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On Modules with Insertion Factor Property

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Abstract. A right ideal I of a ring R is an IFP right ideal if for any $a, b \in R$, if $ab \in I$, then $aRb \subset I$. A ring R is called an IFP ring if 0 is an IFP ideal of R. In this paper, we will introduce the notion of IFP modules as a generalization of IFP rings. Many good properties of IFP rings can be transferred to IFP modules. We also give a generalization of Anderson's Theorem.

Keywords: IFP modules; IFP rings; Fully IFP modules; Fully IFP rings; Anderson's Theorem.

1. Introduction

Prime ideals play an interesting role in studying the structure of rings. Many authors want to transfer this notion to modules. However, with these notions they do not get some properties similar to that of prime ideals in rings. Recently, we have successfully introduced a new notion of prime submodules for a given

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module. Many good properties of prime ideals can be transferred to prime submodules.

The main purpose of this paper is to investigate the class of IFP submodules for a given module and we use it to generalize the Anderson's Theorem, following that, if R is a commutative ring and if every minimal prime ideal over an ideal I of R is finitely generated, then there are only a finitely many minimal prime ideals over I. In 2008, C. Huh, N.K. Kim and Y. Lee [7], generalized Anderson's Theorem for non-commutative rings. They proved that this result is also true for homomorphically IFP-rings. In this paper, we introduce the notion of IFP-modules and fully IFP-modules and prove that if M is a finitely generated quasi-projective and fully IFP right R-module which is a self-generator and if every minimal prime submodules over a fully invariant submodule U of M is finitely M-generated, then there are only finitely many minimal prime submodules over U.

Throughout this paper, all rings are associative ring with identity $1 \neq 0$ and all modules are unitary right R-modules. We write M_R (resp., ${}_RM$) to indicate that M is a right (resp., left) R-module and $S = End(M_R)$, its endomorphism ring. A proper submodule X of M is called a *fully invariant* submodule of M if for any $f \in S$, we have $f(X) \subset X$. Following [7], a fully invariant proper submodule X of M is called *prime submodule* of M if for any ideal I of S and any fully invariant submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$. The fully invariant submodule X of M is called a *strongly prime submodule* of M if for any $\varphi \in S$ and $m \in M$, $\varphi(m) \in X$ imples that $\varphi(M) \subset X$ or $m \in X$. Clearly, every strongly prime submodule of M is prime.

A prime submodule X of M is called a *minimal prime submodule* if there are no prime submodules of M properly contained in X. A right R-module M is called a *self-generator* if it generates all its submodules. General background can be found in [2, 3, 10, 4, 5].

2. IFP Modules and Their Endomorphism Rings

Definition 2.1. A submodule of a right R-module M is said to have *"insertion factor property"* (briefly, an IFP-submodule) if for any endomorphism φ of M and any element $m \in M$, if $\varphi(m) \in X$, then $\varphi Sm \in X$. A right ideal I is an IFP-right ideal if it is an IFP-submodule of R_R, that is for any $a, b \in R$, if $ab \in I$, then $aRb \subset I$. A right R-module M is called an IFP-module if 0 is an IFP-submodule of M. A ring R is IFP if 0 is an IFP-ideal.

By definition, we can see that any intersection of a family of IFP-submodules is again IFP. Clearly, every ideal in a commutative ring is IFP. For a non-commutative ring, we will show that an ideal I of the ring R is strongly prime if and only if it is prime and IFP.

We start with the following proposition.

Proposition 2.2. *Let X be a submodule of a right R-module M. If X is an-IFP submodule and M is quasi-projective,then M/X is an IFP-module. Conversely, if M/X is IFP and X is fully invariant, then X is an IFP-submodule of M.*

Proof. Suppose that X is an IFP-submodule of M and $\overline{\varphi}(\bar{m}) = 0$, where $\bar{\varphi} \in \bar{S} = \mathrm{End}(M/X)$ and $\bar{m} \in M/X$. By the quasi-projectivity of M, there is a $\varphi \in S$ such that $\nu\varphi = \overline{\varphi}\nu$, where $\nu : M \to M/X$ is the natural epimorphism. It follows that $\varphi(m) \in X$. Let $\overline{\xi}$ be any endomorphism of M/X. Then as above, there is $\xi \in S$ such that $\overline{\xi}\nu = \nu\xi$. By assumption, $\varphi\xi(m) \in X$. This leads to $\bar{\varphi}\bar{\xi}(\bar{m}) = 0$, proving that M/X is an IFP module.

