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A Note on Noetherian Modules

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Abstract. In this note, we introduce a class of nearly prime submodules and prove that a finitely generated right R-module M is Noetherian if and only if every nearly prime submodule is finitely generated.

Keywords: Nearly prime; Noetherian module; Cohen's theorem.

1. Introduction and Preliminaries

In 1950, Cohen [4] proved that a commutative ring R is Noetherian if every prime ideal is finitely generated. This result is not true for associative rings with or without identity. In 1971, Koh [7] introduced the class of prime right ideals and proved that if every prime right ideal is finitely generated, then R is right Noetherian.

In 2010, Sanh [15] introduced the definition of prime submodules based on the properties that every R-homomorphism from R_R to R_R is a left multiplication

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and that $End(R_R)$ is isomorphic to R. A fully invariant proper submodule X is prime in M if for any ideal I of S, and any fully invariant U of M, $I(U) \subset X$ implies that $I(M) \subset X$ or $U \subset X$. As we mentioned above, for a finitely generated right R-module M, if every prime submodule is finitely generated, we could not conclude that M is Noetherian. We now introduce a class of submodules which is larger than that of prime submodules. We call them nearly prime submodules and we prove that a finitely generated right R-module M is Noetherian if every nearly prime submodule is finitely generated. From this fact, we can see that any associative ring with or without indentity provided it is finitely generated as a right R-module, is right Noetherian if every nearly prime right ideal is finitely generated.

Throughout this paper, all rings are associative with identity and all modules are unitary right R-modules. For a right R-module M, we denote $S = End(M_R)$ for its endomorphism ring. A submodule X of M is called a *fully invariant* submodule of M if for any $f \in S$, we have $f(X) \subset X$.

2. On Nearly Prime Submodules

In this section, we introduce the class of nearly prime submodules by reducing the condition "fully invariant" as defined in [15] and we investigate some basic properties. Let M be a right R-module, $S = End(M_R)$, its endomorphism ring. For subset $I \subset S$ and $X \subset M$, denote $I(X) = \sum_{\varphi \in I} \varphi(X)$. With $\varphi \in S, m \in M$, by notation $\varphi S(m)$, we means $(\varphi S)(m)$. For convenience, we write $X \leq M$ to indicate that X is a submodule of M.

Definition 2.1. *A submodule X of a right R-module M is called a nearly prime submodule if for any $\varphi \in S$ and for any $m \in M$, if $\varphi S(m) \subset X$ and $\varphi S(X) \subset X$, then either $m \in X$ or $\varphi(M) \subset X$. Note that, in this definition, we do not require X to be fully invariant, but invariant under φS.*

A proper right ideal P of R is a nearly prime right ideal if for any $a, b \in R$ such that $aRb \subset P$ with $aRP \subset P$, then either $a \in P$ or $b \in P$. Without of confusions, we call P a prime right ideal.

Theorem 2.2. *Let X be a proper submodule of M. Then the following conditions are equivalent:*

1. *X is a nearly prime submodule of M;*
2. *For any right ideal I of S, any submodule U of M, if $I(U) \subset X$ and $I(X) \subset X$, then either $I(M) \subset X$ or $U \subset X$;*
3. *For any $\varphi \in S$ and fully invariant submodule U of M, if $\varphi(U) \subset X$ and $\varphi S(X) \subset X$, then either $\varphi(M) \subset X$ or $U \subset X$.*

Proof. (1) $\Rightarrow$ (2). Let $I \subset S$, $U \subset M$ such that $I(U) \subset X$ and $I(X) \subset X$. Suppose that $I(M) \not\subset X$, then we can find $\varphi \in I$ such that $\varphi(M) \not\subset X$. Since

$I(U) = IS(U) \subset X$, then for any $u \in U$, we have $\varphi S(u) \subset X$. By assumption, $u \in X$, proving that $U \subset X$.

(2) $\Rightarrow$ (3). Let $\varphi \in S$, U, a fully invariant with $\varphi(U) \subset X$ and $\varphi S(X) \subset X$. We can see that $\varphi S(U) \subset X$ and $\varphi S(X) \subset X$ and by (2), we have $(\varphi S)(M) \subset X$ or $U \subset X$. This shows that $\varphi(M) \subset X$ or $U \subset X$.

(3) $\Rightarrow$ (1). Let $\varphi \in S$, $m \in M$ with $\varphi S(m) \subset X$ and $\varphi S(X) \subset X$. From $\varphi S(m) \subset X$, we have $\varphi S(mR) \subset X$. Hence $mR \subset X$ or $\varphi(M) \subset X$, by assumption. This shows that either $m \in X$ or $\varphi(M) \subset X$. ■

Corollary 2.3. *Let P be a proper right ideal of R. Then the following conditions are equivalent:*

(1) *P is a prime right ideal of R;*
(2) *For any right ideals I, J of R, if $IJ \subset P$ and $I(P) \subset P$, then either $I \subset P$ or $J \subset P$;*
(3) *For any $a \in R$ and fully invariant ideal I of R, if $aI \subset P$ and $aP \subset P$, then either $a \subset P$ or $I \subset P$.*

Example and Remark.

(1) *Following Sanh [15], a fully invariant is a prime submodule if for any ideal I of $S = End(M)$, any fully invariant submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$. By our definition, any prime submodule of M is nearly prime.*
(2) *It is well-known that if X is a maximal fully invariant submodule of M, then X is prime. If X is a maximal submodule of M, then X is nearly prime. In fact, let $\varphi(U) \subset X$ where U is a submodule of M and $\varphi \in S$ with $\varphi S(X) \subset X$. Suppose that $U \not\subset X$. Then, there is an $u \in U$ such that $X + uR = M$. It follows that $\varphi(M) = \varphi(X) + \varphi(uR) = \varphi(X) + \varphi(u)R \subset X$ since $\varphi(U) \subset X$. This shows that X is nearly prime. Note that, in general, a maximal submodule of a right R-module M needs not to be fully invariant. Therefore the class of nearly prime submodules of a given right R-module M is larger than that of prime submodules.*

By this example, every maximal right ideal is a prime right ideal. The following Theorem is the main result in this note.

