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Global Stability, Periodicity and Boundedness Behavior of a Difference Equation

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Abstract. In this paper, we study the asymptotic behavior and boundedness of the solutions of the difference equation

$$y_{n+1} = \frac{\alpha + \beta\lambda^{-y_n}}{\gamma + y_{n-1}}, \quad n = 0, 1, 2, \dots \tag{1}$$

where $\lambda > 1$ and α, β, γ are positive real numbers, initial conditions y_{-1}, y_0 are arbitrary non-negative numbers.

Keywords: Equilibrium point; Local asymptotic stability; Global asymptotic stability; Positive solutions; Boundedness.

1. Introduction

Difference equation containing exponential terms have many applications in biology. Evolution of a perennial grass depends on the biomass, the litter mass and the total soil nitrogen was described by the difference equations

$$B_{t+1} = cN\frac{e^{a-bL_t}}{1+e^{a-bL_t}},\ L_{t+1} = \frac{L_t^2}{L_t+d} + ckN\frac{e^{a-bL_t}}{1+e^{a-bL_t}} \tag{2}$$

where B is the living biomass, L the litter mass, N the total soil nitrogen, t the time and constants $a, b, c, d > 0$ and $0 < k < 1$. Oscillatory and chaotic nature of (2) was discussed in [17].

Global stability, boundedness nature and periodic character of the positive solution of the difference equation

$$x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n}, \qquad n = 0, 1, 2, \ldots$$

was investigated by El-Metwally et all [6], where $\alpha > 0$ and $\beta > 0$ are the immigration rate and population growth respectively and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers.

Boundedness and global asymptotic behavior of the solution of the difference equations

$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + x_{n-1}}, \qquad n = 0, 1, 2, \ldots \tag{3}$$

and

$$x_{n+1} = \frac{\alpha e^{-(nx_n+(n-k)x_{n-k})}}{\beta + x_n + (n-k)x_{n-k}}, \qquad n = 0, 1, 2, \ldots$$

were studied by Ozturk et all [14, 15], where α and β are positive numbers $k \in \{1, 2, 3, \ldots\}$ and the $x_{-k}, x_{-(k-1)}, \ldots, x_{-1}, x_0$ are arbitrary numbers.

Boundedness and the persistence of the positive solutions, the existence, the attractivity and the global asymptotic stability of the unique positive equilibrium and the existence of periodic solutions concerning the biological model

$$x_{n+1} = \frac{ax_n^2}{x_n+b} + c\frac{e^{k-dx_n}}{1+e^{k-dx_n}}$$

was established in [16], where $0 < a < 1, b, c, d, k$ are positive constants and x_0 is a real number.

Stability analysis of a nonlinear difference equation

$$y_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha y_n + \beta y_{n-1}},\ n = 0, 1, 2, \ldots$$

was established in [3], where α, β and initial conditions are arbitrary positive numbers.

Properties of solutions of various types of second and third order rational difference equation was discussed in [11, 4]. Stability properties and conditions for boundedness of nonlinear difference equations $x_{n+1} = f(x_n)g(x_{n-k})$ was studied in [13] and asymptotic properties of solutions of the difference equation $y_n = \frac{f(y_{n-1},\ldots,y_{n-k})}{g(y_{n-1},\ldots,y_{n-k})}, n = 0, 1, 2, \ldots$ was studied in [2].

Motivated by above studies, we generalize (3) and investigate the global attractivity and boundedness of the solutions of the difference equations (1) for $\lambda > 1$.

2. Preliminaries

Definition 2.1. [11] *Let $I \in \mathbb{R}$ and let $f : I \times I \to I$ be a continuous function. Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-1}), n = 0, 1, 2, \cdots \tag{4}$$

for the initial conditions $y_0, y_{-1} \in I$. We say that $\bar{y}$ is an equilibrium of (1) if $\bar{y} = f(\bar{y}, \bar{y})$.

