Southeast Asian Bulletin of Mathematics (2021) 45: 189–196

Southeast Asian Bulletin of Mathematics

Studies on Generalized Differential-Difference Operator of Normalized Analytic Functions*

Rabha W. Ibrahim
Informetrics Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Faculty of Mathematics & Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Email: rabhaibrahim@yahoo.com; rabhaibrahim@tdtu.edu.vn

Suzan J. Obaiys
School of Mathematical and Computer Sciences, Heriot-Watt University, Malaysia
Email: S.Obaiys@hw.ac.uk

Maslina Darus
Centre of Modelling and Data Sciences, Faculty of Sciences and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia
Email: maslina@ukm.edu.my

Received 15 March 2019
Accepted 22 January 2020

Communicated by H.M. Srivastava

AMS Mathematics Subject Classification(2000): 30C45, 44C55

Abstract. In this paper, we define a new differential-difference operator in the open unit disk. This operator describes the analytic geometric representation of the solution of the well known Beltrami differential equation in a complex domain, which this type of equations usually employs in image processing. Moreover, topological properties such as boundedness and compactness are introduced in different spaces such as Hardy space and $B_{\log}$ space.

Keywords: Beltrami equation; Differential operator; Unit disk; Analytic function; Subordination and superordination; Univalent function.

*The work is partially supported by the Universiti Kebangsaan Malaysia grant: GUP-2017-064.

1. Introduction

Traditional Hardy spaces of the open unit disk or the half-plane recline are the connections between complex and Fourier analysis, and several developments in spectral theory and harmonic analysis variety with them. From a spectral-theoretic point of view, the shift operator (such as Dunkl operator) and its numerous firmness play an critical role and anxiety deep connections between function theory on the one hand, control, approximation, and prediction theory on the other hand [1]–[9].

It is well known that the divergence and gradient operators are formulated in the sense of weak derivatives. This property implies a weak solution for various classes of partial differential equations in a complex domain. Such a problem is ill-posed in the meaning of Hadamard and Sobolev spaces. Therefore, many authors suggested these operators in Hardy spaces and $B_{\log}$. In this note, we shall derive a differential-difference operator in the open unit disk and show that its close in some Hardy spaces and $B_{\log}$ based on per-Schwarzian derivative. A function $f \in \mathcal{H}$ (the space of analytic functions) is called in the class Σ if and only if it has the norm (see [6])

$$\|f\| = sup_{z\in U}(1-|z|^2)|\frac{f''(z)}{f'(z)}| < \infty, \ \ (z \in U). \tag{1}$$

Note that the fraction $T_f := \frac{f''(z)}{f'(z)}$ is called per-Schwarzian derivative which is usually used to discuss the univalency of analytic functions (see [7, 5, 10]).

Let Λ be the class of analytic function formulated by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U = \{z : |z| < 1\}. \tag{2}$$

For a function $f \in \Lambda$, we present the following difference operator

$$\begin{aligned}
D^0_\kappa f(z) =& f(z)\\
D^1_\kappa f(z) =& zf'(z) + \frac{\kappa}{2}\left(f(z) - f(-z) - 2z\right), \quad \kappa \in \mathbb{R}\\
&\vdots\\
D^m_\kappa f(z) =& D_\kappa(D^{m-1}_\kappa f(z))\\
=& z + \sum_{n=2}^{\infty}[n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^m\, a_n z^n.
\end{aligned} \tag{3}$$

It is clear that when $\kappa = 0$, we have the Sàlàgeans differential operator [11]. We call D^m_κ the Sàlàgean-difference operator. Moreover, D^m_κ is a modified Dunkl operator of complex variables [2] and for recent work [4]. Dunkl operator describes a major generalization of partial derivatives and realizes the commutative law in $\mathbb{R}^n$. In geometry, it attains the reflexive relation, which is plotting the space into itself as a set of fixed points.

Example 1.1.

