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Characterization of Modified Weibull Distribution Based on Dual Generalized Order Statistics

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Abstract. In this paper we derive some new recurrence relations for single and product moments of Dual generalized order statistics from Modified Weibull distribution (MWD). Also, using a recurrence relation for single moment to characterize this distribution. Further, recurrence relations for marginal and joint moment generating functions of Dual generalized order statistics from MWD are established. The results for order statistics and lower record values are deduced. Finally, characterization by using joint moment generating functions of Dual generalized order statistics is discussed.

Keywords: Modified Weibull distribution; Dual generalized order statistics; Recurrence relations; Single and product moments; Moment generating function; Characterization.

1. Introduction

The author in [3] presented new model named Dual generalized order statistics (Dgos) since it describes the random variables arranged in descending order of magnitudes. Dgos is considered a generalization of different models which depends on ascending arrangement for random variables like; order statistics, record value and generalized order statistics. Many authors have studied Dgos; see for example: [10, 8].

Recurrence relations are useful in reducing the number of operations necessary to obtain a general form for the function under consideration. In addition, they are used in characterizing the distributions, which is an important area, permitting the identification of population distribution from the properties of the sample. Recurrence relations and identities have attained importance as it reduces the amount of direct computation, time and labour. The authors in [28, 17, 22, 29, 9, 7] investigated the importance of recurrence relations of order statistics in characterization. The authors in [24, 25, 26] established recurrence relations for single and product moments of Dgos based on different distributions as: general class of distributions, inverse Weibull, linear-exponential and Burr distributions.

Let $F(x)$ denote absolutely continuous distribution function with density function $f(x)$. Let $n \in N, k > 0, m_1, ..., m_{n-1} \in R$, $M_r = \sum_{j=r}^{n-1} m_j$, $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, ..., n-1\}$. And let $\widetilde{m} = (m_1, m_2, ..., m_{n-1})$, if $n \geqslant 2$. If the random variables $X^l(r, n, m, k), r = 1, ..., n$, possess a joint density function of the form

$$\begin{aligned} & f_{X(1,n,\widetilde{m},k),...,X(n,n,\widetilde{m},k)}(x_1, ..., x_n) \\ = & k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{j=1}^{n-1} [F(x_i)]^{m_i} f(x_i) \right) [F(x_n)]^{k-1} f(x_n), \end{aligned} \quad (1)$$

on $F^{-1}(1) > x_1 \geq ... \geq x_n > F^{-1}(0)$, they are called Dgos.

When $m_1 = m_2 = ... = m_{n-1} = m$, we have

$$\begin{aligned} & f_{X^l(1,n,m,k),...,X^l(n,n,m,k)}(x_1, ..., x_n) \\ = & k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{j=1}^{n-1} [F(x_i)]^{m} f(x_i) \right) [F(x_n)]^{k-1} f(x_n), \end{aligned} \quad (2)$$

The marginal density function and the joint probability density function of $X^l(r, n, m, k)$, $X^l(r, n, m, k)$ and $X^l(s, n, m, k)$ are given by:

$$f_{X^l(r,n,m,k)}(x) = \frac{C_{r-1}}{\Gamma(r)} [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1} [F(x)] \, dx, \quad (3)$$

and

$$\begin{aligned} & f_{X^l(r,n,m,k),X^l(s,n,m,k)}(x, y) \\ = & \frac{C_{s-1}}{\Gamma(r)\Gamma(s-r)} [F(x)]^m f(x) g_m^{r-1} [F(x)] \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} \\ & \times [F(y)]^{\gamma_s - 1} f(y), \quad x > y, \end{aligned} \quad (4)$$

where

$$C_{r-1} = \prod_{j=1}^{r} \gamma_j, \ \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} \frac{-1}{m+1} x^{m+1} & \text{if} \quad m \neq -1, \\ -\ln(x) & \text{if} \quad m = -1, \end{cases} \quad (5)$$

$$\Gamma(r) = \int_0^\infty x^{r-1} \exp(-x) dx,$$

and

$$g_m(x) = h_m(x) - h_m(1), x \in [0,1).$$

For $\gamma_i \neq \gamma_j, i, j = 1, 2, ..., n-1$, the marginal density function and the joint probability density function of $X^l(r,n,m,k)$, $X^l(r,n,m,k)$ and $X^l(s,n,m,k)$ are given by:

$$C_{r-1} \sum_{i=1}^{r} a_i(r) [1 - F(x)]^{\gamma_i - 1} f(x), \tag{6}$$

and

$$C_{s-1} \left[\sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{1-F(y)}{1-F(x)} \right)^{\gamma_i} \right] \left[\sum_{i=1}^{r} a_i(r) (1-F(x))^{\gamma_i} \right] \frac{f(x)}{1-F(x)} \frac{f(y)}{1-F(y)}, \tag{7}$$

where

$$a_i^{(r)}(s) = \prod_{j=r+1, j\neq i}^{s} \frac{1}{\gamma_j - \gamma_i}, \qquad r+1 \le i \le s \le n,$$
$$a_i(r) = \prod_{j=1, j\neq i}^{r} \frac{1}{\gamma_j - \gamma_i} \qquad \text{and} \qquad a_i(s) = a_i^{(o)}(s). \tag{8}$$

