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Some Recent Results of Fibonacci Numbers, Fibonacci Words and Sturmian Words*

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To Geltrude Mingrone for her wise and precious medical advice

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Abstract. The discovery of incommensurability is a fundamental accomplishment of the Pythagorean School, active in Crotone (Southern Italy) between the sixth and fourth centuries BC. A strong relation can be pointed out between the discovery and several concepts studied today in the field of discrete mathematics (Fibonacci Numbers, Fibonacci Word and Sturmian Words). In this paper, we will cover some of our previous results on Sturmian Words (a class of infinite words containing Fibonacci Word as main example) published in theoretical computer science journals, as well as some geometric constructions published in the Italian journal Nuova Secondaria, in 2005. Also, we will present some ideas that may be useful in mathematical education. Lastly, arguments supporting our thesis that Fibonacci Numbers were known by the Pythagorean School will be given.

Keywords: Incommensurability; Cutting sequence; Golden ratio; Fibonacci numbers; Fibonacci word.

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1. Introduction

Two fortuitous events such as our recent visit to the Ancient Theatre of Epidaurus, in Greece, and the recent restauration of *Medieval Church of San Nicola*, in Pisa, allow us to reconsider two old questions.

The *Ancient Theatre of Epidaurus* is situated within the archaeological site of the Sanctuary of Asklepios, in the Argolis prefecture of the Peloponnese. It is famous and admired for its symmetry, beauty and its exceptional acoustics and it is still hosting for dramatic performances. The cavea was built at the end of the fourth century BC with two seating sections: the lower one has 34 rows of benches, while the upper one has 21 of them. Now, 21 and 34 are Fibonacci Numbers

A marble intarsia on the ancient main entrance of the *Medieval Church of San Nicola* in Pisa attracted the attention of Pietro Armienti, Professor at the department of *Scienze della Terra* at the *Università* of Pisa. Professor Armienti says: *The intarsia reveals the direct influence of the great Pisan mathematician Leonardo Fibonacci due to the presence of circles whose radii represent the first nine elements of the Fibonacci's Sequence and which were arranged to depict some properties of the sequence.* See [2]. These deep observations are very strong arguments in support to the thesis that the intarsia contains a geometric representation of the sequence of Fibonacci Numbers 1, 1, 2, 3, 5, 8, 13, 21, 34 and 55.

Antonio Albano, which is as well a Professor at the *Università* of Pisa, with regards to the tarsia says: *It was not until 1611 that J. Kepler showed that the ratio of two successive numbers in the sequence is alternately higher and lower than the golden section and that, with the progress of sequence, the golden section is the limit of this ratio. Therefore, without the designers of the lunette even being aware of it, they created a fascinating relationship between the tarsia devoted to the Fibonacci Sequence and the linear geometric decoration devoted to the golden section*, as discussed in [1].

Therefore, after visiting the Ancient Theatre of Epidaurus and the reading of the very interesting papers of Professor Armienti and Professor Albano, we reconsidered the following two old questions: when were for the first time the Fibonacci Numbers mathematically well defined and who defined them?

Conventional wisdom suggests that the Fibonacci Numbers were first introduced in 1202 by Leonardo of Pisa, better known today as Fibonacci, in his book Liber abbaci ... The intent of this article is to offer a plausible conjecture as to the origin of the Fibonacci Numbers. See [16].

On the other hand, in [30] we can read that the sequence of the Fibonacci Numbers in India is *at least as old as the origin of the metrical sciences of Sanskrit and Prakrit poetry* in the 6^{th} century A.D.

Our paper contains comments on the relationship between golden ratio and the Fibonacci Numbers. We try to imagine the work of the Pythagorean School and the first steps that led to their discovery of the irrational number Φ, the golden ratio. We suppose that before discovering that no common measure was possible for the side and diagonal of a regular pentagon, in particular they verified that: i) the side was not a common measure, ii) the half of the side was not a common measure, iii) the third of the side was not a common measure and so on. We analyze these "unsuccessful" attempts, during this analysis we realize that the Fibonacci Numbers appear and we conclude that probably the Pythagorean School also noticed ... the Fibonacci Numbers (the analysis of these failed attempts naturally leads Diophantine equations: we limit ourselves here to simply quote [17] and we explicitly mention the article [32] and the exercise 6.44 of [11]).