(i) Let $f(z) = z/(1-z)$. Then $D_3^1 f(z) = z + 2z^2 + 6z^3 + 4z^4 + 8z^5 + 6z^6 + ...$

(ii) Let $f(z) = z/(1-z)^2$. Then $D_3^1 f(z) = z+4z^2+18z^3+16z^4+40z^5+36z^6+...$

The Hadamard product or convolution of two power series is denoted by $(*)$ achieving

$$\begin{aligned} f(z) * h(z) &= \Big(z + \sum_{n=2}^{\infty} a_n z^n\Big) * \Big(z + \sum_{n=2}^{\infty} \eta_n z^n\Big) \\ &= z + \sum_{n=2}^{\infty} a_n \eta_n z^n. \end{aligned} \tag{4}$$

Thus, we have

$$D_\kappa^m f(z) = \mathfrak{D}(z) * f(z),$$

where $\mathfrak{D}(z) := z + \sum_{n=2}^{\infty} [n + \kappa(1 + (-1)^{n+1})]^m \, z^n$.

Consider the space $\mathcal{B}_{\log}$ of all functions $f \in \mathcal{H}$ (the space of all holomorphic functions) which are satisfying

$$\|f\|_{\mathcal{B}_{\log}} = \sup_{z \in U} (1 - |z|^2) \Big| \frac{\partial f(z)}{f(z)} \Big| \ln \frac{1}{(1 - |z|^2)} < \infty, \quad (z \in U),$$

where ∂ and $\bar{\partial}$ respectively denote the derivation operators with respect to complex variables $z = x + iy$ and its conjugate $\bar{z}$. Note that $\partial\bar{\partial} = \bar{\partial}\partial = div(\triangledown)$. The aim of this effort is to show that the operator D_κ^m is closed under the above space by applying the complex Beltrami differential equation

$$\bar{\partial} f = \nu \partial f, \quad |\nu| < 1, \, f \in \Lambda. \tag{5}$$

And the conjugate Beltrami equation

$$\bar{\partial} f = \nu \bar{\partial} \bar{f}, \quad |\nu| < 1, \, f \in \Lambda. \tag{6}$$

Also, some topological properties are discussed in the sequel. The connection between Eq.s (5) and (6) with the operator (3) is that for any solution f for Eq.s (5) and (6), the operator $D_\kappa^m f$ indicates the lower solution in some spaces such as Hardy and $B_{\log}$. Therefore, we show that it is bounded and compact in these spaces.

2. Results

In this section, we illustrate our results.

Theorem 2.1. *Let $f \in \Lambda$. Then $D_\kappa^m : \mathcal{B}_{\log} \to \mathcal{B}_{\log}$ is compact.*

Proof. By applying properties of the convolution product we obtain

$$\begin{aligned}(1-|z|^2)\Big|\frac{(D_\kappa^m f)'(z)}{(D_\kappa^m f)(z)}\Big|\ln\frac{1}{(1-|z|^2)} &= (1-|z|^2)\Big|\frac{(\mathfrak{D}*f)'(z)}{(\mathfrak{D}*f)(z)}\Big|\ln\frac{1}{(1-|z|^2)}\\ &= (1-|z|^2)\Big|\frac{(\mathfrak{D}(z)/z*f'(z)}{(\mathfrak{D}(z)*f(z)}\Big|\ln\frac{1}{(1-|z|^2)}\\ &\le C(\kappa)(1-|z|^2)\Big|\frac{f'(z)}{f(z)}\Big|\ln\frac{1}{(1-|z|^2)}\\ &\le C(\kappa)\|f\|_{\mathcal{B}_{\log}},\end{aligned}$$

where $C(\kappa)>0$. By taking the supremum for the last assertion over U, we obtain $D_\kappa^m:\mathcal{B}_{\log}\to\mathcal{B}_{\log}$. We proceed to show that D_κ^m is compact. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}_{\log}$ such that $f_n\to 0$ uniformly on $\overline{U}$ as $n\to\infty$. A calculation implies that

$$\begin{aligned}\|D_\kappa^m f_n\|_{\mathcal{B}_{\log}} &= \sup_{z\in U}(1-|z|^2)|\frac{(D_\kappa^m f_n)'(z)}{D_\kappa^m f_n(z)}|\ln\frac{1}{(1-|z|^2)}\\ &= \sup_{z\in U}(1-|z|^2)\Big|\frac{(\mathfrak{D}*f_n)'(z)}{(\mathfrak{D}*f_n)(z)}\Big|\ln\frac{1}{(1-|z|^2)}\\ &= \sup_{z\in U}(1-|z|^2)\Big|\frac{(\mathfrak{D}(z)/z*f_n'(z)}{(\mathfrak{D}(z)*f_n(z)}\Big|\ln\frac{1}{(1-|z|^2)}\\ &\le \sup_{z\in U}(1-|z|^2)|\frac{f_n'(z)}{f_n(z)}|\ln\frac{1}{(1-|z|^2)}\\ &\le C(\kappa)\|f_n\|_{\mathcal{B}_{\log}}.\end{aligned}$$