For $m_i = m_j = m; i, j = 1, 2, ..., n-1$, the single and product moments are given by:

$$E\left[X^{l\,j}(r,n,m,k) \right] = \frac{C_{r-1}}{\Gamma(r)} \int_0^\infty x^j \left[F(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1} \left[F(x)\right] dx, \; 0 < x < \infty, \tag{9}$$

and

$$\begin{aligned} &\left[X^{l\,j}(r,n,m,k) X^{l\,j}(s,n,m,k) \right] \\ =& \frac{C_{s-1}}{\Gamma(r)\Gamma(s-r)} \int_0^\infty \int_0^x x^i y^j \left[F(x)\right]^m f(x) g_m^{r-1} \left[F(x)\right] \\ & \times \left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1} \left[F(y)\right]^{\gamma_s - 1} \\ & \times f(y) dy dx, \; 0 < y < x < \infty. \end{aligned} \tag{10}$$

For $\gamma_i \neq \gamma_j, i \neq j = 1, 2, ..., n-1$, the single and product moments are given by:

$$E\left[X^{l_i}(r,n,m,k) \right] = C_{r-1} \int_0^\infty x^i \sum_{i=1}^{r} a_i(r) \left[1 - F(x)\right]^{\gamma_i - 1} f(x) dx, \tag{11}$$

and

$$\begin{aligned} &E\left[X^{l\,j}(r,n,m,k)X^{l\,j}(s,n,m,k)\right] \\ =&C_{s-1}\int\limits_{0}^{\infty}\int\limits_{0}^{x}x^{i}y^{j}\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right] \\ &\times\left[\sum_{i=1}^{r}a_i(r)\left(1-F(x)\right)^{\gamma_i}\right]\frac{f(x)}{1-F(x)}\frac{f(y)}{1-F(y)}dxdy, \end{aligned} \tag{12}$$

where $C_{s-1}, a_i^{(r)}(s)$ and $a_i(r)$ are defined in Eq.(5) and Eq.(8).

The pdf, cdf, reliability function $S(x)$, and hazard function $h(x)$ of the MWD (b,c,d) are given respectively, by

$$f(x) = \left(b+cdx^{d-1}\right)\exp\{-(bx+cx^{d})\}, \quad b,c\geq 0,\; d>0, x>0, \tag{13}$$

$$F(x) = 1-\exp\{-(bx+cx^{d})\}, \tag{14}$$

$$S(x) = \exp\{-(bx+cx^{d})\}, \tag{15}$$

$$h(x) = b+cdx^{d-1}, \tag{16}$$

also, the characterizing differential equation is given by

$$F(x) = \frac{1}{(b+cdx^{d-1})}\left[(\exp\{bx+cx^{d}\})-1\right]f(x), \tag{17}$$

where b is a scale parameter, while c and d are the shape parameters. From Eq. (13), it should be noted as the following

(i) If $b=0, c=c$ and $d=a$ then MWD reduces to $Weibull(c,a)$. See [5].

(ii) If $b=0, c=\theta$ and $d=1$, then MWD reduces to $Exponential(\theta)$. See [11].

(iii) If $b=0$ and $d=2$ then MWD reduces to $Rayleigh(c)$. See [21].

(iv) If $c=\frac{\theta}{2}$ and $d=2$ then MWD reduces to LFRD (b,θ). See [18].

It can be seen that the shape of the hazard function in Eq. (16) depends only on d in which its monotonically increasing when $d>1$; decreasing when $d<1$ and constant when $d=1$. In [27], the authors also plotted the hazard function of the MWD and discussed the change of the value parameters.

In this paper, we derive some new recurrence relations for single and product moments of Dgos from MWD and its characterizations. Many authors have studied Dgos, see for example [12, 15, 14, 23, 2, 16, 20].

In Section 2, we establish recurrence relations for single of Dgos from MWD. Also, recurrence relations for product moment of Dgos from MWD are discussed in Section 3. In Section 4, using a recurrence relation for single moment to characterize this distribution. Further, characterization by using single moment generating functions joint moment generating functions of Dgos is presented in Sections (5,6). Finally, the conclusion in Section 7.

2. Recurrence Relations for the Single Moments

For $m_i = m_j = m, i, j = 1, 2, ..., n-1$.

Lemma 2.1. *For* $2 \le r \le n,\ n \ge 2$ *and* $k = 1, 2, ...$

$$E\left[X^{l_j}(r,n,m,k)\right] - E\left[X^{l_j}(r-1,n,m,k)\right] = -\frac{jC_{r-1}}{\gamma_r \Gamma(r)} \int_{\alpha}^{c} x^{j-1}\left[F(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right] dx. \tag{18}$$

Relation 2.2. For the MWD and $2 \le r \le n,\ n \ge 2$ and $k = 1, 2, ...$

$$E\left[X^{l_j}(r,n,m,k)\right] - E\left[X^{l_j}(r-1,n,m,k)\right] = -\frac{j}{\gamma_r}\left\{E\left[\psi_2(X^l(r,n,m,k)\right] - E\left[\psi_1(X^l(r,n,m,k))\right]\right\}, \tag{19}$$

where $\psi_1(x) = \frac{x^{j-1}}{b+cdx^{d-1}}$ and $\psi_2(x) = \psi_1(x)\exp\{bx + cx^d\}$.

