Southeast Asian Bulletin of Mathematics (2021) 45: 197–216

Southeast Asian
Bulletin of
Mathematics

Positivity of Sums and Integrals for n-Convex Functions via Abel-Gontscharoff's Interpolating Polynomial and Green Functions*

Asif R. Khan
Department of Mathematics, Faculty of Science, University of Karachi, University Road, Karachi-75270, Pakistan
Email: asifrk@uok.edu.pk

Josip E. Pečarić
RUDN University, Miklukho-Maklaya str. 6, 117198 Moscow, Russia
Email: pecaric@hazu.hr

Marjan Praljak
Faculty of Food Technology and Biotechnology, University of Zagreb, Zagreb, Croatia
Email: mpraljak@pbf.hr

Sanja Varosanec
Department of Mathematics, University of Zagreb, 10000 Zagreb, Croatia
Email: varosans@math.hr

Received 9 April 2019
Accepted 22 January 2020

Communicated by W.S. Cheung

AMS Mathematics Subject Classification(2000): 26A51, 26D15, 26D20

Abstract. We consider positivity of sum $\sum_{i=1}^{n} p_i f(x_i)$ involving convex functions of higher order. Analogous for integral $\int_a^b p(x)f(g(x))dx$ is also given. Representation of a function f via the Abel-Gontscharoff's Interpolating Polynomial and Green functions leads us to identities for which we obtain conditions for positivity of the mentioned sum

*This work was supported by the Ministry of Education and Science of the Russian Federation (the Agreement number No. 02.a03.21.0008.)

and integral. We obtain bound for integral remainders which occur in those identities as well as corresponding mean value theorem.

Keywords: n-convex functions; Abel-Gontscharoff's interpolating polynomial; Green function; Čebyšev functional.

1. Introduction

The Abel-Gontscharoff interpolation problem in the real case was introduced in 1935 by Whittaker [18] and subsequently by Gontscharoff [5] and Davis [4]. The Abel-Gontscharoff interpolating polynomial for two points with integral remainder is given in [1]:

Proposition 1.1. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$ and $f \in C^n[a,b]$. Then we have*

$$f(t) = Q_{n-1}(a,b,f,t) + R(f,t), \tag{1}$$

where Q_{n-1} is the Abel-Gontscharoff interpolating polynomial for two-points of degree $n-1$, i.e.,

$$\begin{aligned} & Q_{n-1}(a,b,f,t) \\ = & \sum_{i=0}^{k} \frac{(t-a)^i}{i!} f^{(i)}(a) + \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{(t-a)^{k+1+i}(a-b)^{j-i}}{(k+1+i)!(j-i)!} f^{(k+1+j)}(b) \end{aligned}$$

and the remainder is given by

$$R(f,t) = \int_a^b G_n(t,s) f^{(n)}(s)\, ds,$$

where $G_n(t,s)$ is the Green function in [3, p. 177] *given as*

$$G_n(t,s) = \frac{1}{(n-1)!} \begin{cases} \displaystyle\sum_{i=0}^{k} \binom{n-1}{i} (t-a)^i (a-s)^{n-i-1} & a \leq s \leq t, \\ \displaystyle -\sum_{i=k+1}^{n-1} \binom{n-1}{i} (t-a)^i (a-s)^{n-i-1} & t \leq s \leq b. \end{cases} \tag{2}$$

Further, for $a \leq s, t \leq b$ the following inequality hold:

$$(-1)^{n-k-1} \frac{\partial^i G_n(t,s)}{\partial t^i} \geq 0, \quad 0 \leq i \leq k, \tag{3}$$

$$(-1)^{n-i} \frac{\partial^i G_n(t,s)}{\partial t^i} \geq 0, \quad k+1 \leq i \leq n-1. \tag{4}$$

For further results related to higher order convex functions one can see [11].

The following result is due to Popoviciu [16, 17] (see also [14, 15]).