Suppose that X is a fully invariant submodule of M, with M/X is IFP. Let $\varphi(m) \in X$, where $\varphi \in S$ and $m \in M$. Since X is fully invariant, there is $\bar{\varphi} \in \bar{S}$ such that $\bar{\varphi}\nu = \nu\varphi$. It follows that $\bar{\varphi}(\bar{m}) = 0$. By assumption, we get $\bar{\varphi}\bar{\xi}(\bar{m}) = 0$, for any $\xi \in S$, where $\bar{\xi}\nu = \nu\xi$. This leads to the fact that $\varphi S(m) \subset X$, proving our proposition. ∎

Lemma 2.3. *If X is an IFP submodule of M, then I_X is an IFP right ideal of S. The converse is true if M is a self-generator.*

Proof. Let $\varphi\psi \in I_X$. Then, $\varphi(\psi(m)) \in X$ for any $m \in M$. By hypothesis, $\varphi\xi(\psi(m)) \in X$, for any $\xi \in S$ and any $m \in M$. It follows that $\varphi S\psi \subset I_X$, showing that I_X is IFP.

Conversely, let $\varphi(m) \in X$, where $\varphi \in S$, and $m \in M$. Let $U = \varphi^{-1}(X)$. Since M is a self generator, we get $U = \sum_{i \in A} \psi_i(M)$, for some subset A of S. Hence $m = \sum_{k=1}^{n} \psi_{i_k}(m_{i_k})$, where $i_k \in A$ and $m_{i_k} \in M$. Thus $\varphi(m) = \sum_{k=1}^{n} \varphi\psi_{i_k}(m_{i_k})$. For any $\xi \in S$, we get $\varphi\xi\psi_{i_k} \in I_X$ for $k = 1, \ldots, n$, by assumption. It follows that $\varphi S(m) \subset X$, proving our lemma. ∎

Proposition 2.4. *Let X be a fully invariant proper submodule of a right R-module M. Then X is a strongly prime submodule of M if and only if it is prime and IFP.*

Proof. Suppose that X is a strongly prime submodule of M. For any $\varphi \in S$ and for any $m \in M$, if $\varphi S(m) \subset X$, then $\varphi(m) \in X$. Since X is a strongly prime submodule, we have either $\varphi(M) \subset X$ or $m \in X$. This implies that X is a prime submodule. We assume that $\varphi(m) \in X$. We need to prove that $\varphi S(m) \subset X$. Since $\varphi(m) \in X$, we can see that either $\varphi(M) \subset X$ or $m \in X$. If $m \in X$, then we have $g(m) \in g(X) \subset X$, for all $g \in S$. This means that $S(m) \subset X$. Therefore $\varphi S(m) \subset X$. Suppose that $\varphi(M) \subset X$. We can see that $\varphi S(M) = \varphi(M) \subset X$. This follows that $\varphi S(m) \subset X$, as desired.

Suppose that X is a prime submodule and has IFP. If $\varphi(m) \in X$, then we want to show that either $\varphi(M) \subset X$ or $m \in X$. Since X has IFP, we have $\varphi S(m) \subset X$. By the primeness of X, we can see that either $\varphi(M) \subset X$ or $m \in X$. This shows that X is a strongly prime submodule, as required. ∎

We now study the relationship between an IFP module and its endomorphism ring.

Theorem 2.5. *Let M be a right R-module and S its endomorphism ring. If M is an IFP-module, then S is an IFP-ring. The converse is true if M is a self-generator.*

Proof. Let $\varphi\psi = 0 \in S$. Then $\varphi(\psi(m)) = 0$ for all $m \in M$. If M is IFP, then for any $\xi \in S$, we have $\varphi\xi(\psi(m)) = 0$ for all $m \in M$. It follows that $\varphi S\psi = 0$, showing that S is an IFP-ring.