Proof. From Lemma 2.1, and Eq. (17), we have

$$E\left[X^{l_j}(r,n,m,k)\right] - E\left[X^{l_j}(r-1,n,m,k)\right] = -\frac{jC_{r-1}}{\gamma_r \Gamma(r)} \int_{\alpha}^{c} x^{j-1}\left[F(x)\right]^{\gamma_r - 1} \frac{1}{b+cdx^{d-1}} \times \left[(\exp\{bx+cx^d\}) - 1\right] f(x) g_m^{r-1}\left[F(x)\right] dx, \tag{20}$$

then,

$$E\left[X^{l_j}(r,n,m,k)\right] - E\left[X^{l_j}(r-1,n,m,k)\right] = -\frac{jC_{r-1}}{\gamma_r \Gamma(r)} \int_{0}^{\infty} x^{j-1}\left[F(x)\right]^{\gamma_r - 1} \frac{1}{b+cdx^{d-1}} \left[\exp\{bx+cx^d\}\right] f(x) g_m^{r-1}\left[F(x)\right] dx + \frac{jC_{r-1}}{\gamma_r \Gamma(r)} \int_{0}^{\infty} x^{j-1}\left[F(x)\right]^{\gamma_r - 1} \frac{1}{b+cdx^{d-1}} f(x) g_m^{r-1}\left[F(x)\right] dx, \tag{21}$$

hence, Eq. (19) is obtained. ■

Remark 2.3.

(i) For $m = 0$, $k = 1$, the recurrence relation of Dgos reduces to the recurrence relation of ordinary order statistics form MWD as follows:

$$E\left[X^j_{n-r+1:n}\right] - E\left[X^j_{n-r+2:n}\right] = -\frac{j}{(n-r+1)}\left[E\left[\psi_2(X_{n-r+1:n})\right] - E\left[\psi_1(X_{n-r+1:n})\right]\right]. \tag{22}$$

(ii) For $m=-1, k=1$, the recurrence relation for single moments of lower record value (L) form MWD can be obtained as follows:

$$E\left[X_{L(r)}^{j}\right]-E\left[X_{L(r-1)}^{j}\right]=-j\left[E\left[\psi_{2}(X_{L(r)})\right]-E\left[\psi_{1}(X_{L(r)})\right]\right]. \quad (23)$$

(iii) For $m=0, k=b-n+1$, $b\in\mathbb{R}^{+}$, the recurrence relation for single moment of dual order statistics with non-integral sample size form MWD is:

$$\begin{aligned} &E\left[X_{b-r+1:n}^{j}\right]-E\left[X_{b-r+2:n}^{j}\right]\\ &=-\frac{j}{(b-r+1)}\left[E\left[\psi_{2}(X_{b-r+1:b})\right]-E\left[\psi_{1}(X_{n-b+1:b})\right]\right]. \end{aligned} \quad (24)$$

(iv) For $m=b-1$, $k=b$, the recurrence relation for sequential order statistics from MWD is:

$$\begin{aligned} &E\left[X^{j}(r,n,b-1,b)\right]-E\left[X^{j}(r-1,n,b-1,b)\right]\\ &=-\frac{j}{b(n-r+1)}\left[E\left[\psi_{2}(X(r,n,b-1,b))\right]-E\left[\psi_{1}(X(r,n,b-1,b))\right]\right]. \end{aligned} \quad (25)$$

(v) For $m=0, k=1, b=0, c=b$ and $d=n$, the results of Weibull distribution given in [22] are obtained.

(vi) For $m=0, k=1$ and $b=0$, our results agree with the results in [20].

(vii) For $m=0, k=1$ and $c=0$, the results for exponential distribution in [19] are are deduced.

(viii) For $m=0, k=1, b=0$ and $d=2$, the results in [21] are deduced.

(ix) For $m=0, k=1, c=\frac{\theta}{2}$ and $d=2$, our results agree with the results in [18].

Lemma 2.4. *For* $2\le r\le n, n\ge 2$ *and* $k=1,2,...,$

(i)

$$\begin{aligned} &E\left[X^{l_j}(r,n,m,k)\right]-E\left[X^{l_j}(r-1,n-1,m,k)\right]\\ &=-\frac{jC_{r-1}}{\gamma_1\Gamma(r)}\int_{\alpha}^{c}x^{j-1}\left[F(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]dx. \end{aligned} \quad (26)$$

(ii)

$$\begin{aligned} &E\left[X^{l_j}(r-1,n,m,k)\right]-E\left[X^{l_j}(r-1,n-1,m,k)\right]\\ &=\frac{j(m+1)C_{r-2}}{\gamma_1\Gamma(r-1)}\int_{\alpha}^{c}x^{j-1}\left[F(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]dx. \end{aligned} \quad (27)$$

Relation 2.5. For the MWD and $2\le r\le n$, $n\ge 2$ and $k=1,2,...$

(i)

$$E\left[X^{l_j}(r,n,m,k)\right]-E\left[X^{l_j}(r-1,n-1,m,k)\right]$$
$$=-\frac{j}{\gamma_1}\left[E\left[\psi_2(X^l(r,n,m,k)\right]-E\left[\psi_1(X^l(r,n,m,k))\right]\right]. \qquad (28)$$

(ii)

$$E\left[X^{l_j}(r,n,m,k)\right]-E\left[X^{l_j}(r-1,n-1,m,k)\right]$$
$$=\frac{(m+1)j(r-1)}{\gamma_1\gamma_r}\left[E\left[\psi_2(X^l(r,n,m,k))\right]-E\left[\psi_1(X^l(r,n,m,k)\right]\right]. \qquad (29)$$

Where

$$\psi_1(x)=\frac{x^{j-1}}{b+cdx^{d-1}} \text{ and } \psi_2(x)=\psi_1(x)\exp\{bx+cx^d\}.$$

Proof. By using Eq. (5) and Lemma 2.4, Relation (2.5) is proved. ■

Case II: $\gamma_i \neq \gamma_j$, $i \neq j$, $i,j=1,2,...,n-1$.

