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A Note on Presentation of General Linear Groups over a Finite Field

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Abstract. In this article we have given Lie regular generators of linear group $GL(2, \mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field with $q = p^n$ elements. Using these generators we have obtained presentations of the linear groups $GL(2, \mathbb{F}_{2^n})$ and $GL(2, \mathbb{F}_{p^n})$ for each positive integer n.

Keywords: Lie regular units; General linear group; Presentation of a group; Finite field.

1. Introduction

Suppose $\mathbb{F}$ is a finite field and $GL(n, \mathbb{F})$ is the general linear the group of $n \times n$ invertible matrices and $SL(n, \mathbb{F})$ is special linear group of $n \times n$ matrices with determinant 1. We know that $GL(n, \mathbb{F})$ can be written as a semidirect product, $GL(n, \mathbb{F}) = SL(n, \mathbb{F}) \rtimes \mathbb{F}^*$, where $\mathbb{F}^*$ denotes the multiplicative group of $\mathbb{F}$. Let H and K be two groups having presentations $H = \langle X \mid R \rangle$ and $K = \langle Y \mid S \rangle$. Then a presentation of semidirect product of H and K is given by,

$$H \rtimes_\eta K = \langle X, Y \mid R, S, xyx^{-1} = \eta(y)(x) \ \ \forall x \in X, y \in Y \rangle,$$

where $\eta : K \to Aut(H)$ is a group homomorphism. Now we summarize some literature survey related to the presentation of groups. In 1977, S.M. Green established a presentation of $SL(n, R)$ for, $n \geq 3$ (see [3]) and in 1994, T.A. Fransis has found a presentation of $GL(n, R)$, where R is a division ring (see [2]). In case R is a field, T.A. Fransis has also provided a presentation of $SL(n, R)$.

Generators for the semigroup has been provided by J. Konieczny (see [5]). A presentation of $SL(n,\mathbb{F})$ has also been given by G. Chiaselotti (see [1]). We have seen that a presentation of $SL(n,\mathbb{F})$ has been found, so we can always find one presentation of $GL(n,\mathbb{F})$ by using semidirect product. In that case, the cardinality of the generating set of $GL(n,\mathbb{F})$ is dependent on $\mathbb{F}$. It motivates us to find a presentation of $GL(n,\mathbb{F})$ with fix number of generators. In this article, we establish a presentation of $GL(2,\mathbb{F}_q)$ with fix number of generators, where generators are Lie regular units. These elements have been first introduced by R.K. Sharma et al. in 2012 (see [8, 4]). A generating set for $GL(4,\mathbb{Z}_n)$ has been found by S. Maheshwari and R.K. Sharma in 2016 (see [7]). This article is arranged in the following way: Section 2 contains some basic definitions and well known results. In Section 3, we have shown that Lie regular units generate $GL(2,\mathbb{F}_q)$ (see Theorems 3.1 and 3.2). In the last Section, we have established a presentation for $GL(2,\mathbb{F}_{2^n})$ and $GL(2,\mathbb{F}_{p^n})$ (see Theorems 3.3 and 3.2) with generating set of cardinality 3.

Throughout this paper, ϕ denotes the Euler's totient function and $\mathcal{U}(R)$ denotes the unit group of the ring R. Suppose G be a group then $o(G)$ denotes the order of the group G.

2. Preliminaries

Here we record some well known results and basic definitions, which we shall use frequently in this note.

Lemma 2.1. *The order of the special linear group $SL(2,\mathbb{F}_q)$ is $\frac{o(GL(2,\mathbb{F}_q))}{q-1}$.*

Definition 2.2. *An element 'a' of a ring R is said to be Lie regular if $a = [e,u] = eu - ue$, where e is an idempotent of R and u is a unit of R. Further, a unit in R is said to be Lie regular unit if it is Lie regular as an element of R.*

In the following lemma $e_{ij}(r)$ for $1 \leq i,j \leq n$ and $r \in \mathbb{F}_q$ denotes elementary matrix of the form $e_{ij} = I + r\epsilon_{ij}$, where ϵ_{ij} denotes the matrix with 1 on the (i,j)-th position and 0 elsewhere and I is the $n \times n$ identity matrix.