Conversely, since $I_0 = \{f \in S | f(M) = 0 \subset M\} = 0$, is an IFP-ideal, it follows that 0 is an IFP-submodule of M by Lemma 2.3, proving our theorem. ■

The following theorem gives some characterizations of IFP modules.

Theorem 2.6. *Let M be a right R-module and $S = \mathrm{End}(M)$. The following conditions are equivalent:*

(i) *M is an IFP-module;*
(ii) *For any $m \in M$, $l_S(m)$ is an ideal of S;*
(iii) *For any $\varphi \in S$, $\ker(\varphi)$ is a fully invariant submodule of M;*

If M is quasi-projective, then the above conditions are equivalent to:

(iv) *For any $\varphi \in S$, $\ker(\varphi)$ is an IFP-ideal of S;*
(v) *$M/\ker(I)$ is an IFP-module for any subset I of S;*

If M is a self-generator, then the above conditions (i), (ii) *and* (iii) *are equivalent to:*

(vi) *For any $m \in M$, $l_S(m)$ is an IFP-ideal of S;*
(vii) *$S/l_S(A)$ is an IFP-ring for any subset $A \subset M$.*

Proof. (i)⇒(ii). Let $\varphi \in l_S(m)$ and $\xi \in S$, where $m \in M$. The $\varphi(m) = 0$. By (i), we have $\varphi\xi(m) = 0$. It follows that $\varphi\xi \in l_S(m)$, proving that $l_S(m)$ is a two-sided ideal of S.

(ii)⇒(i). Let $\varphi(m) = 0$ where $\varphi \in S$ and $m \in M$. Since $l_S(m)$ is an ideal, for any $\xi \in S$, we have $\varphi\xi \in l_S(m)$. This shows that M is an IFP-module.

(i)⇒(iii). Let $\varphi \in S$. For any $m \in \ker(\varphi)$, we get $\varphi(m) = 0$. By assumption, $\varphi\xi(m) = 0$ for all $\xi \in S$. This shows that $\xi(\ker(\varphi)) \subset \ker(\varphi)$, i.e., $\ker(\varphi)$ is a fully invariant submodule of M.

(iii)⇒(i). With $\varphi(m) = 0$ we have $m \in \ker(\varphi)$, which is fully invariant in M. Thus for any $\xi \in S$, $\xi(m) \in \ker(\varphi)$, and hence $\varphi\xi(m) = 0$, proving that M is an IFP-module.

(i)⇒(iv). Let $\psi \in S$, $m \in M$ such that $\psi(m) \in \ker(\varphi)$. Then $(\varphi\psi)(m) = 0$. By (i), we get that $\varphi\psi\xi(m) = 0$ for all $\xi \in S$. This shows that $\psi S(m) \subset \ker(\varphi)$, showing that $\ker(\varphi)$ is an IFP-submodule of M.

(iv)⇒(v). We note that $\ker(I) = \bigcap_{f\in I}\ker(f)$, and each $\ker(f)$ is an IFP-submodule of M, and hence $\ker(I)$ is an IFP-submodule of M. Since$M$ is quasi-projective, by applying Proposition 2.2, we can see that $M/\ker(I)$ is an IFP-module.

(v)⇒(i). This part is clear by taking $I = \{1_M\}$, 1_M is the identity map of M.

(i)⇒(vi). Let $m \in M$ and $\varphi\psi \in l_S(m)$, where $\varphi, \psi \in S$. Then $\varphi(\psi(m)) = 0$. By assumption, $\varphi S\psi(m) = 0$. It follows that $\varphi S\psi \subset l_S(m)$, as required.

(vi)⇒(vii). Since $l_S(A) = \bigcap_{a\in A} l_S(a)$ for any subset A of M, we see that $l_S(A)$ is an IFP ideal of S, and therefore $S/l_S(A)$ is an IFP-ring.

(vii)⇒(i). Taking $A = M$, then it is clear that S is an IFP-ring. Since M is a self-generator, by applying Theorem 2.5 we can see that M is an IFP-module.