Proposition 1.2. *Let $n \geq 2$. Inequality*

$$\sum_{r=1}^{m} p_r f(x_r) \geq 0 \tag{5}$$

holds for all n-convex functions $f : [a,b] \to \mathbb{R}$ if and only if the $m-$tuples $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ satisfy

$$\sum_{r=1}^{m} p_r x_r^k = 0, \qquad \textit{for all } k \in \{0,1,\ldots,n-1\}, \tag{6}$$

$$\sum_{i=1}^{m} p_r (x_r - t)_+^{n-1} \geq 0, \qquad \textit{for every } t \in [a,b]. \tag{7}$$

Remark 1.3. If $n = 2$, then conditions (6) and (7), i.e.,

$$\sum_{r=1}^{m} p_r = 0, \qquad \sum_{r=1}^{m} p_r x_r = 0$$

and

$$\sum_{r=1}^{m} p_r (x_r - x_i)_+ \geq 0, \qquad i \in \{1,\ldots,m-1\}$$

can be replaced by

$$\sum_{r=1}^{m} p_r = 0 \quad \text{and} \quad \sum_{r=1}^{m} p_r |x_r - x_i| \geq 0 \quad \text{for} \quad i \in \{1,\ldots,m\},$$

and vice versa.

The integral analogue is given in the next proposition.

Proposition 1.4. *Let $n \geq 2$, $p : [\alpha,\beta] \to \mathbb{R}$ and $g : [\alpha,\beta] \to [a,b]$. Then, the inequality*

$$\int_{\alpha}^{\beta} p(x) f(g(x))\, dx \geq 0 \tag{8}$$

holds for all n-convex functions $f : [a,b] \to \mathbb{R}$ if and only if

$$\begin{aligned} &\int_{\alpha}^{\beta} p(x) g(x)^k\, dx = 0, \qquad \textit{for all } k \in \{0,1,\ldots,n-1\} \\ &\int_{\alpha}^{\beta} p(x) \left(g(x) - t\right)_+^{n-1}\, dx \geq 0, \qquad \textit{for every } t \in [a,b]. \end{aligned} \tag{9}$$

After this introductory section, we follow with Sections 2 and 3 where identities for $\sum_{i=1}^{n} p_i f(x_i)$ and $\int_a^b p(x)f(g(x))dx$ are given using Abel-Gontscharoff's Interpolating Polynomial and the Green functions respectively. Also we consider inequalities for n-convex functions which based on these identities. Section 4 is devoted to estimations of functions A_k by using Čebyšev and Ostrowski type inequalities. In last we give idea to prove related mean value theorems and study of n-exponential convexity related to functions defined in up coming sections. It is worth mentioning that some of its results can also be found in [8].

2. Popoviciu Type Identities and Inequalities and Abel-Gontscharoff's Interpolating Polynomial

We start this section with the identities of generalizations of Popoviciu type inequality using Abel-Gontscharoff interpolating polynomial for two points.

Theorem 2.1. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, $\boldsymbol{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}^m$. Let $f \in C^n[a,b]$ and G_n be the Green function defined as in* (2). *Then*

$$\sum_{r=1}^{m} p_r\, f\left(x_r\right) \,=\, \theta_1(f) + \int_a^b \left(\sum_{r=1}^{m} p_r G_n\left(x_r, s\right)\right) f^{(n)}(s)ds, \tag{10}$$

where

$$\begin{aligned}\theta_1(f) \,=\, &\sum_{i=0}^{k} \frac{f^{(i)}(a)}{i!} \sum_{r=1}^{m} p_r\left(x_r - a\right)^i \\ &+ \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \left(\sum_{r=1}^{m} p_r (x_r - a)^{k+1+i}\right) \frac{(-1)^{j-i}\left(b-a\right)^{j-i}}{(k+1+i)!(j-i)!} f^{(k+1+j)}(b).\end{aligned} \tag{11}$$

Proof. Consider the expression

$$\sum_{r=1}^{m} p_r f(x_r). \tag{12}$$

By using Proposition 1.1, we have

$$\begin{aligned}f(t) =&\sum_{i=0}^{k} \frac{\left(t-a\right)^i}{i!} f^{(i)}(a) \\ &+ \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \frac{\left(t-a\right)^{k+1+i}\left(-1\right)^{j-i}\left(b-a\right)^{j-i}}{(k+1+i)!\,(j-i)!} f^{(k+1+j)}(b) \\ &+ \int_a^b G_n\left(t,s\right) f^{(n)}(s)ds.\end{aligned} \tag{13}$$

Substituting this value of f in (12) and some arrangements, we get (10). ∎

Integral version of the above theorem can be stated as:

