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Local Properties of Generalized Absolute Matrix Summability of Factored Fourier Series

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Abstract. In this paper, a theorem on local property of $\varphi - |A, p_n|_k$ summability of factored Fourier series, which generalizes a known result, has been proved. This new theorem also includes several new and known results.

Keywords: Summability factors; Absolute matrix summability; Fourier series; Local property; Infinite series; Hölder inequality; Minkowski inequality.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). By (u_n) and (t_n) we denote the n-th $(C, 1)$ means of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geq 1$, if (see [17])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1)\,. \tag{2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [18]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta\sigma_{n-1}|^k < \infty, \tag{4}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1.$$

In the special case, when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\bar{N}, p_n|$) summability. Also, if we take $k = 1$ and $p_n = 1/(n+1)$, then summability $|\bar{N}, p_n|_k$ is equivalent to the summability $|R, logn, 1|$.

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{5}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [36])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{6}$$

and it is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [35])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\bar{\Delta} A_n(s)|^k < \infty, \tag{7}$$

where $\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s)$.

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A, p_n|_k$, $k \geq 1$, if (see [33])

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |\bar{\Delta} A_n(s)|^k < \infty. \tag{8}$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |A, p_n|_k$ summability reduces to $|A, p_n|_k$ summability. Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|\bar{N}, p_n|_k$ summability. Furthermore, if we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, $\varphi - |A, p_n|_k$ reduces to $|C, 1|_k$ summability. Finally, if we take $\varphi_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|R, p_n|_k$ summability (see [7]).

A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{9}$$

and

$$f(t) \sim \sum_{n=1}^{\infty}(a_n cosnt + b_n sinnt) = \sum_{n=1}^{\infty} C_n(t), \tag{10}$$

where (a_n) and (b_n) denote the Fourier coefficients. It is well known that the convergence of the Fourier series at $t = x$ is a local property of the generating function f (i.e. it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of the generating function f (see [37]).

2. Known Results

Mohanty [24, Theorem B] has demonstrated that the summability $|R, logn, 1|$ of

$$\sum \frac{C_n(t)}{log(n+1)}, \tag{11}$$

at $t = x$, is a local property of the generating function of $\sum C_n(t)$. Later on Matsumoto [22, Theorem 2] improved this result by replacing the series (11) by

$$\sum \frac{C_n(t)}{\{loglog(n+1)\}^{1+\epsilon}}, \quad \epsilon > 0. \tag{12}$$

Generalizing the above result Bhatt proved the following theorem.

Theorem 2.1. [2, Theorem] *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $|R, logn, 1|$ of the series $\sum C_n(t)\lambda_n logn$ at a point can be ensured by a local property.*

Also, Mishra has proved the following most general theorem.

Theorem 2.2. [23, Theorem 3] *Let the sequence (p_n) be such that*

$$P_n = O(np_n), \tag{13}$$

$$P_n\Delta p_n = O(p_n p_{n+1}). \tag{14}$$

Then the summability $|\bar{N}, p_n|$ of the series

$$\sum \frac{C_n(t)\lambda_n P_n}{np_n} \tag{15}$$

at a point can be ensured by local property, where (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent.

Many works dealing with summability theory have been done (see [1]-[2], [5]-[6], [8]-[15], [19]-[32], [34]). Few of them are given above. Furthermore, Bor has proved the following theorem.

Theorem 2.3. [4, Theorem] *Let $k \geq 1$ and (p_n) be a sequence such that the conditions (13) and (14) of Theorem 2.2 are satisfied. Then the summability $|\bar{N}, p_n|_k$ of the series (15) at a point can be ensured by local property, where (λ_n) is as in Theorem 2.2.*

3. Main Result

The aim of this paper is to generalize Theorem 2.3 for $\varphi - |A, p_n|_k$ summability.

Before stating the main theorem, we must first introduce some further notations.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{16}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{17}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{18}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{19}$$

Now, we shall prove the following theorem.

Theorem 3.1. *Let $k \geq 1$ and $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \; n = 0, 1, ..., \tag{20}$$

$$a_{n-1,v} \geq a_{nv}, \; for \; n \geq v+1, \tag{21}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{22}$$

$$|\hat{a}_{n,v+1}| = O\left(v \left|\Delta_v \hat{a}_{nv}\right|\right), \tag{23}$$

and $\left(\frac{\varphi_n p_n}{P_n}\right)$ be a non-increasing sequence. If all the conditions of Theorem 2.3 are satisfied and (φ_n) is any sequence of positive constants such that

$$\sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} \frac{1}{v}(\lambda_v)^k = O(1) \quad as \quad m \to \infty, \tag{24}$$

$$\sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} \Delta\lambda_v = O(1) \quad as \quad m \to \infty, \tag{25}$$

$$\sum_{v=1}^{m} \left(\frac{\varphi_v p_v}{P_v}\right)^{k-1} \frac{1}{v}(\lambda_{v+1})^k = O(1) \quad as \quad m \to \infty, \tag{26}$$

then the summability $\varphi - |A, p_n|_k$ of the series $\sum \frac{C_n(t)\lambda_n P_n}{np_n}$ at a point can be ensured by local property.

It should be noted that if we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 2.3. In this case, the conditions (20)-(26) are obvious and the condition "$\left(\frac{\varphi_n p_n}{P_n}\right)$ is a non-increasing sequence" automatically satisfied.

We need the following lemmas for the proof of Theorem 3.1.