Lemma 2.3. [2, p. 944] *Suppose $\mathbb{F}_q$ is a finite field. Then $SL(n,\mathbb{F}_q)$ has a presentation with generators $e_{ij}(r)$ and relations:*

(i) $e_{ij}(r)e_{ij}(s) = e_{ij}(r+s)$,
(ii) $[e_{ij}(r), e_{kl}(s)] = 1 \quad \text{if} \quad i \neq l,\ j \neq k$,
(iii) $[e_{ij}(r), e_{jk}(s)] = e_{ik}(rs) \quad \text{if} \quad i,j$ and k are distinct, *and*
(iv) $ee_{ji}(r)e^{-1} = e_{ij}(-trt) \quad for\ e = e_{ij}(t)e_{ji}(-t^{-1})e_{ij}(t), t \in \mathbb{F}_q^*$.

Let α be a primitive element of $\mathbb{F}_q$, where $q = p^n$ and $1 \leq i,j \leq q-1$. For convenience, we rewrite the above presentation in new symbols as follows:

Corollary 2.4. *Let*

$$a_i = \begin{pmatrix} 1 & \alpha^i \\ 0 & 1 \end{pmatrix}, b_j = \begin{pmatrix} 1 & 0 \\ \alpha^j & 1 \end{pmatrix}.$$

Then $SL(2, \mathbb{F}_{p^n})$ is generated by a_i and b_j with these relations:

(i) $a_i^p = 1$ *and* $b_j^p = 1$,

(ii) *for $i \neq j$, we have $a_i a_j = a_k$, $b_i b_j = b_k$, where k is such that $\alpha^i + \alpha^j = \alpha^k$, and $1 \leq k \leq q-1$,*

(iii) $(a_i b_{q-1-i}^{-1} a_i)(b_j)(a_i b_{q-1-i}^{-1} a_i)^{-1} = a_{2i+j}^{-1}$,

(iv) $(b_i a_{q-1-i}^{-1} b_i)(a_j)(b_i a_{q-1-i}^{-1} b_i)^{-1} = b_{2i+j}^{-1}$.

Theorem 2.5. [6, Theorem 1.1] *Two groups having same presentation are isomorphic.*

3. Lie Regular Generators of General Linear Groups

Observe that the element

$$a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = \left[\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right]$$

is a Lie regular unit in $M_2(\mathbb{F}_q)$ and $b = \begin{pmatrix} 0 & k \\ 1 & 0 \end{pmatrix}$, where k is invertible in $\mathbb{F}_q$, is also a Lie regular unit in $M_2(\mathbb{F}_q)$ (see [8, Proposition 2.14]).

Theorem 3.1. *Suppose $\mathbb{F}_{2^n}$ is a finite field and $\alpha \in \mathbb{F}_{2^n}^*$ is a primitive element i.e. $o(\alpha) = 2^n - 1$. Then the linear group $GL(2, \mathbb{F}_{2^n})$ is generated by Lie regular units a, b* and *c, where*

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}.$$

Proof. Let G be a subgroup of $GL(2, \mathbb{F}_{2^n})$ generated by a, b and c. Set

$$a_i = (bc)^{-i}(bab)(bc)^i = \begin{pmatrix} 1 & \alpha^i \\ 0 & 1 \end{pmatrix}, b_j = (bc)^j a (bc)^{-j} = \begin{pmatrix} 1 & 0 \\ \alpha^j & 1 \end{pmatrix},$$

where $1 \leq i, j \leq 2^n - 1$. By using Corollary 2.4, a_i and b_j generate $SL(2, \mathbb{F}_q)$. Let $x = bc = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, then order of x is $2^n - 1$. Consider a subgroup of G,

$$H = \langle x \mid x^{2^n - 1} \rangle.$$

We see that $H \cap SL(2, \mathbb{F}_{2^n}) = \{I_2\}$, thus

$$o(HSL(2,\mathbb{F}_{2^n})) = (2^n-1)o(SL(2,\mathbb{F}_{2^n})).$$

Since $HSL(2,\mathbb{F}_{2^n}) \subseteq G \leq GL(2,\mathbb{F}_{2^n})$ and by lemma 2.1, $o(GL(2,\mathbb{F}_{2^n})) = (2^n - 1)o(SL(2,\mathbb{F}_{2^n}))$, as a consequence, we have $HSL(2,\mathbb{F}_{2^n}) = GL(2,\mathbb{F}_{2^n})$. ∎