Definition 2.2. [11, Definition 1.1.1]

- (i) *The equilibrium $\bar{y}$ of (1) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that y_{-1}, $y_0 \in I$ with $|y_0 - \bar{y}| + |y_{-1} - \bar{y}| < \delta$, then $|y_n - \bar{y}| < \epsilon$ for all $n \geq -1$.*
- (ii) *The equilibrium $\bar{y}$ of (1) is called locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that y_{-1}, $y_0 \in I$ with $|y_0 - \bar{y}| + |y_{-1} - \bar{y}| < \gamma$, then $\lim_{n\to\infty} y_n = \bar{y}$.*
- (iii) *The equilibrium $\bar{y}$ of equilibrium (1) is called a global attractor if for every $y_{-1}, y_0 \in I$ we have $\lim_{n\to\infty} y_n = \bar{y}$.*
- (iv) *The equilibrium $\bar{y}$ of (1) is called globally asymptotically stable if it is locally stable and a global attractor.*
- (v) *The equilibrium $\bar{y}$ of equilibrium (1) is called unstable if it is not stable.*

Definition 2.3. [11] *Let $s = \frac{\partial f}{\partial u}(\bar{y}, \bar{y})$ and $t = \frac{\partial f}{\partial v}(\bar{y}, \bar{y})$ denote the partial derivatives of $f(u, v)$ evaluated at an equilibrium $\bar{y}$ of (1). Then the equation*

$$y_{n+1} = sy_n + ty_{n-1}, n = 0, 1, 2, \cdots \tag{5}$$

is called the linearized equation associated with (1) about the equilibrium point $\bar{y}$.

The characteristic equation of (5) is the equation

$$\mu^2 - s\mu - t = 0 \tag{6}$$

with characteristic roots $\mu_\pm = \frac{s \pm \sqrt{s^2+4t}}{2}$.

Definition 2.4. [11, Definition 1.1.2] *The sequence $\{y_n\}$ is said to be periodic with period p if $y_{n+p} = y_n$ for $n = 0, 1, 2, ..$*

Theorem 2.5. [5, Theorem 1] *Consider the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), \quad n = 0, 1, 2, \cdots \tag{7}$$

where $k \in \{1, 2, 3, \cdots\}$. Let $I = [a, b]$ be some interval of real numbers and assume that $f : [a, b] \times [a, b] \to [a, b]$ is a continuous function satisfying the following properties:

(i) *$f(u, v)$ is non-increasing in each argument.*
(ii) *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system $m = f(M, M)$ and $M = f(m, m)$ then $m = M$.*

Then (7) has a unique positive equilibrium $\bar{y}$ and every solution of (7) converges to $\bar{y}$.

Theorem 2.6. [11, Theorem 1.1.1] *(Linearized stability)*

(i) *If both roots of quadratic equation (6) lie in unit disk $|\lambda| < 1$ then equilibrium $\bar{y}$ of (1) is locally asymptotically stable.*
(ii) *If at least one of the roots of (6) has absolute value greater than one, then the equilibrium $\bar{y}$ of (1) is unstable.*
(iii) *A necessary and sufficient condition for both roots of (6) to lie in the open unit disk $|\lambda| < 1$, is*
$$|s| < 1 - t < 2. \tag{8}$$
In this case the locally asymptotically stable equilibrium point $\bar{y}$ is also called a sink.
(iv) *A necessary and sufficient condition for both roots of (6) to have absolute value greater than one is*
$$|t| > 1 \text{ and } |s| < |1 - t|. \tag{9}$$
In this case $\bar{y}$ is called a repeller.
(v) *A necessary and sufficient condition for one root of (6) to have absolute value greater than one and for the other to have absolute value less than one is*
$$s^2 + 4t > 0 \text{ and } |s| > |1 - t|. \tag{10}$$
In this case unstable equilibrium point $\bar{y}$ is called a saddle point.

3. Main Results

In this section, we discuss the stability of the solutions of equation (1). We show that the positive equilibrium point of equation (1) is a global attractor with a basin that depend on the conditions posed on the coefficients and on the variable n.

The equilibrium points of equation (1) are the solutions of the equation

$$\bar{y} = \frac{\alpha + \beta\ \lambda^{-\bar{y}}}{\gamma + \bar{y}}. \tag{11}$$

Set $f(y) = \frac{\alpha+\beta\lambda^{-y}}{\gamma+y} - y$.

When $y = 0$, $f(0) = \frac{\alpha+\beta}{\lambda} > 0$ and $\lim_{y\to\infty}(\frac{\alpha+\beta\lambda^{-y}}{\gamma+y} - y) = -\infty$.