Lemma 3.2. [23, Lemma 2] *If the sequence (p_n) is such that the conditions (13) and (14) of Theorem 2.2 are satisfied, then*

$$\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right). \tag{27}$$

Lemma 3.3. [16, Lemma 4] *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then (λ_n) is non-negative and decreasing, and $n\Delta\lambda_n \to 0 \;$ as $\; n \to \infty$.*

Lemma 3.4. *Let $k \geq 1$. If (s_n) is bounded and all conditions of Theorem 3.1 are satisfied, then the series*

$$\sum_{n=1}^{\infty} \frac{a_n \lambda_n P_n}{np_n}, \tag{28}$$

is summable $\varphi - |A, p_n|_k$, *where* (λ_n) *is as in Theorem 2.2.*

Remark 3.5. Since (λ_n) is a convex sequence, therefore $(\lambda_n)^k$ is also convex sequence and

$$\sum \frac{1}{n}(\lambda_n)^k < \infty. \tag{29}$$

Proof of Lemma 3.4. Let (I_n) denote the A-transform of the series $\sum \frac{a_n \lambda_n P_n}{n p_n}$. Then, by (18) and (19), we have $\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v \lambda_v P_v}{v p_v}$. Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} \lambda_v P_v}{v p_v} \right) \sum_{r=1}^{v} a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n p_n} \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} \lambda_v P_v}{v p_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \\
&= \sum_{v=1}^{n-1} \frac{P_v \lambda_v \Delta_v(\hat{a}_{nv})}{v p_v} s_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} \Delta \lambda_v P_v}{v p_v} s_v \\
&+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta \left(\frac{P_v}{v p_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Lemma 3.4, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |I_{n,r}|^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{30}$$

First, by applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned}
&\sum_{n=2}^{m+1} \varphi_n^{k-1} |I_{n,1}|^k \\
&= \sum_{n=2}^{m+1} \varphi_n^{k-1} \left| \sum_{v=1}^{n-1} \frac{P_v \lambda_v \Delta_v(\hat{a}_{nv})}{v p_v} s_v \right|^k \\
&\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{v p_v} \left| \Delta_v(\hat{a}_{nv}) \right| (\lambda_v) |s_v| \right\}^k \\
&\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{v p_v} \right)^k |\Delta_v(\hat{a}_{nv})| (\lambda_v)^k |s_v|^k \right\} \times \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right\}^{k-1}
\end{aligned}$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}|\Delta_v(\hat{a}_{nv})|(\lambda_v)^k\right\}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}(\lambda_v)^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\Delta_v(\hat{a}_{nv})|$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}(\lambda_v)^k a_{vv}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}(\lambda_v)^k\frac{p_v}{P_v}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\left(\frac{P_v}{p_v}\right)^{k-1}\frac{1}{v^k}(\lambda_v)^k$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}v^{k-1}\frac{1}{v^k}(\lambda_v)^k$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\frac{1}{v}(\lambda_v)^k$$
$$= O(1) \quad as \quad m\to\infty,$$

by virtue of the hypotheses of Lemma 3.4.

Now, using the fact that $P_v = O(vp_v)$ by (13), and Hölder's inequality we have that

$$\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,2}|^k$$
$$= \sum_{n=2}^{m+1}\varphi_n^{k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}\Delta\lambda_v P_v}{vp_v}s_v\right|^k$$
$$\leq \sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}\frac{P_v}{vp_v}|\hat{a}_{n,v+1}|\Delta\lambda_v|s_v|\right\}^k$$
$$\leq \sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}|\hat{a}_{n,v+1}|\Delta\lambda_v|s_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\right\}^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}|\hat{a}_{n,v+1}|\Delta\lambda_v\times\left\{\sum_{v=1}^{n-1}\Delta\lambda_v\right\}^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}|\hat{a}_{n,v+1}|\Delta\lambda_v\right\}$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}\Delta\lambda_v\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\hat{a}_{n,v+1}|$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\Delta\lambda_v\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\Delta\lambda_v \\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Lemmas 3.3 and 3.4.

Now, since $\Delta\left(\frac{P_v}{vp_v}\right)=O\left(\frac{1}{v}\right)$ by Lemma 3.2, we have that

$$
\begin{aligned}
&\sum_{n=2}^{m+1}\varphi_n^{k-1}|I_{n,3}|^k \\
&= \sum_{n=2}^{m+1}\varphi_n^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\Delta\left(\frac{P_v}{vp_v}\right)s_v\right|^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})|s_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})^k|s_v|^k\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})^k \\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}(\lambda_{v+1})^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\frac{1}{v}(\lambda_{v+1})^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\frac{1}{v}(\lambda_{v+1})^k \\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Lemma 3.4.

Finally, we have that

$$
\begin{aligned}
\sum_{n=1}^{m}\varphi_n^{k-1}|I_{n,4}|^k &= \sum_{n=1}^{m}\varphi_n^{k-1}\left|\frac{a_{nn}P_n\lambda_n}{np_n}s_n\right|^k \\
&= O(1)\sum_{n=1}^{m}\varphi_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{n^k}\left(\frac{P_n}{p_n}\right)^k(\lambda_n)^k|s_n|^k \\
&= O(1)\sum_{n=1}^{m}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\frac{1}{n}(\lambda_n)^k
\end{aligned}
$$

$$= O(1) \quad as \quad m \to \infty,$$

by virtue of the hypotheses of Lemma 3.4.

This completes the proof of Lemma 3.4. ■

Proof of Theorem 3.1. The convergence of the Fourier series at $t = x$ is a local property of f (i.e., it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Theorem 3.1 is a consequence of Lemma 3.4. ■

4. Conclusions

If we take $\varphi_n = \frac{P_n}{p_n}$, then we get a theorem dealing with $|A, p_n|_k$ summability. If we take $a_{nv} = \frac{p_v}{P_n}$, then we have another result dealing with $\varphi - |\bar{N}, p_n|_k$ summability. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result dealing with $\varphi - |C, 1|_k$ summability. If we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for $|C, 1|_k$ summability.

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