Lemma 2.6. *For* $2 \le r \le n, n \ge 2$ *and* $k=1,2,...$

$$E\left[X^{l_j}(r,n,\widetilde{m},k)\right]-E\left[X^{l_j}(r-1,n,\widetilde{m},k)\right]$$
$$=-jC_{r-2}\sum_{i=1}^{r}a_i(r)\int_{\alpha}^{c}x^{j-1}\left[F(x)\right]^{\gamma_r}dx. \qquad (30)$$

Relation 2.7. For the MWD and $2 \le r \le n$, $n \ge 2$ and $k=1,2,...$

$$E\left[X^{l_j}(r,n,\widetilde{m},k)\right]-E\left[X^{l_j}(r-1,n,\widetilde{m},k)\right]$$
$$=-\frac{j}{\gamma_r}\left\{E\left[\psi_2(X^l(r,n,\widetilde{m},k))\right]-E\left[\psi_1(X^l(r,n,\widetilde{m},k))\right]\right\}, \qquad (31)$$

where

$$\psi_1(x)=\frac{x^{j-1}}{b+cdx^{d-1}} \text{ and } \psi_2(x)=\psi_1(x)\exp\{bx+cx^d\}.$$

Proof. By using Lemma 2.4 and Eq. (17), we get

$$E\left[X^{l_j}(r,n,\widetilde{m},k)\right]-E\left[X^{l_j}(r-1,n,\widetilde{m},k)\right]$$
$$=-jC_{r-2}\sum_{i=1}^{r}a_i(r)\int_{\alpha}^{c}x^{j-1}[F(x)]^{\gamma_r-1}\left[\frac{(\exp\{bx+cx^d\})-1}{b+cdx^{d-1}}\right]f(x)dx, \qquad (32)$$

after some simplifications, we obtain Eq. (31). ■

Remark 2.8. Relation (2.2) can be deduced from Relation (2.7) by replacing $\widetilde{m}$ with m, $m \neq -1$.

3. Recurrence Relations for the Product Moments

For $m_i = m_j = m, i, j = 1, 2, ..., n-1$.

Lemma 3.1. *For* $1 \leq r \leq s \leq n-1, n \geq 2$ *and* $k = 1, 2, ...$

$$E\left[X^{l_i}(r,n,m,k)X^{l_j}(s,n,m,k)\right] - E\left[X^{l_i}(r,n,m,k)X^{l_j}(s-1,n,m,k)\right]$$
$$= -\frac{jC_{s-1}}{\gamma_s \Gamma(r)\Gamma(s-r)} \int_{\alpha}^{c}\int_{\alpha}^{x} x^i y^{j-1} \left[F(x)\right]^m f(x) g_m^{r-1}\left[F(x)\right]$$
$$\times \left[h_m\left(F(y)\right) - h_m\left(F(x)\right)\right]^{s-r-1} \left[F(y)\right]^{\gamma_s} f(y) dy dx. \qquad (33)$$

Relation 3.2. For the MWD and $2 \leq r < s \leq n-1$, $n \geq 2$ and $k = 1, 2, \ldots$,

$$E\left[X^{l_i}(r,n,m,k)X^{l_j}(s,n,m,k)\right] - E\left[X^{l_i}(r,n,m,k)X^{l_j}(s-1,n,m,k)\right]$$
$$= -\frac{j}{\gamma_s}\Big\{E\left(X^{l_i}(r,n,m,k)\right)\psi_2(X^l(s,n,m,k)) \qquad (34)$$
$$+ E\left(X^{l_i}(r,n,m,k)\right)\psi_1(X^l(s,n,m,k))\Big\},$$

where $\psi_1(y) = \frac{y^{j-1}}{b+cdy^{d-1}}$ and $\psi_2(y) = \psi_1(y)\exp\{by + cy^d\}$.

Proof. By using Eq. (17) and Lemma 3.1, we have

$$E\left[X^{l_i}(r,n,m,k)X^{l_j}(s,n,m,k)\right] - E\left[X^{l_i}(r,n,m,k)X^{l_j}(s-1,n,m,k)\right]$$
$$= -\frac{jC_{s-1}}{\gamma_s \Gamma(r)\Gamma(s-r)} \int_{\alpha}^{c}\int_{\alpha}^{x} x^i y^{j-1} \left[F(x)\right]^m f(x) g_m^{r-1}\left[F(x)\right] \Big[h_m\left(F(y)\right) \qquad (35)$$
$$- h_m\left(F(x)\right)\Big]^{s-r-1} \left[F(y)\right]^{\gamma_s - 1} \frac{1}{b+cdy^{d-1}} \left[(\exp\{by+cy^d\})-1\right] f(y) dy dx,$$

then, after some simplifications, Eq. (34) is obtained. ■

Remark 3.3.