Now $f'(y) = \frac{-[(\gamma+y)\beta\lambda^{-y}\ln\lambda+\alpha+\beta\lambda^{-y}]}{(\gamma+y)^2} - 1 < 0$. Therefore the equilibrium point is unique.

Theorem 3.1. *The equilibrium point $\bar{y}$ is locally asymptotically stable if*

$$\begin{aligned}\beta <& \lambda^{\frac{(2-\delta)\gamma+\gamma\sqrt{(\delta-2)^2+4\delta(\frac{\alpha\delta}{\gamma^2}+1)}}{2\delta}} \\ &\times\left[\frac{\gamma^2(2-\delta)+\gamma^2\sqrt{(\delta-2)^2+4(\frac{\alpha\delta}{\gamma^2}+1)}}{\delta^2}+\frac{\gamma^2}{\delta}\right]\end{aligned} \tag{12}$$

and is unstable if

$$\begin{aligned}\beta >& \lambda^{\frac{(2-\delta)\gamma+\gamma\sqrt{(\delta-2)^2+4\delta(\frac{\alpha\delta}{\gamma^2}+1)}}{2\delta}} \\ &\times\left[\frac{\gamma^2(2-\delta)+\gamma^2\sqrt{(\delta-2)^2+4(\frac{\alpha\delta}{\gamma^2}+1)}}{\delta^2}+\frac{\gamma^2}{\delta}\right]\end{aligned} \tag{13}$$

where $\delta = \gamma\ln\lambda$.

Proof. The linearized equation of (1) is $x_{n+1} = sx_n + tx_{n-1},\ n = 0,1,2,\ldots,$ where $s = \frac{\partial f}{\partial u}(\bar{y},\bar{y})$, $t = \frac{\partial f}{\partial v}(\bar{y},\bar{y})$. We have $f(u,v) = \frac{\alpha+\beta\lambda^{-u}}{\gamma+v}$, $s = f_u(\bar{y},\bar{y}) = -\frac{\beta\lambda^{-\bar{y}}\ln\lambda}{\gamma+\bar{y}}$, $t = f_v(\bar{y},\bar{y}) = -\frac{(\alpha+\beta\lambda^{-\bar{y}})}{(\gamma+\bar{y})^2} = \frac{-\bar{y}}{\gamma+\bar{y}}$. Therefore the linearized equation is

$$x_{n+1} + \frac{\beta\gamma^{-\bar{y}}\ln\lambda}{\gamma+\bar{y}}x_n + \frac{\bar{y}}{\gamma+\bar{y}}x_{n-1} = 0.$$

Suppose that the equilibrium point is locally asymptotically stable.

From (8), we get

$$\begin{aligned}&\frac{\beta\lambda^{-\bar{y}}\ln\lambda}{\gamma+\bar{y}} < 1 + \frac{\bar{y}}{\gamma+\bar{y}} < 2\\ \Rightarrow&\beta\lambda^{-\bar{y}}\ln\lambda < 2\bar{y}+\gamma.\end{aligned} \tag{14}$$

We have

$$\begin{aligned}&\bar{y}(\gamma+\bar{y}) = \alpha+\beta\lambda^{-\bar{y}}\\ \Rightarrow&\gamma\bar{y}+\bar{y}^2-\alpha = \beta\lambda^{-\bar{y}}.\end{aligned} \tag{15}$$

Substituting (15) in (14), we get

$$\bar{y}^2 + \left(\gamma - \frac{2}{\ln\lambda}\right)\bar{y} - \left(\alpha + \frac{\gamma}{\ln\lambda}\right) < 0. \tag{16}$$

(16) is in quadratic form and has two real equilibrium points. Solving it, we get

$$\bar{y} = \frac{-(\delta-2) \pm \sqrt{(\delta-2)^2 + 4\delta\left(\frac{\alpha}{\gamma}\ln\lambda + 1\right)}}{2\ln\lambda}.$$