Theorem 2.2. *Let $n,k \in \mathbb{N}$, $n \geq 2$, $0 \leq k \leq n-1$, and $x : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ be continuous functions. Let $f \in C^n[a,b]$ and G_n be the Green function defined as in* (2). *Then*

$$\int_{\alpha}^{\beta} p(\tau)\, f\left(x(\tau)\right) d\tau = \theta_2(f) + \int_a^b \left(\int_{\alpha}^{\beta} p(\tau) G_n\left(x(\tau), s\right) d\tau \right) f^{(n)}(s) ds, \tag{14}$$

where

$$\begin{aligned} \theta_2(f) =& \sum_{i=0}^{k} \frac{f^{(i)}(a)}{i!} \int_{\alpha}^{\beta} p(\tau)\left(x(\tau)-a\right)^i dt \\ &+ \sum_{j=0}^{n-k-2} \sum_{i=0}^{j} \left(\int_{\alpha}^{\beta} p(\tau)\left(x(\tau)-a\right)^{k+1+i} d\tau \right) \\ &\times \frac{(-1)^{j-i}\left(b-a\right)^{j-i}}{(k+1+i)!(j-i)!} f^{(k+1+j)}(b). \end{aligned} \tag{15}$$

If $\mathbf{x}$ and $\mathbf{p}$ satisfy additional conditions, then we get generalization of Popoviciu type inequality for $n-$convex functions, i.e., we give a lower bound for the sum $\sum p_r f(x_r)$ which depends only on nodes $x_1,\ldots,x_m$, weights $p_1,\ldots,p_m$ and values of higher derivatives of a function f at points a and b.

Theorem 2.3. *Let all the assumptions of Theorem 2.1 be valid. In addition, if for all $s \in [a,b]$*

$$0 \leq \sum_{r=1}^{m} p_r G_n\left(x_r, s\right), \tag{16}$$

then for every n-convex function $f : [a,b] \to \mathbb{R}$, the following inequality holds

$$\sum_{r=1}^{m} p_r\, f\left(x_r\right) \geq \theta_1(f), \tag{17}$$

where $\theta_1(f)$ is given in (11).

If the reverse inequality in (16) *holds, then also the reverse inequality in* (17) *holds.*

Proof. Since the function f is n-convex, therefore without loss of generality we can assume that f is n-times differentiable and $f^{(n)}(x) \geq 0$, for all $x \in [a,b]$. Hence we can apply Theorem 2.1 to get (17). ∎

Integral version of the above theorem can be stated as:

Theorem 2.4. *Let all the assumptions of Theorem 2.2 be valid. In addition, if for all $s \in [a,b]$*

$$0 \leq \int_{\alpha}^{\beta} p(\tau)\, G_n\left(x(\tau), s\right) d\tau, \tag{18}$$

then for every n-convex function $f : [a,b] \to \mathbb{R}$, it holds

$$\int_{\alpha}^{\beta} p(\tau)\, f\left(x(\tau)\right) d\tau \geq \theta_2(f), \tag{19}$$

where $\theta_2(f)$ is defined in (15).

If the reverse inequality in (18) holds, then also the reverse inequality in (19) holds.

In some cases the assumption $\sum_{r=1}^{m} p_r G_n\left(x_r, s\right) \geq 0, \quad s \in [a,b]$ can be replaced with more simpler conditions in which we recognize assumptions from Popoviciu's theorem about positivity of sum $\sum p_r f(x_r)$ for a convex function f. Namely we have the following statement.

Theorem 2.5. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $1 \leq k \leq n-1$, $\mathbf{x} \in [a,b]^m$ $\mathbf{p} \in \mathbb{R}^m$ be m-tuples such that $\sum_{r=1}^{m} p_r = 0$, $\sum_{r=1}^{m} p_r |x_r - x_s| \geq 0$, for $s = 1, 2, \ldots, m$ and let G_n be the Green function defined as in (2).*

(i) *If k is odd and n is even or k is even and n is odd, then for every n-convex function $f : [a,b] \to \mathbb{R}$, it holds*

$$\sum_{r=1}^{m} p_r\, f\left(x_r\right) \geq \theta_1(f), \tag{20}$$

where $\theta_1(f)$ is given in (11).
Moreover, if $f^{(i)}(a) \geq 0$ for $i \in \{2, \ldots, k\}$ and $f^{(k+1+j)}(b) \geq 0$ if $j - i$ is even and $f^{(k+1+j)}(b) \leq 0$ if $j - i$ is odd for $i \in \{0, \ldots, j\}$ and $j \in \{0, \ldots, n-k-2\}$, then $\sum_{r=1}^{m} p_r\, f\left(x_r\right) \geq 0$.