Since for $f_n\to 0$ on $\overline{U}$ we have $\|f_n\|_{\mathcal{B}_{\log}}\to 0$, and that ε is an arbitrary positive number, by letting $n\to\infty$ in the last inequality, we receive that $lim_{n\to\infty}\|D_\kappa^m f_n\|=0$. Thus, D_κ^m is compact. ■

Theorem 2.2. *Let $f\in\Lambda$. Then $D_\kappa^m:\Sigma\to\Sigma$ is compact.*

Proof. A computation implies that

$$\begin{aligned}(1-|z|^2)\Big|\frac{(D_\kappa^m f)''(z)}{(D_\kappa^m f)'(z)}\Big| &= (1-|z|^2)\Big|\frac{((\mathfrak{D}*f)')'(z)}{(\mathfrak{D}*f)'(z)}\Big|\\ &= (1-|z|^2)\Big|\frac{(\mathfrak{D}(z)/z*f'(z))'}{(\mathfrak{D}(z)/z*f'(z)}\Big|\\ &\le C(\kappa)(1-|z|^2)\Big|\frac{f''(z)}{f'(z)}\Big|\\ &\le C(\kappa)\|f\|_\Sigma.\end{aligned}$$

By taking the supremum for the last assertion over U, we get $D_\kappa^m\in\Sigma$. We aim to show that D_κ^m is compact. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in Σ and that $f_n\to 0$

uniformly on $\overline{U}$ as $n \to \infty$. Then we have

$$\begin{aligned}\|D_\kappa^m f_n\| &= \sup_{z\in U}(1-|z|^2)|\frac{(D_\kappa^m f_n)''(z)}{(D_\kappa^m f_n)'(z)}| \\ &\le \sup_{z\in U}(1-|z|^2)|\frac{f_n''(z)}{f_n'(z)}| \\ &\le C(\kappa)\|f_n\|_\Sigma.\end{aligned}$$

Since for $f_n \to 0$ on $\overline{U}$ we have $\|f_n\|_\Sigma \to 0$, and that ε is an arbitrary positive number, by letting $n \to \infty$ in the last inequality, we reach to limit $lim_{n\to\infty}\|D_\kappa^m f_n\| = 0$. Thus, D_κ^m is compact in Σ. ■

Definition 2.3. *For spaces of holomorphic functions on the open unit disk, the Hardy space H^p involves functions f whose mean square value on the circle of radius $|z| = |re^{i\theta}| = r$ remains bounded as $r \longrightarrow 1$ from below. In general it satisfies the integral*

$$\sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p \mathrm{d}\theta\right)^{\frac{1}{p}} < \infty.$$

The norm is symbolic by $\|f\|_{H^p}, p \ge 1$.

Theorem 2.4. *Let $f \in \Lambda$. Then $D_\kappa^m : H^p \to H^p$, $p \ge 1$ is compact.*

Proof. Let $r \longrightarrow 1$. By applying the convolution product, we have

$$\begin{aligned}&\sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|D_\kappa^m f\left(re^{i\theta}\right)\right|^p \mathrm{d}\theta\right)^{\frac{1}{p}} \\ &= \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\mathfrak{D}\left(re^{i\theta}\right) * f\left(re^{i\theta}\right)\right|^p \mathrm{d}\theta\right)^{\frac{1}{p}} \\ &\le \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p \mathrm{d}\theta\right)^{\frac{1}{p}} * \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p \mathrm{d}\theta\right)^{\frac{1}{p}} \\ &\le \Omega(\kappa)\|f\|_{H^p} < \infty,\end{aligned}$$

where $\Omega(\kappa) := \|\mathfrak{D}(r)\|$. Hence, $D_\kappa^m \in H^p$. In the similar manner of Theorems 2.1 and 2.2, we can show that for every sequence of analytic functions converge in H^p. This completes the proof. ■

Next result shows that the lower solution of (5) is convex in $\mathbb{U}$.