Finally we would like to point out that this paper is solely based on some remarks about the arguments used by the Pythagorean School and not on historical documents.

2. Structure of the Paper

After this section, the following pages contain six other sections. In the *Preliminaries* we recall the necessary notions of combinatorics on words: we use the definitions and the notations of theoretical computer science; see for example [15]. In Section 4, we recall the definitions of *Fibonacci Numbers and Words* while some of our results concerning them are in Section 5. In Section 6 we recall that *Fibonacci Word* is the main example of *Sturmian Words* and, remembering the content of some of our previous articles, we try to show that "reasonable" properties of the Fibonacci Word can be extended to all Sturmian Words.

For a long time we have been concerned with mathematical education (see, for example, [21] and [22]), which continues today. In Section 7 of this paper we recall some geometric constructions published in Nuova Secondaria; we recall some notions of theoretical computer science (mainly the notion of cutting sequence) and some hints to professors of secondary school in order to prepare a better presentation of *irrational numbers.* In Section 8 we recall how the Pythagorean School discovered the irrationality of Φ and we will give some arguments supporting our thesis that Fibonacci Numbers was known by the Pythagorean School.

3. Preliminaries

A *finite word* (or, in short, a *word*) $u = a_1 a_2 \cdots a_k$ over an *alphabet* $\mathcal{A}$ is a finite sequence of elements of $\mathcal{A}$.

The length of u is k and is denoted by $|u|$. For example, $u = aabba$ is a word over the alphabet $\{a, b\}$ and $|u| = 5$.

An *infinite word* over an *alphabet* $\mathcal{A}$ is an infinite sequence of elements of $\mathcal{A}$. Concerning the importance of the notion of infinite word see, for example, [12], [25] and [26].

An *infinite periodic word* is an infinite word s such that there exists a finite word u such that $s = uu \cdots u \cdots$ and s is denoted by u^ω. For example $aabaab \cdots aab \cdots$, the word obtained by the juxtaposition of infinitely many copies of the finite word aab, is an infinite periodic word and is denoted by $(aab)^\omega$.

An *infinite ultimately periodic word* is an infinite word s such that there exists finite words p and u such that $s = puu \cdots u \cdots$ and s is denoted by pu^ω. For example $paabaab \cdots aab \cdots$, the word obtained by the juxtaposition of the finite word p and of infinitely many copies of the finite word aab, is an infinite ultimately periodic word and is denoted by $p(aab)^\omega$.

A *recurrent word* is an infinite word s such that when it contains a factor u then it contains infinitely many occurrences of u.

A *uniformly recurrent word* is an infinite word s such that for each factor u of s there exists an integer $N(u)$ such that each factor of s having length at least $N(u)$ contains an occurrence of u.

Clearly each uniformly recurrent word is also a recurrent word.

A finite word v is a *prefix* of a finite (resp., infinite) word w if there exists a finite (resp., infinite) word v' such that $w = vv'$.

4. Fibonacci Numbers and Words

Let $\{a, b\}^+$ be the free semigroup on $\{a, b\}$ and φ be the morphism from $\{a, b\}^+$ into $\{a, b\}^+$ defined as follows $\varphi(a) = ab$ and $\varphi(b) = a$. The n–th (finite) Fibonacci Word f_n is defined in the following way: $f_0 = b$ and, for each $n \geq 0$, $f_{n+1} = \varphi(f_n)$. In particular, we have: $f_1 = a$, $f_2 = ab$, $f_3 = aba$, $f_4 = abaab$, $f_5 = abaababa$, $f_6 = abaababaabaab$, $f_7 = abaababaabaababaababa \ldots$. It is clear that, for each $n \geq 2$, f_n is the product (juxtaposition) $f_{n-1}f_{n-2}$ of f_{n-1} and f_{n-2}.

Moreover, for each $n \geq 0$, $|f_n|$ is the n-th element F_n of the Fibonacci Sequence $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, $F_5 = 8$, $F_6 = 13$, $F_7 = 21 \ldots$. See [9] (and, also, [6], [7], [18], [20], [23], [33], ...).

Now, consider a fixed regular pentagon having side s and diagonal d; in this paper we propose a particular interpretation of the sequence of Fibonacci Numbers: to each $i \geq 0$ we associate a unit of measure such that, with respect to it, F_i is exactly the measure of s and F_{i+1} is quite exactly the measure of d. Recall that $\frac{F_{n+1}}{F_n}$ tends to $\Phi = \frac{\sqrt{5}+1}{2}$, the *golden ratio*, when n goes to infinity. The golden ratio is exactly the ratio between diagonal and side of regular pentagon.