(ii) *If k and n both are even or odd, then for every n-convex function $f : [a,b] \to \mathbb{R}$, the reverse inequality in (20) holds.*
Moreover, if $f^{(i)}(a) \leq 0$ for $i = 0, \ldots, k$ and $f^{(k+1+j)}(b) \leq 0$ if $j - i$ is even, and $f^{(k+1+j)}(b) \geq 0$ if $j - i$ is odd for $i \in \{0, \ldots, j\}$ and $j \in \{0, \ldots, n-k-2\}$, then $\sum_{r=1}^{m} p_r\, f\left(x_r\right) \leq 0$.

Proof. (i) Let us consider properties (3) and (4) for $i = 2$. If k is odd and n is even, then for $k = 1$ we get $(-1)^{n-2} \frac{\partial^2 G_n(t,s)}{\partial t^2} \geq 0$ from (4), i.e. $\frac{\partial^2 G_n(t,s)}{\partial t^2} \geq 0$, i.e. G_n is convex. For $k > 1$, from (3) we get the same inequality. If k is even and

n is odd, then $k \geq 2$ and from (3) we get that G_n is convex in the first variable. By Remark 1.1, applied on the function G_n we get

$$\sum_{r=1}^{m} p_r G_n\left(x_r, s\right) \geq 0,$$

i.e., the assumptions of Theorem 2.3 are fulfilled and inequality (20) holds. If further assumptions on $f^{(i)}(a)$ and $f^{(k+1+j)}(b)$ are valid, then the right-hand side of (20) is nonnegative.

The case (ii) is proved in a similar manner. ■

An integral analogue of the previous theorem is the following theorem.

Theorem 2.6. *Let $n, k \in \mathbb{N}$, $n \geq 2$, $1 \leq k \leq n-1$, $x : [\alpha, \beta] \to [a, b]$ and $p : [\alpha, \beta] \to \mathbb{R}$ be continuous functions satisfying*

$$\int_{\alpha}^{\beta} p(\tau) = 0, \qquad \int_{\alpha}^{\beta} p(\tau) x(\tau) = 0,$$

and

$$\int_{\alpha}^{\beta} p(\tau)(x(\tau) - s)_+ \geq 0 \quad \text{for } s \in [a, b],$$

and let G_n be the Green function defined as in (2).

(i) *If k is odd and n is even or k is even and n is odd, then for every n-convex function $f : [a, b] \to \mathbb{R}$, then*

$$\int_{\alpha}^{\beta} p(\tau)\, f\left(x(\tau)\right) d\tau \geq \theta_2(f). \tag{21}$$

Moreover, if $f^{(i)}(a) \geq 0$ for $i \in \{0, \ldots, k\}$ and $f^{(k+1+j)}(b) \geq 0$ if $j - i$ is even and $f^{(k+1+j)}(b) \leq 0$ if $j - i$ is odd for $i \in \{0, \ldots, j\}$ and $j \in \{0, \ldots, n-k-2\}$, then $\int_{\alpha}^{\beta} p(t) f\left(x(t)\right) dt \geq 0$.

(ii) *If k and n both are even or odd, then for every n-convex function $f : [a, b] \to \mathbb{R}$, then the reverse inequality holds in (21).*
Moreover, if $f^{(i)}(a) \leq 0$ for $i \in \{0, \ldots, k\}$ and $f^{(k+1+j)}(b) \leq 0$ if $j - i$ is even, and $f^{(k+1+j)}(b) \geq 0$ if $j - i$ is odd for $i \in \{0, \ldots, j\}$ and $j \in \{0, \ldots, n-k-2\}$, then $\int_{\alpha}^{\beta} p(t) f\left(x(t)\right) dt \leq 0$.

3. Results Obtained by Green Functions and Abel-Gontscharoff's Interpolating Polynomial

Now we recall the definition of Green function G which would be used in some of our results. The function $G : [a,b] \times [a,b]$ is defined by

$$G(s,t) = \begin{cases} \dfrac{(s-b)(t-a)}{b-a} & \text{for } a \le t \le s, \\ \dfrac{(t-b)(s-a)}{b-a} & \text{for } s \le t \le b. \end{cases} \tag{22}$$

The function G is convex and continuous with respect to both s and t.