Theorem 2.5. *Let f be a solution of (5) in Λ. If $h(z), z \in \mathbb{U}$ is univalent convex in $\mathbb{U}$ then there occurs a lower solution achieving the subordination relation*

$$D_\kappa^m f(z) \prec z\exp\Big(\int_0^z \frac{h(\eth(w))-1}{w}dw\Big), \tag{7}$$

where $\eth(z)$ *is analytic in* $\mathbb{U}$, *with* $\eth(0) = 0$ *and* $|\eth(z)| < 1$.

Proof. Let $f \in \Lambda$. Then we get the following conclusion

$$\begin{aligned}
&\Re\left(\frac{z(D_\kappa^m f(z))'}{D_\kappa^m f(z)}\right) > 0\\
\Leftrightarrow&\Re\left(\frac{z+\sum\limits_{n=2}^{\infty} n\left(n+\frac{\kappa}{2}(1+(-1)^{n+1})\right)^m a_n z^n}{z+\sum\limits_{n=2}^{\infty}\left(n+\frac{\kappa}{2}(1+(-1)^{n+1})\right)^m a_n z^n}\right) > 0\\
\Leftrightarrow&\Re\left(\frac{1+\sum\limits_{n=2}^{\infty} n\left(n+\frac{\kappa}{2}(1+(-1)^{n+1})\right)^m a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty}\left(n+\frac{\kappa}{2}(1+(-1)^{n+1})\right)^m a_n z^{n-1}}\right) > 0\\
\Leftrightarrow&\left(\frac{1+\sum\limits_{n=2}^{\infty} n\left(n+\frac{\kappa}{2}(1+(-1)^{n+1})\right)^m a_n}{1+\sum\limits_{n=2}^{\infty}\left(n+\frac{\kappa}{2}(1+(-1)^{n+1})\right)^m a_n}\right) > 0, \quad z \to 1^+\\
\Leftrightarrow&\left(1+\sum_{n=2}^{\infty} n\left(n+\frac{\kappa}{2}(1+(-1)^{n+1})\right)^m a_n\right) > 0.
\end{aligned}$$

Moreover, by the definition of $D_\kappa^m f(z)$, we indicate that $(D_\kappa^m f(0) = 0$. Consequently, yields that

$$\frac{z(D_\kappa^m f(z))'}{D_\kappa^m f(z)} \in \mathcal{P} := \{P \in \mathbb{U} : P(z) = 1 + p_1 z + p_2 z^2 + ...\},$$

which implies that $D_\kappa^m f$ is starlike solution. Then there occurs a univalent convex function h such that $D_\kappa^m f \prec h$ $(D_\kappa^m f \in S^*(h))$. We conclude that

$$\Big(\frac{z(D_\kappa^m f(z))'}{D_\kappa^m f(z)}\Big) \prec h(z), \quad z \in \mathbb{U},$$

which implies that there is a Schwarz function with $\eth(0) = 0$ and $|\eth(z)| < 1$ fulfilling the equality:

$$\Big(\frac{z(D_\kappa^m f(z))'}{D_\kappa^m f(z)}\Big) = h(\eth(z)), \quad z \in \mathbb{U}.$$

A calculation gives us

$$\Big(\frac{(D_\kappa^m f(z))'}{D_\kappa^m f(z)}\Big) - \frac{1}{z} = \frac{h(\eth(z)) - 1}{z}.$$

By integrating both sides, we have

$$\log D_\kappa^m f(z) - \log z = \int_0^z \frac{h(\eth(w)) - 1}{w} dw.$$

Hence, we obtain

$$\log \frac{D_\kappa^m f(z)}{z} = \int_0^z \frac{h(\eth(\xi)) - 1}{w} dw. \tag{8}$$

By utilizing the meaning of subordination, we conclude that

$$D_\kappa^m f(z) \prec z \exp\Big(\int_0^z \frac{h(\eth(w)) - 1}{w} dw \Big).$$

■

Remark 2.6.

(i) In Theorems 2.1 and 2.4, the constant $C(\kappa)$ is called the Dirichlet regularization (DR), which is defined as a limit of the formula of $\mathfrak{D}(z)$. For example, (by using Mathematica-Wolfram 11.2) when $\kappa = 1/2$, we have $|DR| = 1/12$, $\kappa = 1$ and $|DR| = 7/12$, when $k = 2, |DR| = 19/12, k = 3$, we get $|DR| = 31/12$. Therefore, $C(\kappa)$ increases whenever κ increases.