Note that, for each $n \geq 1$, the n–th (finite) Fibonacci Word f_n is a prefix of f_{n+1}, the $(n+1)$–th (finite) Fibonacci Word. There exist a unique infinite word, namely the (infinite) Fibonacci Word f, such that, for each $n \geq 1$, f_n is a prefix of f and we have

$$f = abaababaabaababaababaabaababaabaababaababa \ldots .$$

5. Some Results on Fibonacci Words

In the Lothaire book the sequence of Fibonacci Words is presented in Chapter 1 (see [15]), because of its central role in Combinatorics on words. See also [28] and [29].

For each $n \geq 2$, we denote g_n the product $f_{n-2}f_{n-1}$ and h_n the longest common prefix of f_n and g_n. In particular, we have: $g_2 = ba$, $g_3 = aab$, $g_4 = ababa$, $g_5 = abaabaab$, ... and $h_2 = 1$ (= the empty word), $h_3 = a$, $h_4 = aba$, $h_5 = abaaba \ldots$.

The following "*near commutativity property*" (see [13], [14]) is very interesting and intriguing.

Lemma. *For each $n \geq 2$, we have the following statements:*

(i) *$f_n = h_n xy$ and $g_n = h_n yx$, where $x, y \in \{a, b\}$, $x \neq y$, and if n is even then $xy = ab$, and if n is odd then $xy = ba$;*

(ii) *h_n is a palindrome.*

It is easy but important to prove the following result.

The infinite Fibonacci Word is uniformly recurrent and not ultimately periodic.

These two just recalled results can be usefully presented to secondary school students.

If w is a finite non empty word, we can consider the infinite periodic word p_w, defined as follows: $p_w = www \cdots w \cdots = w^\omega$. We say that u is a *fractional power* of the (finite) word w if u is a prefix of p_w. We say that w is the *base* and $k = \frac{|u|}{|w|}$ is the *exponent* of the fractional power and we write $u = w^k$. In general k is a rational number, but if k is an integer we have the usual notion of the power of a word.

For the two following Propositions, see [19].

Proposition 5.1. *No fractional power with exponent greater than $1 + \frac{\sqrt{5}+1}{2}$ can be a left factor of the Fibonacci Word f. More precisely, if vvu is a fractional power which is a left factor of f then $v = f_n$ for some n and $|vvu| \leq |f_n| + |f_n| + |f_{n-1}| - 2$*

Proposition 5.2. *The Fibonacci Word f contains no fractional power with exponent greater than $2 + \frac{\sqrt{5}+1}{2}$ and, for any real number $\epsilon > 0$, it contains a fractional power with exponent greater than $2 + \frac{\sqrt{5}+1}{2} - \epsilon$.*

This study of the fractional powers of the Fibonacci Word has been very useful for the analogous study of the fractional powers of the Sturmian Words.

The Fibonacci Word is also strictly related to the theory of codes. For the definitions of prefix codes and bifix codes, see the book of Berstel and Perrin [5]. For the following three propositions see [24].

Proposition 5.3. *Let*
$$f = u_0 u_1 \dots u_i \dots$$
be the factorization of f *such that* $|u_i| = F_{2i+1}$. *Then* $\{u_i \mid i \geq 0\}$ *is a prefix code.*

Proposition 5.4. *Let*
$$f = v_0 v_1 \dots v_i \dots$$
be the factorization of f *such that* $|v_i| = F_{2(i+1)}$. *Then* $\{v_i \mid i \geq 0\}$ *is a prefix code.*

Proposition 5.5. *Let* $n \geq 4$. *Let* $f = w_0 w_1 \dots w_i \dots$ *be the factorization of* f *such that* $|w_0| = F_n + F_{n-2} - 1$ *and, for each* $i \geq 1$, $|w_i| = F_{n+2(i-2)-1} + 2F_{n+2(i-1)}$. *Then* $\{w_i \mid i \geq 0\}$ *is a bifix code.*

6. Fibonacci Words and Sturmian Words

For the definition of the Sturmian Words, see [4], and for the definition of Episturmian Words, see [8].