(i) Setting $m = 0, k = 1$ in Eq. (34), we get the recurrence relation for product moments of ordinary order statistics from MWD as follows:

$$E\left[X^i_{n-r+1:n}X^j_{n-s+1:n}\right] - E\left[X^i_{n-r+2:n}X^j_{n-s+2:n}\right]$$
$$= -\frac{j}{(n-s+1)}\Big[E\left[X^i_{n-r+1:n}\psi_2(X_{n-s+1:n})\right] \qquad (36)$$
$$-E\left[X^i_{n-r+1:n}\psi_1(X_{n-r+1:n})\right]\Big].$$

(ii) Putting $m = -1, k = 1$ in Eq. (34), we get the recurrence relation for product moments of lower order statistics (L) from MWD as follows:

$$E\left[X^i_{L(r)}X^j_{L(s)}\right] - E\left[X^i_{L(r)}X^j_{L(s-1)}\right] = -j\left[E\left[X^i_{L(r)}\psi_2(X_{L(r)})\right] - E\left[X^i_{L(r)}\psi_1(X_{L(s)})\right]\right], \tag{37}$$

for $\gamma_i \neq \gamma_j, i \neq j \ , i,j = 1,2,...,n-1$.

Relation 3.4. For the MWD and $2 \leq r < s \leq n-1$, $n \geq 2$ and $k = 1,2,\ldots$,

$$E\left[X^{l_i}(r,n,\widetilde{m},k)X^{l_j}(s,n,\widetilde{m},k)\right] - E\left[X^{l_i}(r,n,\widetilde{m},k)X^{l_j}(s-1,n,\widetilde{m},k)\right]$$
$$= -\frac{j}{\gamma_s}\Big\{E\left(X^{l_i}(r,n,\widetilde{m},k)\right)\psi_2(X^l(s,n,\widetilde{m},k))$$
$$+E\left(X^{l_i}(r,n,\widetilde{m},k)\right)\psi_1(X^l(s,n,\widetilde{m},k))\Big\}. \tag{38}$$

Proof. The result can be established by replacing m with $\widetilde{m}$ in Relation (3.2). ■

4. Characterizations Based on Recurrence Relation for Single Moments

Theorem 4.1. *Let X be a non-negative random variable having an absolutely continuous distribution $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$. Then, for $m_i = m_j = m$*

$$E\left[X^{l_j}(r,n,m,k)\right] - E\left[X^{l_j}(r-1,n,m,k)\right] = -\frac{j}{\gamma_r}\left\{E\left[\psi_2(X^l(r,n,m,k))\right] - E\left[\psi_1(X^l(r,n,m,k))\right]\right\}, \tag{39}$$

and for $\gamma_i \neq \gamma_j, i \neq j, i,j = 1,2,...,n-1$.

$$E\left[X^{l_j}(r,n,\widetilde{m},k)\right] - E\left[X^{l_j}(r-1,n,\widetilde{m},k)\right] = -\frac{j}{\gamma_r}\left\{E\left[\psi_2(X^l(r,n,\widetilde{m},k))\right] - E\left[\psi_1(X^l(r,n,\widetilde{m},k))\right]\right\}, \tag{40}$$

where

$$\psi_1(x) = \frac{x^{j-1}}{b + cdx^{d-1}} \text{ and } \psi_2(x) = \psi_1(x)\exp\{bx + cx^d\}.$$

if and only if X has MWD(b,c,d).

Proof. For $m_i = m_j = m$, the necessary part follows immediately from Eq. (17) and Lemma 2.1. On the other hand if the recurrence relation in Eq. (39) is satisfied, then from Eq. (9), we have

$$\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty x^j\left[F(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$
$$-\frac{C_{r-2}}{\Gamma(r-1)}\int_0^\infty x^j\left[F(x)\right]^{\gamma_r+m}f(x)g_m^{r-2}\left[F(x)\right]dx$$
$$=-\frac{jC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty \frac{x^{j-1}}{b+cdx^{d-1}}\left[\exp\{bx+cx^d\}\right]\left[F(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$
$$+\frac{jC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty \frac{x^{j-1}}{b+cdx^{d-1}}\left[F(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx. \quad (41)$$

Integrating by parts the left hand side of Eq. (41), we have

$$\int_0^\infty x^j\left[F(x)\right]^{\gamma_r+m}f(x)g_m^{r-2}\left[F(x)\right]dx$$
$$=\frac{j}{(r-1)}\int_0^\infty x^{j-1}\left[F(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]dx$$
$$+\frac{\gamma_r}{(r-1)}\int_0^\infty x^j\left[F(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx, \quad (42)$$

by substituting Eq. (42) into Eq. (41) and after some simplifications, we get

$$\frac{jC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty x^{j-1}\left[F(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]dx$$
$$\times\left[F(x)+\frac{f(x)}{b+cdx^{d-1}}-\frac{\exp\{bx+cx^d\}}{b+cdx^{d-1}}f(x)\right]dx=0. \quad (43)$$

Applying Muntz-Szasz theorem in [4] to Eq. (43), we obtain

$$F(x)=\frac{1}{b+cdx^{d-1}}\left[(\exp\{bx+cx^d\})-1\right]f(x),$$

which proves X has MWD(b,c,d).

For $\gamma_i\neq\gamma_j, i\neq j, i,j=1,2,...,n-1$, the necessary part follows immediately from Eq. (17) and Lemma 3.1. On the other hand if the recurrence relation in

Eq. (40) is satisfied, then from Eq. (40), we have

$$C_{r-1}\int_0^\infty f(x)\sum_{i=1}^{r} a_i(r)\,[F(x)]^{\gamma_i-1}\,x^j\,dx$$
$$-C_{r-2}\int_0^\infty f(x)\sum_{i=1}^{r-1} a_i(r-1)\,[F(x)]^{\gamma_i-1}\,x^j\,dx$$
$$=-\frac{jC_{r-1}}{\gamma_r}\int_0^\infty f(x)\sum_{i=1}^{r} a_i(r)\,[F(x)]^{\gamma_i-1}\,\frac{x^{j-1}}{b+cdx^{d-1}}\exp\left(bx+cx^d\right)dx$$
$$+\frac{jC_{r-1}}{\gamma_r}\int_0^\infty f(x)\sum_{i=1}^{r} a_i(r)\,[F(x)]^{\gamma_i-1}\,\frac{x^{j-1}}{b+cdx^{d-1}}dx. \qquad (44)$$

