitute a strict chain of sheaf topoi

$$Sh_{\{1\}\Rightarrow(-)}(PRO) \subset Sh_{\{1,2\}\Rightarrow(-)}(PRO) \subset \ldots \subset Sh_{\mathbb{N}\Rightarrow(-)}(PRO).$$

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