The Fibonacci Word f is the main example of a Sturmian Words and some of its "reasonable" properties hold for all Sturmian Words and can even be extended to all Episturmian Words:

1) For each $n \geq 2$, h_n is a prefix of f. Being h_n the longest common prefix of f_n and g_n one easily has
$$h_n ab \sim h_n ba,$$
i.e. $h_n ab$ is a conjugate of $h_n ba$. For example, $abaab \sim ababa, \dots$. In [25] we proved that the relation
$$wab \sim wba$$
characterizes the palindromic prefixes of standard Sturmian Words.

2) It is clear that the first occurrence of the palindrome a is in central position in the prefix a of f. It is also clear that the first occurrence of the palindrome b is in central position in the prefix $h_4 = aba$ of f. Similarly, the first occurrence of aa is central in $h_5 = abaaba$ and so on. These very easy remarks were at the origin of the foundamental definition of Episturmian Word [8].

3) It is again easy to verify that aa, aab, $aaba$ are respectively the smallest words in the lexicographic order among the factors of f of lenght 2, 3, 4 .... These remarks are, for example, at the origin of the results of [26] on Sturmian and Episturmian Words.

7. Irrational Numbers

The following is one of the most beautiful definitions in mathematics. *Those magnitudes measured by the same measure are said to be commensurable, but those which admit no common measure are said to be incommensurable* (definition 1 of Book X of *Elements* of Euclid). Incommensurability is also one of the most difficult arguments for high school students and, even, for university students.

Perhaps the difficulty stems from its abstract character and the absence of supporting images.

We have given some suggestions to secondary school teachers, for example in [27] published in Italian in *Nuova Secondaria*. We suggested to present the notion of *cutting sequence*.

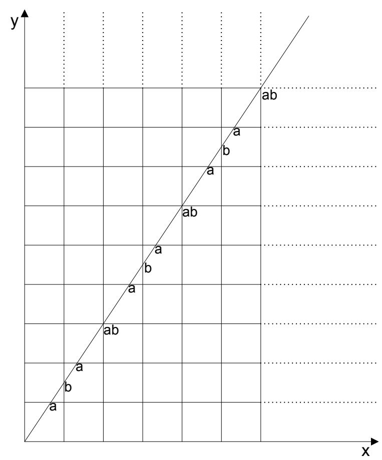

Figure 1: A periodic cutting sequence.

In Figure 1 we see the cutting sequence

$$abaababaababaab = (abaab)^3$$

which is the prefix of the periodic cutting sequence $(abaab)^\omega$.

In Figure 2 we see the prefix

$$abaababaabaaba$$

of the cutting sequence corresponding to $y = \Phi x$ where Φ is the *golden ratio*, $\frac{\sqrt{5}+1}{2}$, the ratio between diagonal and side of regular pentagon. Note that, since Φ is an irrational number, the line $y = \Phi x$ will never cross a point with integer coordinates different by $(0, 0)$.

We also recall that in [27] we presented four new examples of geometric constructions of incommensurable segments. One of them concerns the regular octagon (see Figure 3). We proved that the sides of the external octagon and of the internal octagon are incommensurable. Another of them concerns the regular

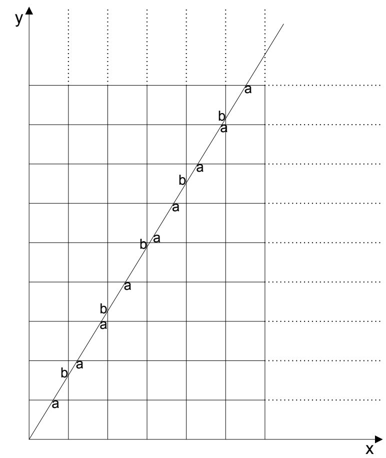

Figure 2: A non-periodic cutting sequence, the Fibonacci Word.

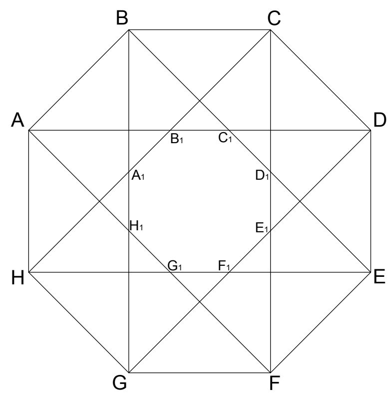

Figure 3: The sides of the two regular octagons are incommensurable.

dodecagon (see Figure 4). We proved that the sides of the external dodecagon and of the internal dodecagon are incommensurable.