Setting $a_i(r-1)=(\gamma_r-\gamma_i)\,a_i(r)$, and after some simplifications, we obtain

$$C_{r-2}\int_0^\infty f(x)\sum_{i=1}^{r} a_i(r)\gamma_i\,[F(x)]^{\gamma_i-1}\,x^j\,dx$$
$$=-\frac{jC_{r-1}}{\gamma_r}\int_0^\infty \frac{x^{j-1}}{b+cdx^{d-1}}f(x)\sum_{i=1}^{r} a_i(r)\,[F(x)]^{\gamma_i-1}\exp\left(bx+cx^d\right)dx$$
$$+\frac{jC_{r-1}}{\gamma_r}\int_0^\infty \frac{x^{j-1}}{b+cdx^{d-1}}f(x)\sum_{i=1}^{r} a_i(r)\,[F(x)]^{\gamma_i-1}\,dx. \qquad (45)$$

Integrating by parts the left hand side of Eq. (45), we have

$$\begin{aligned}&C_{r-2}\int_0^\infty f(x)\sum_{i=1}^{r} a_i(r)\gamma_i\,[F(x)]^{\gamma_i-1}\,x^j\,dx\\ &=-\,jC_{r-2}\int_0^\infty \sum_{i=1}^{r} a_i(r)\gamma_i\,[F(x)]^{\gamma_i}\,x^{j-1}dx.\end{aligned} \qquad (46)$$

Then, by substituting Eq. (46) into Eq. (45), we obtain

$$-\frac{jC_{r-1}}{\gamma_r}\int_0^\infty x^{j-1}\sum_{i=1}^{r} a_i(r)\gamma_i\,[F(x)]^{\gamma_i-1}$$
$$\times\left[F(x)-\frac{\exp\{bx+cx^d\}}{b+cdx^{d-1}}f(x)+\frac{f(x)}{b+cdx^{d-1}}\right]dx=0. \qquad (47)$$

Applying Muntz Szasz theorem in [4] to Eq. (47), we obtain

$$F(x)=\frac{1}{b+cdx^{d-1}}\left[(\exp\{bx+cx^d\})-1\right]f(x),$$

which leads to X has MWD(b,c,d). ∎

5. Characterization Based on Recurrence Relations for the Single Moments Generating Function of Dgos

The single moment generating function of Dgos can be obtained, for integers a such that $a \geq 1$,we have when $m \neq -1$:

$$\begin{aligned} M^{(a)}_{(r;n,m,k)}(t) &= E\left[\exp\left(tX^{(a)}_{(r,n,m,k)}\right)\right] \\ &= \frac{C_{r-1}}{\Gamma(r)}\int_0^\infty [\exp\{tx^a\}]\,[F(x)]^{\gamma_r-1}\, f(x) g_m^{r-1}\,[F(x)]\,dx, \end{aligned} \tag{48}$$

for simplicity $X_{r,n,m,k} = X$.

Relation 5.1. Let X be a random variable. Then for integers a such that $\mathbf{a} \geq 1$, the following recurrence relation is satisfied

$$\begin{aligned} &M^{(a)}_{(r;n,m,k)}(t) - M^{(a)}_{(r-1;n,m,k)}(t) \\ &= -\frac{at}{\gamma_r}\left\{E\left[\frac{X^{a-1}\exp\{tX^a+bX+cX^d\}}{b+cdX^{d-1}}\right] - E\left[\frac{X^{a-1}\exp\{tX^a\}}{b+cdX^{d-1}}\right]\right\}, \end{aligned} \tag{49}$$

if and only if X has MWD (b,c,d).

Proof. By using Eq. (48) and integrating by parts, we obtain

$$\begin{aligned} M^{(a)}_{(r;n,m,k)}(t) &= -\frac{atC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty x^{a-1}[\exp\{tx^a\}]\,[F(x)]^{\gamma_r}\, f(x) g_m^{r-1}\,[F(x)]\,dx \\ &\quad + \frac{(r-1)\,C_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty [\exp\{tx^a\}]\,[F(x)]^{\gamma_{r-1}-1}\, f(x) g_m^{r-2}\,[F(x)]\,dx. \end{aligned} \tag{50}$$

Then,

$$\begin{aligned} &M^{(a)}_{(r;n,m,k)}(t) - M^{(a)}_{(r-1;n,m,k)}(t) \\ &= -\frac{atC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty x^{a-1}[\exp\{tx^a\}]\,[F(x)]^{\gamma_r}\, g_m^{r-1}\,[F(x)]\,dx. \end{aligned} \tag{51}$$

By using Eq. (17), we get

$$\begin{aligned}
&M^{(a)}_{(r;n,m,k)}(t) - M^{(a)}_{(r-1;n,m,k)}(t)\\
&= -\frac{atC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty x^{a-1}[\exp\{tx^a\}]\,[F(x)]^{\gamma_r-1}\,\frac{\left[\exp\{bx+cx^d\}\right]}{(b+cdx^{d-1})}f(x)g_m^{r-1}\,[F(x)]\,dx\\
&\quad+\frac{atC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty x^{a-1}[\exp\{tx^a\}]\,[F(x)]^{\gamma_r-1}\,\frac{f(x)}{(b+cdx^{d-1})}g_m^{r-1}\,[F(x)]\,dx, \qquad (52)
\end{aligned}$$

after some simplifications, Eq. (49) is obtained.