Theorem 3.2. *Suppose $\mathbb{F}_q$ is a finite field with $q = p^n$, where p is an odd prime and $\alpha \in \mathbb{F}_q^*$ is a primitive element i.e. $o(\alpha) = q-1$. Then the linear group $GL(2,\mathbb{F}_q)$ is generated by Lie regular units a, b and c, where*

$$a = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}.$$

Proof. Let G be a subgroup of $GL(2,\mathbb{F}_q)$ generated by a, b and c. Set

$$a_i = (bc)^{-i}(b(bc)^{\frac{q-1}{2}}ab)(bc)^i = \begin{pmatrix} 1 & \alpha^i \\ 0 & 1 \end{pmatrix},$$
$$b_j = (bc)^j((bc)^{\frac{q-1}{2}}a)(bc)^{-j} = \begin{pmatrix} 1 & 0 \\ \alpha^i & 1 \end{pmatrix},$$

where $1 \leq i, j, \leq q-1$. By using Corollary 2.4, a_i and b_j generate $SL(2,\mathbb{F}_q)$. Remaining proof is same as the proof of Theorem 3.1. ∎

4. Presentation of $GL(2,\mathbb{F}_{2^n})$

In the following theorem we assume that $1 \leq i, j \leq 2^n - 1$ and $n > 1$.

Theorem 4.1. *Let*

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix},$$

where α is a primitive element of $\mathbb{F}_{2^n}$. Then a presentation of $GL(2,\mathbb{F}_{2^n})$ is

$$\langle a, b, c \mid \ a^2, b^2, c^{2(2^n-1)}, c^2 \in center, (ab)^3, (bc)^{2^n-1},$$
$$aa_ia = (bab)b_i(bab)\ , a_1a_2 = a_{k_0}\rangle,$$

where $k_0 \in \mathbb{N}$ is such that $\alpha + \alpha^2 = \alpha^{k_0}$ and

$$a_i = (bc)^{-i}(bab)(bc)^i = \begin{pmatrix} 1 & \alpha^i \\ 0 & 1 \end{pmatrix}, b_j = (bc)^j a(bc)^{-j} = \begin{pmatrix} 1 & 0 \\ \alpha^j & 1 \end{pmatrix}.$$

Proof. $\alpha + \alpha^2$ is a non-zero element in $\mathbb{F}_{2^n}$ for $n > 1$. So there exists k_0 such that $\alpha + \alpha^2 = \alpha^{k_0}$. Let G be a group generated by a, b, c. Then G has the presentation,

$$\langle a, b, c \mid \ a^2, b^2, c^{2(2^n-1)}, c^2 \in center, (ab)^3, (bc)^{2^n-1},$$
$$aa_ia = (bab)b_i(bab), a_1a_2 = a_{k_0}\rangle.$$

First, we shall show that G is finite. Consider a group H having the following presentation

$$H = \langle a_i,\ b_j \mid \ a_i^2, b_j^2, a_i a_j = a_k, b_i b_j = b_k,\ a_i b_{2^n-1-i} a_i b_j (a_i b_{2^n-1-i} a_i)^{-1} = a_{2i+j},$$
$$(b_i a_{q-1-i}^{-1} b_i)(a_j)(b_i a_{q-1-i}^{-1} b_i)^{-1} = b_{2i+j}^{-1}\rangle,$$

where $k \in \mathbb{N}$ is such that $\alpha^i + \alpha^j = \alpha^k$. We have some observations,

(i) If $a_1 a_2 = a_{k_0}$, then $a_k a_{k+1} = a_{k_0+(k-1)}$ for $1 \leq k, k_0 \leq 2^n - 1$ and k_0 is such that $\alpha + \alpha^2 = \alpha^{k_0}$. We shall show this by induction, for $k = 1$ it holds. Assume for $k = m$, $a_m a_{m+1} = a_{k_0+(m-1)}$, from here we get

$$bab(bc)^{-1} = (bc)^{1-k_0} bab(bc)^{k_0-2} bab. \tag{4.1}$$

Let $k = m + 1$. Then

$$\begin{aligned} a_{m+1} a_{m+2} =& (bc)^{-1-m} bab(bc)^{-1} bab(bc)^{m+2} \\ =& (bc)^{-m-k_0} bab(bc)^{m+k_0} \quad \text{using (4.1)} \\ =& a_{k_0+m}. \end{aligned}$$

(ii) $ba_i b = b_i$.

(iii) $(ab_i)^2 = 1$.
The statement holds by using the relation $aa_i a = (bab)b_i(bab)$ and the observation (1).

(iv) $ca_i c^{-1} = b_{i-1}$.