We choose the positive solution, substituting in (14) we get

$$\begin{aligned}
&\beta < \lambda^{\frac{(2-\delta)\gamma+\gamma\sqrt{(\delta-2)^2+4\delta\left(\frac{\alpha}{\gamma}\ln\lambda+1\right)}}{2\delta}} \\
&\quad\times\left[\frac{\gamma^2(2-\delta)+\gamma^2\sqrt{(\delta-2)^2+4\delta\left(\frac{\alpha}{\gamma}\ln\lambda+1\right)}}{\delta^2} + \frac{\gamma^2}{\delta}\right] \\
\Rightarrow\ &\beta < \lambda^{\frac{(2-\delta)\gamma+\gamma\sqrt{(\delta-2)^2+4\delta(\frac{\alpha\delta}{\gamma^2}+1)}}{2\delta}}\left[\frac{\gamma^2(2-\delta)+\gamma^2\sqrt{(\delta-2)^2+4(\frac{\alpha\delta}{\gamma^2}+1)}}{\delta^2} + \frac{\gamma^2}{\delta}\right].
\end{aligned}$$

This completes proof of first part.

Suppose that the equilibrium point is unstable. By (10)

$$\begin{aligned}
&\frac{1}{(\gamma+\bar{y})^2}\beta^2\lambda^{-2\bar{y}}\ln\lambda^2 + 4\left(\frac{-\bar{y}}{\gamma+\bar{y}}\right) > 0 \\
\Rightarrow &\frac{\beta^2\lambda^{-2\bar{y}}\ln\lambda^2}{(\gamma+\bar{y})^2} > \frac{4\bar{y}}{\gamma+\bar{y}}
\end{aligned} \tag{17}$$

and

$$\beta\lambda^{-\bar{y}}\ln\lambda > \gamma + 2\bar{y}. \tag{18}$$

Using (15), we get

$$\begin{aligned}
&(\gamma\bar{y} + \bar{y}^2 - \alpha)\ln\lambda > 2\bar{y} + \gamma \\
\Rightarrow &\bar{y}^2 + \left(\gamma - \frac{2}{\ln\lambda}\right)\bar{y} - \left(\alpha + \frac{\gamma}{\ln\lambda}\right) > 0.
\end{aligned} \tag{19}$$

The positive solution is

$$\bar{y} = \frac{-(\delta-2) + \sqrt{(\delta-2)^2 + 4\delta\left(\frac{\alpha}{\gamma}\ln\lambda + 1\right)}}{2\ln\lambda}. \tag{20}$$

Substituting (20) in (18) we get,

$$\beta > \lambda^{\frac{(2-\delta)\gamma+\gamma\sqrt{(\delta-2)^2+4\delta(\frac{\alpha\delta}{\gamma^2}+1)}}{2\delta}} \times \left[\frac{\gamma^2(2-\delta)+\gamma^2\sqrt{(\delta-2)^2+4(\frac{\alpha\delta}{\gamma^2}+1)}}{\delta^2}+\frac{\gamma^2}{\delta}\right].$$

Similarly we can show that (17) is also satisfied. ∎

Lemma 3.2. *Let* $f(u,v)=\frac{\alpha+\beta\lambda^{-u}}{\gamma+v}$ *and* $u,v\epsilon[0,\infty)$. *Then* $f(u,v)$ *is a non-increasing function both in* u *and* v.

Proof. We have

$$f_u(u,v)=\frac{-\beta\lambda^{-u}\ln\lambda}{\gamma+v}\leq 0 \text{ and } f_v(u,v)=\frac{-(\alpha+\beta\lambda^{-u})}{(\gamma+v)^2}\leq 0.$$

This completes the proof. ∎

The following theorem gives a sufficient condition for the boundedness of the positive solutions of (1).

Theorem 3.3. *The following statements are true:*

(i) *Every positive solution of equation* (1) *is bounded if* $\alpha < y_n$.

(ii) *The positive equilibrium point of equation* (1) *is bounded if* $\alpha < \bar{y_1}$.

Proof. (i) Suppose that $\alpha < y_n$. Let $\{y_n\}_{n=1}^{\infty}$ be a positive solution of (1) for $n = 0, 1, 2, \ldots$. We have

$$0 < y_{n+1} = \frac{\alpha+\beta\lambda^{-y_n}}{\gamma+y_{n-1}} < \frac{\alpha+\beta\lambda^{-\alpha}}{\gamma+y_{n-1}} < \frac{\alpha+\beta\lambda^{-\alpha}}{\gamma} \tag{21}$$

which shows that every positive solution of equation (1) is bounded.