8. The Number Φ and the Fibonacci Numbers

Pythagoras probably knew and taught the substance of what is contained in the first two books of Euclid ... (see [3]).

In particular, the four following Propositions (that correspond to the Propositions 18, 19, 20 and 32 of Book 1 of Euclid's Elements, see [10]) were very well known to the Pythagorean School.

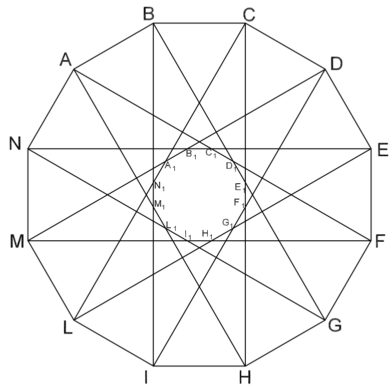

Figure 4: The sides of the two regular dodecagons are incommensurable.

Proposition 8.1. *In any triangle, the greater side subtends the greater angle.*

Proposition 8.2. *In any triangle, the greater angle is subtended by the greater side.*

Proposition 8.3. *In any triangle, the sum of any two sides is greater than the remaining one.*

Proposition 8.4. *In any triangle, the sum of the three interior angles equals two right angles.*

The following proposition is the fundamental argument in one of the Pythagorean possible proofs of the incommensurability of side and diagonal of a regular pentagon. So, we suppose that this proposition also was very well known by the Pythagorean School [27].

Pythagorean Proposition. *A strictly decreasing sequence of positive integers is necessarily finite.*

We now shortly recall how the Pythagorean School could have used the previous proposition in their proof of incommensurability.

Look at Figure 5: a common measure of side and diagonal of the pentagon $ABCDE$ implies the existence of two different integers b and a such that

$$b : a = a : (a + b)$$

and, consequently, that

$$b(a + b) = a^2.$$

Now, two integers b and a satisfying $b(a + b) = a^2$ cannot exist!

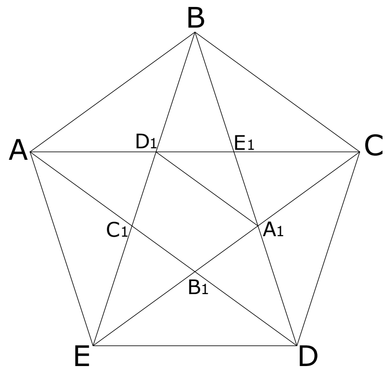

Figure 5: Side and diagonal of a regular pentagon have not a common measure.

Indeed, if a **and** b **are both even** then, by the Pythagorean Proposition, also occurs a **smaller case** where at least one of the two integers a and b is odd. Moreover if a **and** b **are both odd** then $(a+b)b$ is even and a^2 is odd. Contradiction. If a **is even and** b **odd** then $(a+b)b$ is odd and a^2 is even. Contradiction. And, finally, if a **is odd and** b **even** then $(a+b)b$ is even and a^2 is odd. Contradiction.

Therefore a and b cannot be both integers and so *side* and *diagonal* of the *regular pentagon* cannot have a common measure and so the Pythagorean School proved the

Theorem. *Side and diagonal of a regular pentagon are incommensurable.*

Sure, this foundamental result may not be reached at the first attempt. More probably, the Pythagorean School tried several times to find a common measure of the *side* and *diagonal* of the *regular pentagon*, i.e., they tried to find two **integers** k and n such that the segment contained exactly k times in the side was also contained exactly n times in the diagonal. In other words, for a probably long period, they tried to find an exact integer solution of the equation $k(n+k)=n^2$. Clearly we don't know how many attempts they made but at a moment (when they proved that *the side and diagonal of a regular pentagon are incommensurable*, see [31]) they realized that any further attempts would be in vain. So, the Pythagorean School formalized the impossibility to have integer solutions for the equation $k(n+k)=n^2$.

Now, let us see in detail the failed attempts concerning the two above mentioned **integers** k and n.