As relations in the group H can be obtained by using above observations and relations in the group G. Which implies that the group H is a subgroup of G, in fact it is a normal subgroup of the group G. Corollary 2.4 and Theorem 2.5, give that the group H is isomorphic to $SL(2, \mathbb{F}_{2^n})$. Now consider the quotient group G/H, we see that

$$G/H = \langle c \mid c^{2^n-1}\rangle.$$

and $o(G) = o(GL(2, \mathbb{F}_{2^n}))$. This proves that $G \cong GL(2, \mathbb{F}_{2^n})$. ∎

In $\mathbb{F}_2$, we see that $\alpha = 1$, so the element b is same as the element c. Hence we have a remark for presentation of $GL(2, \mathbb{F}_2)$.

Remark 4.2. A presentation of $GL(2, \mathbb{F}_2)$ is

$$\langle a, b \mid a^2, b^2, (ab)^3\rangle.$$

In the following corollary we are giving a presentation of $GL(2, \mathbb{F}_{2^n})$, where $2 \leq n \leq 7, n \neq 5$. In this case, we have a primitive polynomial, $1 + x + x^n$ (see [9]) and hence $\alpha + \alpha^2 = \alpha^{n+1}$.

Corollary 4.3. *A presentation of* $GL(2,\mathbb{F}_{2^n})$ *is*

$$\langle a,b,c \mid \ a^2,b^2,c^{2(2^n-1)},c^2 \in center,(ab)^3,(bc)^{2^n-1},aa_ia=(bab)b_i(bab),\ a_1a_2=a_{n+1}\rangle.$$

5. Presentation of $GL(2,\mathbb{F}_{p^n})$

Theorem 5.1. *Let* p *be an odd prime,* $q=p^n$ *for* $n\geq 1$ *and* α *be a primitive element of* $\mathbb{F}_q$. *Let*

$$a=\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}, b=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, c=\begin{pmatrix}0 & \alpha\\ 1 & 0\end{pmatrix}$$

and $a_i=(bc)^{-i}(b(bc)^{\frac{q-1}{2}}ab)(bc)^i$, $b_i=(bc)^j((bc)^{\frac{q-1}{2}}a)(bc)^{-j}$ for $1\leq i,j\leq q-1$. *Then a presentation of* $GL(2,\mathbb{F}_q)$ *is*

$$\begin{aligned}\langle a,b,c \mid \ & a^2,b^2,c^{2(q-1)},c^2\in center,(ab)^3,((bc)^{\frac{q-1}{2}}a)^p,(bc)^{q-1},\\ & a_1a_2=a_2a_1=a_{k_0},\quad a_1^{i'}=a_{k_{i'}}(2\leq i'\leq p-1),\quad b_{q-1}b_i=b_ib_{q-1},\\ & a_{q-1}^{-1}b_{q-1}a_{q-1}^{-1}=b(bc)^{\frac{q-1}{2}},\quad aa_ia=a_{q-1}^{-1}b_ia_{q-1},\\ & (a_{q-1}b_{q-1}^{-1}a_{q-1})b_m(a_{q-1}b_{q-1}^{-1}a_{q-1})^{-1}=a_m^{-1}\rangle,\end{aligned}$$

where k_0 *and* $k_{i'}$ *are such that* $\alpha+\alpha^2=\alpha^{k_0}$, $i'\alpha=\alpha^{k_{i'}}$ *and* $1\leq k_0,k_{i'},m\leq q-1$.

Proof. Let $\alpha+\alpha^2$ be not an invertible element in $\mathbb{F}_q$. Then $\alpha+\alpha^2=0$, it provides that $\alpha=-1$. This is the possible case when $q=3$. We will discuss this case separately. Hence $\alpha+\alpha^2$ is invertible in $\mathbb{F}_q$, $q\neq 3$, so there exist k_0 such that $\alpha+\alpha^2=\alpha^{k_0}$. Since i' and α are non-zero elements of $\mathbb{F}_q$, there exist $k_{i'}$ such that $\imath'\alpha=\alpha^{k_{i'}}$ as $\imath'\alpha$ is a non-zero element. Suppose G is a subgroup of $GL(2,\mathbb{F}_q)$ generated by a,b,c and having presentation

$$\begin{aligned}\langle a,b,c \mid \ & a^2,\ \ b^2,\ \ c^{2(q-1)},\ \ c^2\in center,(ab)^3,((bc)^{\frac{q-1}{2}}a)^p,(bc)^{q-1},\\ & a_1a_2=a_2a_1=a_{k_0},\quad a_1^{i'}=a_{k_{i'}},\quad b_{q-1}b_i=b_ib_{q-1},\\ & a_{q-1}^{-1}b_{q-1}a_{q-1}^{-1}=b(bc)^{\frac{q-1}{2}},\quad aa_ia=a_{q-1}^{-1}b_ia_{q-1},\\ & (a_{q-1}b_{q-1}^{-1}a_{q-1})b_m(a_{q-1}b_{q-1}^{-1}a_{q-1})^{-1}=a_m^{-1}\rangle.\end{aligned}$$