(ii) Suppose $\bar{y}$ is an equilibrium point of equation (1). Let $\alpha < \bar{y_1}$. Then

$$\bar{y_1} = \frac{\alpha+\beta\lambda^{-\bar{y_1}}}{\gamma+\bar{y_1}} < \frac{\alpha+\beta\lambda^{-\alpha}}{\gamma}. \tag{22}$$

Therefore $\bar{y_1}$ is bounded. ∎

Theorem 3.4. *Equation* (1) *has no positive solutions of prime period two.*

Proof. Let $\ldots, \phi, \psi, \phi, \psi, \ldots$ be a period two solution of (1). Then

$$\phi = \frac{\alpha+\beta\lambda^{-\psi}}{\gamma+\phi} \text{ and } \psi = \frac{\alpha+\beta\lambda^{-\phi}}{\gamma+\psi}$$
$$\Rightarrow \phi^2+\gamma\phi-\beta\lambda^{-\psi}-\alpha=0 \;\text{ and }\; \psi^2+\gamma\psi-\beta\lambda^{-\phi}-\alpha=0.$$

Equating we get, $\phi^2+\gamma\phi-\beta\lambda^{-\psi} = \psi^2-\gamma\psi-\beta\lambda^{-\phi}$. Set $F(z) = z^2+\gamma z-\beta\lambda^{-z}-\alpha$. Now $F(\bar{y}) = \bar{y}^2 + \gamma\bar{y} - \beta\lambda^{-y} - \alpha = 0$. $F'(z) = 2z + \gamma + \beta\lambda^{-z}\ln\lambda > 0$ which shows that $F(z)$ is increasing. ■

Theorem 3.5. *Suppose that* (12) *holds and* $\beta < \gamma$. *Then the equilibrium point* $\bar{y}$ *of* (1) *is globally asymptotically stable.*

Proof. From Lemma 3.2, $f(u, v)$ is non-increasing in each of its arguments. Then for any $u, v \in [0, \infty)$, we have $0 < f(u, v) < \frac{\alpha+\beta\lambda^{-u}}{\gamma+v} < \frac{\alpha+\beta\lambda^{-\alpha}}{\gamma}$.

Let $m = \lim_{n\to\infty} \inf y_n$ and $M = \lim_{n\to\infty} \sup y_n$ and $\epsilon > 0$ such that $\epsilon < \min\{\frac{\alpha+\beta\lambda^{-\alpha}}{\gamma} - M, m\}$. Then there exist $n_o \in N$ such that $m - \epsilon \leq y_n \leq M + \epsilon$ for all $n > n_0$. Since f is non-increasing, we get

$$\frac{\alpha + \beta\lambda^{-(M+\epsilon)}}{\gamma + (M + \epsilon)} \leq y_{n+1} \leq \frac{\alpha + \beta\lambda^{-(m-\epsilon)}}{\gamma + (m - \epsilon)}, n \geq n_0 + 1.$$

Therefore

$$\frac{\alpha + \beta\lambda^{-(M+\epsilon)}}{\gamma + (M + \epsilon)} \leq m \leq M \leq \frac{\alpha + \beta\lambda^{-(m-\epsilon)}}{\gamma + (m - \epsilon)}, n \geq n_0 + 1.$$

Since ϵ is arbitrary, we get

$$\frac{\alpha + \beta\lambda^{-M}}{\gamma + M} \leq m \leq M \leq \frac{\alpha + \beta\lambda^{-m}}{\gamma + m}, n \geq n_0 + 1$$

Which gives $\alpha + \beta\lambda^{-M} - \gamma m \leq mM \leq \alpha + \beta\lambda^{-m} - \gamma M$. Since $\beta < \gamma$, we get $M \leq m$, hence $M = m = \bar{y_1}$.

From Lemma 2.5 and (1) has a unique equilibrium point and every solution of equation (1) converges to $\bar{y_1}$. This shows that $\lim_{n\to\infty} y_n = \bar{y_1}$ and the proof of the theorem is completed. ■

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