Conversely, if the characterizing condition given in Eq. (49) is satisfied, we have

$$\begin{aligned}
&\frac{C_{r-1}}{\Gamma(r)}\int_0^\infty [\exp\{tx^a\}]\,[F(x)]^{\gamma_r-1}\,f(x)g_m^{r-1}\,[F(x)]\,dx\\
&\quad-\frac{C_{r-2}}{\Gamma(r-1)}\int_0^\infty [\exp\{tx^a\}]\,[F(x)]^{\gamma_{r-1}-1}\,f(x)g_m^{r-2}\,[F(x)]\,dx\\
&= -\frac{atC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty \frac{x^{a-1}\left[\exp\{tx^a+bx+cx^d\}\right]}{(b+cdx^{d-1})}\,[F(x)]^{\gamma_r-1}\,f(x)g_m^{r-1}\,[F(x)]\,dx\\
&\quad+\frac{atC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty \frac{x^{a-1}\,[\exp\{tx^a\}]}{(b+cdx^{d-1})}\,[F(x)]^{\gamma_r-1}\,f(x)g_m^{r-1}\,[F(x)]\,dx. \qquad (53)
\end{aligned}$$

Integrating by parts the second term of in the left hand side of Eq. (53), we obtain

$$\begin{aligned}
&\int_0^\infty [\exp\{tx^a\}]\,[F(x)]^{\gamma_{r-1}-1}\,f(x)g_m^{r-2}\,[F(x)]\,dx\\
&= \int_0^\infty [\exp\{tx^a\}]\,[F(x)]^{\gamma_r+m}\,f(x)g_m^{r-2}\,[F(x)]\,dx\\
&= \frac{\gamma_r}{(r-1)}\int_0^\infty [\exp\{tx^a\}]\,[F(x)]^{\gamma_r-1}\,f(x)g_m^{r-1}\,[F(x)]\,dx\\
&\quad+\frac{at}{(r-1)}\int_0^\infty x^{a-1}[\exp\{tx^a\}]\,[F(x)]^{\gamma_r}\,g_m^{r-1}\,[F(x)]\,dx. \qquad (54)
\end{aligned}$$

By substituting Eq. (54) into Eq. (53) and after some simplifications, we get

$$-\frac{atC_{r-1}}{\gamma_r\Gamma(r)}\int_0^\infty x^{a-1}[\exp\{tx^a\}]\,[F(x)]^{\gamma_r-1}\,g_m^{r-1}\,[F(x)]$$
$$\times\left[F(x)-\frac{\left[\exp\{bx+cx^d\}\right]}{(b+cdx^{d-1})}f(x)+\frac{1}{(b+cdx^{d-1})}f(x)\right]dx=0. \tag{55}$$

Applying Muntz-Suasz theorem [4] to Eq. (55), we have

$$F(x)=\frac{1}{(b+cdx^{d-1})}\left[(\exp\{bx+cx^d\})-1\right]f(x), \tag{56}$$

which leads to X has the MWD(b,c,d). ■

6. Characterization Based on Recurrence Relations for the Joint Moments Generating Function of Dgos

The joint moment generating function of Dgos can be written, from using Eq. (4), we have when $m\neq -1$

$$M_{r,s;n,m,k}(t_1,t_2)=E\left[\exp\{t_1X_{r;n,m,k}+t_2X_{s;n,m,k}\}\right]$$
$$=\frac{C_{k-1}}{\Gamma(r)\Gamma(k-r)}\int_{-\infty}^{\infty}\int_{-\infty}^{x}\left[\exp\left(t_1x+t_2y\right)\right][F(x)]^m f(x)g_m^{r-1}\left[F(x)\right]$$
$$\times\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-1}[F(y)]^{\gamma_k-1}f(y)dydx, \tag{57}$$

for simplicity $X_{r;n,m,k}=X$, $X_{s;n,m,k}=Y$, $C_{k-1}=\prod_{j=1}^{k}\gamma_j$ and $\gamma_i=k+(n-i)(m+1)$.

Relation 6.1. Let X be a random variable, r,s be two integers such that $1\leq r\leq s\leq n$, m and k be real numbers such that $m\neq -1, k>1$. Then, the following recurrence relation is satisfied

$$M_{r;s:n,m,k}(t_1,t_2)-M_{r;s-1:n,m,k}(t_1,t_2)$$
$$=-\frac{t_2}{\gamma_k}\left\{E\left[\frac{\exp\{t_1X+t_2Y+bY+cY^d\}}{b+cdY^{d-1}}\right]-E\left[\frac{\exp\{t_1X+t_2Y\}}{b+cdY^{d-1}}\right]\right\}, \tag{58}$$

if and only if Y has the $MWD(b,c,d)$.