First we shall show that group G is finite. Consider a subgroup H of G which is given by

$$\begin{aligned}H=\langle a_i,\ b_j \mid\ & a_i^p,b_j^p,\quad a_ia_j=a_k,\quad b_ib_j=b_k,\quad (a_ib_{q-1-i}^{-1}a_i)b_j(a_ib_{q-1-i}^{-1}a_i)^{-1}=\\ & a_{2i+j}^{-1},\quad (b_ia_{q-1-i}^{-1}b_i)a_j(b_ia_{q-1-i}^{-1}b_i)^{-1}=b_{2i+j}^{-1}\rangle,\end{aligned}$$

where $k\in\mathbb{N}$ is such that $\alpha^i+\alpha^j=\alpha^k, 1\leq k\leq q-1$. We have some observations,

(i) $ba_ib=b_i$.

(ii) $(ab_i)^2 = 1$.

The statement holds by using relation $b_{q-1}b_i = b_ib_{q-1}$.

(iii) $ca_ic^{-1} = b_{i-1}$.

By using the above observations and relations in the group G, we see that the relations in the group H can be obtained. Hence H is a subgroup of the group G, in fact normal subgroup of G. Corollary 2.4 and Theorem 2.5, give that the group H is isomorphic to $SL(2, \mathbb{F}_q)$. Now we consider the quotient group G/H.

Case 1. When $\frac{q-1}{2}$ is even we have,

$$G/H = \langle c \mid c^{q-1} \rangle.$$

Case 2. When $\frac{q-1}{2}$ is odd we have,

$$G/H = \langle b, c \mid b^2, c^{\frac{q-1}{2}}, bc = cb \rangle.$$

In both cases we see that $o(G/H) = q - 1$. This proves that $G \cong GL(2, \mathbb{F}_q)$. ■

Corollary 5.2. *When $q = 3$, a presentation of $GL(2, \mathbb{F}_q)$ is*

$$\begin{aligned}\langle a, b, c \mid\ & a^2, b^2, c^4, c^2 \in center, (ab)^3, (bca)^p, (bc)^2, \quad a_1^2 = a_2, \quad b_{q-1}b_i = b_ib_{q-1}, \\ & a_{q-1}^{-1}b_{q-1}a_{q-1}^{-1} = b(bc)^{\frac{q-1}{2}}, \quad aa_ia = a_{q-1}^{-1}b_ia_{q-1}, \\ & (a_{q-1}b_{q-1}^{-1}a_{q-1})b_m(a_{q-1}b_{q-1}^{-1}a_{q-1})^{-1} = a_m^{-1}\rangle,\end{aligned}$$

where $1 \leq m \leq 2$.

In the following corollary we are giving a presentation of $GL(2, \mathbb{F}_9)$. In this case, we have a primitive polynomial, $2 + x + x^2$.

Corollary 5.3. *A presentation of $GL(2, \mathbb{F}_9)$ is*

$$\begin{aligned}\langle a, b, c \mid\ & a^2, b^2, c^{16}, c^2 \in center, (ab)^3, ((bc)^4a)^3, (bc)^8, \\ & a_1a_2 = a_2a_1 = a_8, \quad a_1^2 = a_5, \quad b_8b_i = b_ib_8, \quad a_8^{-1}b_8a_8^{-1} = b(bc)^4, \\ & aa_ia = a_8^{-1}b_ia_8, \quad (a_8b_8^{-1}a_8)b_m(a_8b_8^{-1}a_8)^{-1} = a_m^{-1}\rangle.\end{aligned}$$

Thus we have found presentations for $GL(2, \mathbb{F}_{2^n})$ and $GL(2, \mathbb{F}_{p^n})$ in terms of Lie regular units. The exciting thing about these generators is that these elements have a fixed form $\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$ for some $\alpha \in \mathbb{F}_q \backslash \{0\}$, except $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$.

All results are verified with MAGMA software.

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