■

The following Corollary is a direct consequence of the above theorem.

Corollary 2.7. *For a ring R, the following conditions are equivalent:*

(i) *R is an IFP-ring;*
(ii) *For any $a \in R$, $l_R(a)$ is an ideal of R;*
(iii) *For any $a \in R$, $r_R(a)$ is an ideal of R;*
(iv) *For any $a \in R$, $l_R(a)$ is an IFP-ideal of R;*
(v) *For any $a \in R$, $r_R(a)$ is an ideal of R;*
(vi) *For any $a \in R$, $R/r_R(a)$ is an IFP-ring;*
(vii) *For any $a \in R$, $R/l_R(a)$ is an IFP ring.*

3. A Generalization of Anderson's Theorem

In 1994, D.D. Anderson proved that if R be a commutative ring and I an ideal of R, and all the prime ideals minimal over I are finitely generated, then there are only finitely many prime ideals minimal over I. In [6], C. Huh, N.K. Kim and Y. Lee extended Anderson's theorem for noncommutative rings, following that, for a homomorphically IFP ring R and a proper ideal I of R, if every prime ideal minimal over I is finitely generated, then there are only finitely many prime ideals minimal over I. In this section, we introduce the notion of *fully IFP* modules and give a generalization of Anderson's Theorem.

Definition 3.1. *A module M is called a fully IFP module if M/U is IFP for every proper fully invariant submodule U of M. A ring is called a fully IFP ring if R/I is IFP for every proper ideal I of R.*

Note that fully IFP rings is called homomorphically IFP rings in [6]. Next, we study the relationship between a fully IFP module and its endomorphism ring. By the definition, if M is fully IFP, then it is an IFP-module.

Theorem 3.2. *Let M be a finitely quasi-projective right R-module which is a self-generator. Then, M is a fully IFP if and only if S is a fully IFP ring.*

Proof. Suppose that M is a fully IFP module. Let J be a proper ideal of S. It follows from [10, Theorem 18.4] that $J(M)$ is a proper fully invariant submodule of M. By assumption, $M/J(M)$ is IFP. It follows from Proposition 2.2 that $J(M)$ is an IFP-submodule of M. By Lemma 2.3, $I_{J(M)}$ is an IFP right ideal of S. Hence S/J is an IFP, proving that S is a fully IFP ring.

Conversely, suppose that S is a fully IFP ring. Let U be a proper fully invariant submodule of M. Then clearly I_U is a proper ideal of S. By assumption, we have S/I_U is an IFP. Since M is a self-generator, it would imply that M/U is an IFP. Thus M is a fully IFP module. The proof of our theorem is now complete. ∎

Proposition 3.3. [8, Proposition 2.1] *Let M be a right R-module which is a self-generator. Then,*

(i) *If X is a minimal prime submodule of M, then I_X is a minimal prime ideal of S.*

(ii) *If P is a minimal prime ideal of S, then $X := P(M)$ is a minimal prime submodule of M and $I_X = P$.*

For following theorem, we refer to Huh et al. [6].

Theorem 3.4. [6, Theorem 3] *Let R be a homomorphically IFP ring and I a proper ideal of R. If every minimal prime ideal over I is finitely generated then there are only finitely many minimal prime ideals over I.*

Motivating this result we can prove the following theorem as a generalization of Anderson's theorem.

Theorem 3.5. *Let M be a fully IFP which is a self-generator and U, a proper fully invariant of M. Suppose that every minimal prime submodule over U is finitely generated, then there are only finitely many minimal prime submodules over U.*

Proof. Since M is a fully IFP module, by Theorem 3.2, S is a fully IFP ring. By [10, Theorem 18.4], I_U is a proper ideal of S. By Theorem 3.4, there are only finitely many minimal prime ideals over I_U. Applying Proposition 3.3, we can see that there are only finitely many minimal prime submodules over U. This completes our proof. ∎

The following Corollary is a direct consequence of the above theorem.

Corollary 3.6. *Let M be a homomorphically IFP module which is a self-*

generator. If every minimal prime submodule of M is finitely generated, then there are only finitely many minimal prime submodules of M.

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