Theorem 2.4. *Let M be a finitely generated right R-module. Then M is a Noetherian module if and only if every nearly prime submodule of M is finitely generated.*

Proof. Assume that every nearly prime submodule is finitely generated and suppose on the contrary that there is a submodule A of M which is not finitely generated. Then, the set $\mathcal{F} = \{X < M \mid X \text{ is not finitely generated}\}$ is non-empty. Let $X_1 \subset X_2 \subset \dots$ be a chain in $\mathcal{F}$. Since M is finitely generated, $\bigcup_{i=1}^{\infty} X_i$ is a proper submodule of M. By Zorn's Lemma, the set $\mathcal{F}$ has a maximal element, A_0 says. We now prove that A_0 is nearly prime. Suppose on the contrary that there are $\varphi \in S, m \in M$ such that $\varphi Sm \subset A_0$ with $\varphi SA_o \subset A_0$ but $\varphi(M) \not\subset A_0$ and $m \notin A_0$. Then $A_0 + \varphi(M)$ contains properly A_0, and hence it is finitely generated, that is $A_0 + \varphi(M) = x_1R + x_2R + \cdots + x_nR$ for some $x_1, x_2, \ldots, x_n \in M$. Let $K = \{a \in M \mid \varphi(a) \in A_0\}$. By assumption $A_0 \subset K$ and $m \in K$. Since $m \notin A_0$, K contains properly A_0, and hence it is finitely generated. Since each $x_i \in A_0 + \varphi(M), i = 1 \cdots n$, we can write $x_i = b_i + \varphi(m_i)$ where $b_i \in A_0$ and $m_i \in M$. From $\varphi(K) \subset A_0$, it follows that $b_1R + b_2R + \cdots + b_nR + \varphi(K) \subset A_0$. We now prove that $A_0 \subset b_1R + \cdots + b_nR + \varphi(K)$. Take any $w \in A_0$. Then $w \in A_0 + \varphi(M)$. We can write

$$\begin{aligned} w &= \sum_{i=1}^{n} x_i r_i = \sum_{i=1}^{n} (b_i + \varphi(m_i)) r_i \\ &= \sum_{i=1}^{n} b_i r_i + \sum_{i=1}^{n} \varphi(m_i r_i) = \sum_{i=1}^{n} b_i r_i + \varphi(\sum_{i=1}^{n} m_i r_i) \end{aligned}$$

From this we can see that $\sum_{i=1}^{n} m_i r_i \in K$, and hence $w \in b_1R + \cdots + b_nR + \varphi(K)$. This proves that $A_0 = b_1R + \cdots + b_nR + \varphi(K)$. Since K is finitely generated, it would imply that $\varphi(K)$ is finitely generated and so is A_0, which is a contradiction. Thus, every submodule of M is finitely generated and we can conclude that M is Noetherian. ■

References

[1] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.

[2] S.I. Bilavska, B.V. Zabavsky, On the structure of maximal non-finitely generated ideals of a ring and Cohen's theorem, *Buletinul Academiei de Stiinte a Republich Moldova. Matematica* **65** (1) (2011) 33–41.

[3] V.R. Chandran, On two analogues of Cohen's theorem, *Indian. J. Pure and Appl. Math.* **8** (1977) 54–59.

[4] I.S. Cohen, Commutative rings with restricted minimum condition, *Duke Mathematical Journal* **17** (1) (1950) 27–42.

[5] C. Faith, *Algebra II: Ring Theory*, Springer-Verlag, Berlin, 1976.

[6] K.R. Goodearl and R.B. Warfield, *An Introduction to Noncommutative Noetherian Rings,* Cambridge University Press, Cambridge, UK, 2004.

[7] K. Koh, On prime one-sided ideals, *Canad. Math. Bull.* **14** (1971) 259–260.

[8] G.O. Michler, Prime right ideals and right Noetherian rings, In: *Ring Theory (Proc. Conf., Park City, Utah, 1971)*, Academic Press, New York, 1972.

[9] N.D.H. Nghiem, N.V. Sanh, N.T. Bac, C. Somsup, On modules with insertion factor property, *Southeast Asian Bull. Math.* **43** (4) (2019) 501–507.

[10] A.C. Ozcan, A. Harmanci, P.F. Smith, Duo modules, *Glassgow Math. J.* **48** (2006) 533–545.

[11] M.L. Reyes, Noncommutative generalizations of theorems of Cohen and Kaplansky, *Algebras and Representation Theory* **15** (2012) 933–975.

[12] N.V. Sanh, S. Asawasamrit, K.F.U. Ahmed, L.P. Thao, On prime and semiprime Goldie modules, *Asian-Eur J. Math.* **4** (2) (2011) 321–334.

[13] N.V. Sanh, T. Dong, N.T. Bac, A generalization of Cohen's theorem, *East-West J. Math.* **16** (1) (2014) 87–91.

[14] N.V. Sanh and L.P. Thao, A generalization of Hopkins-Levitzki theorem, *Southeast Asian Bull. Math.* **37** (4) (2013) 591–600.

[15] N.V. Sanh, N.A. Vu, K.F.U. Ahmed, S. Asawasamrit, L.P. Thao, Primeness in module category, *Asian-Eur J. Math.* **3** (1) (2010) 145–154.

[16] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Tokyo, 1991.