(ii) The lower solutions of Eq. (6) can be studied in Hardy space and $B_{\log}$ by using the distributional sense of the following $\bar{\partial}$-equation [9]

$$\bar{\partial} f = \nu \bar{\partial} \bar{f} \Longrightarrow \bar{\partial}\omega = \alpha\bar{\omega}, \quad \alpha \in L_q(U),\, q > 2.$$

(iii) The Hardy space $H^p(U)$, $p \geq 1$ is defined as the space of holomorphic functions f on U such that the integral

$$M_r(f) = \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^p \mathrm{d}\theta$$

remain bounded for $r < 1$. The norm on this Hardy space may define by the limit

$$\|f\|_2 = \lim_{r\to 1} \sqrt{M_r(f)}\,.$$

Hardy spaces in the disk are formulated to Fourier series. A function f is in $H^p(U)$ if and only if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty\,. \tag{9}$$

Thus, it involves of those functions that are $L^2(U)$ on the disk, and whose negative frequency Fourier coefficients are equal to zero. Therefore, the operator D_κ^m satisfies (9) for sufficient value $\kappa \in [0, \infty)$.

(iv) For future study, we suggest to investigate the existence of solution of (5) and (6) in a complex Hilbert space by using D_κ^m.

3. Conclusion

For a normalized subclass of analytic functions in the open unit disk, we formulated a deferential-difference operator. We introduced sufficient conditions of this operator to be bounded and compact in some well known spaces such as Hardy space. Moreover, if f is a solution for (6) then the lower bound of f in all the above space is the operator (3). For future works, one can apply the same operator in different classes of analytic functions such harmonic, p-valent and meromorphic classes. Or it can be generalized by utilizing the concept of q-calculus to bring fractional differential operator in the open unit disk.

Acknowledgement. The authors would like to express their thanks to the editor (Professor Hanchun YANG) and the reviewer for his/her deep comments to improve our effort.

References

[1] L. Baratchart, B. Leblond, S. Rigat, E. Russ, Hardy spaces of the conjugate Beltrami equation, *Journal of Functional Analysis* **259** (2) (2010) 384–427.

[2] C.F. Dunkl, Differential-difference operators associated with reflections groups, *Trans. Am. Math. Soc.* **311** (1989) 167–183.

[3] Y. Fischer, J. Leblond, J.R. Partington, E. Sincich, Bounded extremal problems in Hardy spaces for the conjugate Beltrami equation in simply-connected domains, *Applied and Computational Harmonic Analysis* **31** (2) (2011) 264–285.

[4] R.W. Ibrahim, New classes of analytic functions determined by a modified differential-difference operator in a complex domain, *Karbala International Journal of Modern Science* **3** (1) (2017) 53–58.

[5] R.W. Ibrahim, M. Darus, General properties for Volterra-type operators in the unit disk, *International Scholarly Research Network ISRN Mathematical Analysis* (2011), Art. No. 149830, 11 pages. Doi: doi.org/10.5402/2011/149830

[6] Y.C. Kim, S. Ponnusamy, T. Sugawa, Mapping properties of nonlinear integral operators and pre-Schwarzian derivatives, *J. Math. Anal. Appl.* **299** (2004) 433–447.

[7] Y.C. Kim, T. Sugawa, Norm estimates of the per-Schwarzian derivative for certain classes of univalent functions, *Proc. of the Edinburg Mathematical Society* **49** (2006) 131–143.

[8] S. Klimentov, Representations of the second kind for the Hardy classes of solutions to the Beltrami equation, *Siberian Mathematical Journal* **55** (2) (2014) 262–275.

[9] E. Pozzi, Hardy spaces of generalized analytic functions and composition operators, *Concrete Operators* **5** (1) (2018) 9–23.

[10] J.K. Prajapat, Some sufficient conditions for certain class of analyatic and Multivalent functions, *Southeast Asian Bull. Math.* **34** (2) (2010) 357–361.

[11] G.S. Sàlàgean, Subclasses of univalent functions, In: *Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981)*, Lecture Notes in Math. **1013**, Springer Berlin, 1983.