Proof. By using Eq. (57), we have

$$M_{r;s;n,m,k}(t_1,t_2)$$
$$=\frac{C_{k-1}}{\Gamma(r)\Gamma(k-r)}\int_0^\infty(\exp\{t_1x\})\,[F(x)]^m f(x)g_m^{r-1}\left[F(x)\right]D(x)dx, \tag{59}$$

where

$$D(x)=\int_0^x(\exp\{t_2y\})\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-1}\left[F(y)\right]^{\gamma_k-1}f(y)dy. \quad (60)$$

Integrating by parts Eq. (60), we obtain

$$\begin{aligned}D(x)\;=\;&-\frac{t_2}{\gamma_k}\int_0^x\left[\exp\left(t_2y\right)\right]\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-1}\left[F(y)\right]^{\gamma_k}dy \quad (61)\\&+\frac{(k-r-1)}{\gamma_k}\int_0^x\left[\exp\left(t_2y\right)\right]\left[F(y)\right]^{\gamma_k+m}\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-2}f(y)dy.\end{aligned}$$

By substituting Eq. (61) into Eq. (59), and after some simplifications, we get

$$\begin{aligned}&M_{r;s;n,m,k}(t_1,t_2)-M_{r;s-1;n,m,k}(t_1,t_2)\\=\;&-\frac{t_2C_{k-1}}{\gamma_k\Gamma(r)\Gamma\left(k-r\right)}\\&\times\int_0^\infty\int_0^x\frac{\left[\exp\left(t_1x+t_2y+by+cy^d\right)\right]}{b+cdy^{d-1}}\left[F(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]\\&\times\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-1}\left[F(y)\right]^{\gamma_k-1}f(y)dydx\\&+\frac{t_2C_{k-1}}{\gamma_k\Gamma(r)\Gamma\left(k-r\right)}\int_0^\infty\int_0^x\frac{\left[\exp\left(t_1x+t_2y\right)\right]}{b+cdy^{d-1}}\left[F(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]\\&\times\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-1}\left[F(y)\right]^{\gamma_k-1}f(y)dydx. \quad (62)\end{aligned}$$

Then, after some simplifications, Eq. (58) is obtained.

Conversely, if the characterizing condition given in Eq. (58) is satisfied, then, we have

$$\begin{aligned}&\frac{C_{k-1}}{\Gamma(r)\Gamma\left(k-r\right)}\int_0^\infty\int_0^x\frac{\left[\exp\left(t_1x+t_2y+by+cy^d\right)\right]}{b+cdy^{d-1}}\left[F(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]\\&\times\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-1}\left[F(y)\right]^{\gamma_k-1}f(y)dydx\\&-\frac{C_{k-2}}{\Gamma(r)\Gamma\left(k-r-1\right)}\int_0^\infty\int_0^x\frac{\left[\exp\left(t_1x+t_2y+by+cy^d\right)\right]}{b+cdy^{d-1}}\left[F(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]\\&\times\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-2}\left[F(y)\right]^{\gamma_{k-1}-1}f(y)dydx\\=\;&-\frac{t_2C_{k-1}}{\gamma_k\Gamma(r)\Gamma\left(k-r\right)}\int_0^\infty\int_0^x\frac{\left[\exp\left(t_1x+t_2y+by+cy^d\right)\right]}{b+cdy^{d-1}}\left[F(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]\end{aligned}$$

$$\times \left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-1}\left[F(y)\right]^{\gamma_k-1}f(y)dydx$$
$$+\frac{t_2C_{k-1}}{\gamma_k\Gamma(r)\Gamma(k-r)}\int_0^\infty\int_0^x\frac{\left[\exp\left(t_1x+t_2y\right)\right]}{b+cdy^{d-1}}\left[F(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$
$$\times \left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-1}\left[F(y)\right]^{\gamma_k-1}f(y)dydx. \quad (63)$$

Integrating by parts the second term of the left hand side of Eq. (63), we have

$$\int_0^x\left[\exp\left(t_1y\right)\right]\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-2}\left[F(y)\right]^{\gamma_{k-1}-1}f(y)dy$$
$$=\int_0^x\left[\exp\left(t_1y\right)\right]\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-2}\left[F(y)\right]^{\gamma_k+m}f(y)dy \quad (64)$$
$$=\frac{t_2}{(k-r-1)}\int_0^x\left[\exp\left(t_1y\right)\right]\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-1}\left[F(y)\right]^{\gamma_k}dy$$
$$+\frac{\gamma_k}{(k-r-1)}\int_0^x\left[\exp\left(t_1y\right)\right]\left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-1}\left[F(y)\right]^{\gamma_k-1}f(y)dy.$$

By substituting Eq. (64) into Eq. (63), we obtain

$$-\frac{t_2C_{k-1}}{\gamma_k\Gamma(r)\Gamma(k-r)}\int_0^\infty\int_0^x\frac{\left[\exp\left(t_1x+t_2y\right)\right]}{b+cdy^{d-1}}\left[F(x)\right]^m f(x)g_m^{r-1}\left[F(x)\right]$$
$$\times \left[h_m\left[F(y)\right]-h_m\left[F(x)\right]\right]^{k-r-1}\left[F(y)\right]^{\gamma_k-1} \quad (65)$$
$$\times\left[F(y)-\frac{\left[\exp\left(by+cy^d\right)\right]}{b+cdy^{d-1}}f(y)+\frac{f(y)}{b+cdy^{d-1}}\right]dydx=0.$$

Applying [4, Muntz-Szasz theorem] to Eq. (65), we get

$$F(y)=\frac{1}{b+cdy^{d-1}}\left[(\exp\{by+cy^d\})-1\right]f(y), \quad (66)$$

which leads to Y has $MWD(b,c,d)$. ■

7. Conclusion

This paper deals with the Dgos from the MWD. Recurrence relations between the single and product moments are derived. Characterizations of the MWD based on a recurrence relation for single moments, marginal and joint moment generating of Dgos are discussed. Special cases are also deduced.

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