First step: $k=1$ (This attempt concerns the side as a common measure). Now, $1(1+1)$ and 1^2 differ by 1 and the same happens for $1(1+2)$ and 2^2. So, easily the equation $k(n+k)=n^2$ has no integer solutions with $k=1$ and the required common measure cannot be the side. This conclusion arises also by the above recalled Propositions of Book 1 of Euclid's Elements that suggest directly: the measure of the side must be strictly smaller than the measure of the diagonal and the measure of the diagonal must be strictly smaller than two times the side.

Second step: $k = 2$ (This attempt concerns the half of the side as a common measure). Necessarily $2 < n < 4$. So n must be 3. But $2(2+3)$ and 3^2 differ by 1. Therefore, the half of the side also fails as a common measure.

Third step: $k = 3$ (This attempt concerns the third part of the side as a common measure). Necessarily $3 < n < 6$. So n must be 4 or 5. But $3(3+4)$ and 4^2 differ by 5 while $3(3+5)$ and 5^2 differ by 1 and also the third part of the side fails as a common measure.

Fourth step: $k = 4$ (This attempt concerns the fourth part of the side as a common measure). Necessarily $4 < n < 8$. So n must be 5, 6 or 7: when $n = 5$, then $k(n+k)$ and n^2 differ by 11; when $n = 6$, then $k(n+k)$ and n^2 differ by 4 (note that this case is clearly related to a previous one: $k = 2$, $n = 3$), when $n = 7$, then $k(n+k)$ and n^2 differ by 5. So the fourth part of the side fails as a common measure.

Fifth step: $k = 5$ (This attempt concerns the fifth part of the side as a common measure). Necessarily $5 < n < 10$. So n must be 6, 7, 8 or 9: when $n = 6$, then $k(n+k)$ and n^2 differ by 19; when $n = 7$, then $k(n+k)$ and n^2 differ by 11; when $n = 8$, then $k(n+k)$ and n^2 differ by 1; when $n = 9$, then $k(n+k)$ and n^2 differ by 11. So the fifth part of the side fails as a common measure.

We could stop here but we continue with the help of a program prepared by my brother Mario (while the Pythagorean School has continued with many patience and long calculations) and we verify that, for each positive integer k, $1 \leq k \leq 1000$, we never found an integer n, $k+1 \leq n \leq 2k-1$ such that $k(n+k)$ and n^2 are equal (**and this is in perfect accord with the Theorem: side and diagonal of a regular pentagon are incommensurable**).

Now, remark that in some cases (**and only in these cases**) the difference between $k(n+k)$ and n^2 (**that never could be** 0), is the smallest possible, i.e., it is exactly 1.

We report these values in the following table, which was obtained using a program prepared by my brother Mario.

This program selects all the k, $1 \leq k \leq 1000$, such that there exist n such that $k+1 \leq n \leq 2k-1$ and that $k(n+k)$ and n^2 differ just by 1. We suppose that the Pythagorean School (which loved to play with numbers) has remarked these cases.

The Fibonacci Numbers are:

i) in the first column,

ii) in the second column with a shift of one place

and

iii) in the third column with a shift of two places.

Moreover, the squares of Fibonacci Numbers are in the fifth column, whereas in the fourth column there are alternatively the predecessor and the successor

k	n	$n+k$	$k(n+k)$	n^2
1	1	2	1^2+1	1^2
1	2	3	2^2-1	2^2
2	3	5	3^2+1	3^2
3	5	8	5^2-1	5^2
5	8	13	8^2+1	8^2
8	13	21	13^2-1	13^2
13	21	34	21^2+1	21^2
21	34	55	34^2-1	34^2
34	55	89	55^2+1	55^2
55	89	144	89^2-1	89^2
89	144	233	144^2+1	144^2
144	233	377	233^2-1	233^2
233	377	610	377^2+1	377^2
377	610	987	610^2-1	610^2
610	987	1597	987^2+1	987^2
987	1597	2584	1597^2-1	1597^2

of these squares.

Is it possible that they did not notice them? It is certainly possible. On the other hand, even in this case, the previous remarks may be helpful to explain to students of secondary school or of the university the deep link between the Fibonacci Numbers and the irrational number Φ.

Now, there are good reasons to conclude that the Pythagorean School knew the Fibonacci Numbers.

Yet, this is not what we have demonstrated in our paper. We have shown that, *almost certainly*, the Pythagoreans worked with them, but we do not have documents allowing us to retrieve such a conclusion. In fact, we only have the combinatorial argument attributed to the Pythagorean School for the discovery of Φ.

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