For any function $f : [a,b] \to \mathbb{R}$, $f \in C^2[a,b]$, we can obtain the following integral identity by simply using integration by parts

$$f(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) + \int_a^b G(x,s) f''(s) ds, \tag{23}$$

where the function G is defined as above in (22) (see also [19]).

Also, integration by parts easily yields that for any function $f \in C^2[a,b]$ the following identity holds

$$f(x) = f(a) - af'(b) + f'(b)x + \int_a^b G_1(x,s) f''(s) ds, \tag{24}$$

where the function $G_1 : [a,b] \times [a,b] \to \mathbb{R}$ is the Green's function for 'two-point right focal problem' of the boundary value problem

$$z''(\xi) = 0, \;\; z(a) = z(b) = 0$$

and is given by

$$G_1(s,t) = \begin{cases} a-t & \text{if } \; a \le t \le s, \\ a-s & \text{if } \; s \le t \le b. \end{cases} \tag{25}$$

Motivated by Abel-Gontscharoff identity (24) and the related Green's function (25), we would like to recall some new types of Green functions $G_l : [a,b] \times [a,b] \to \mathbb{R}$, $(l = 2,3,4,)$ defined as in [6] (see also [7]):

$$G_2(s,t) = \begin{cases} s-b & \text{if } \; a \le t \le s, \\ t-b & \text{if } \; s \le t \le b, \end{cases} \tag{26}$$

$$G_3(s,t) = \begin{cases} s-a & \text{if } \; a \le t \le s, \\ t-a & \text{if } \; s \le t \le b, \end{cases} \tag{27}$$

$$G_4(s,t) = \begin{cases} b-t & \text{if } \; a \le t \le s, \\ b-s & \text{if } \; s \le t \le b. \end{cases} \tag{28}$$

The functions G_l for $l \in \{1,2,3,4\}$ are continuous, symmetric and convex with respect to both variables s and t.

In similar manner as in equation (24) we can define other equations for G_2, G_3 and G_4 as follows

$$f(x) = f(b) - bf'(a) - f'(a)x + \int_a^b G_2(x,s)f''(s)ds, \tag{29}$$

$$f(x) = f(b) - bf'(b) + (f'(b) - f'(a))a + f'(a)x + \int_a^b G_3(x,s)f''(s)ds, \tag{30}$$

$$f(x) = f(a) - af'(a) - (f'(b) - f'(a))b + f'(b)x + \int_a^b G_4(x,s)f''(s)ds. \tag{31}$$

Now we obtain results using the Green function G, (22), together with the Abel-Gontscharoff polynomials. Here it is worth mentioning that we would use G_0 for Green function G defined in (22).

We begin with some identities related to generalizations of Popoviciu type inequality.

Theorem 3.1. *Let $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $f \in C^n[\alpha, \beta]$ and $\boldsymbol{x} \in [a,b]^m$, $\boldsymbol{p} \in \mathbb{R}^m$. Also let G and G_n be defined by (22) and (2) respectively. Then*

$$\sum_{r=1}^{m} p_r\, f(x_r) = \theta_3(f, G_0) + \int_a^b \int_a^b \left(\sum_{r=1}^{m} p_r G(x_r, s) \right) G_{n-2}(s,t) f^{(n)}(t) dt ds,$$

where $\theta_3(f, G_0)$ is defined as

$$\begin{aligned} \theta_3(f, G_0) \;=\; & \frac{f(b) - f(a)}{b-a} \sum_{r=1}^{m} p_r x_r + \frac{bf(a) - af(b)}{b-a} \sum_{r=1}^{m} p_r \\ & + \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \sum_{r=1}^{m} p_r G(x_r, s)\,(s-a)^i\, ds \\ & + \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}\,(b-a)^{j-i}\, f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\ & \times \int_a^b \sum_{r=1}^{m} p_r G(x_r, s)\,(s-a)^{k+1+i}\, ds. \end{aligned} \tag{32}$$

Proof. Putting $x = x_r$ in (23), multiplying it with p_r where $r \in \{1, 2, \ldots m\}$, adding all the identities we get

$$\begin{aligned} \sum_{r=1}^{m} p_r f(x_r) =& \frac{f(b) - f(a)}{b-a} \sum_{r=1}^{m} p_r x_r + \frac{bf(a) - af(b)}{b-a} \sum_{r=1}^{m} p_r \\ & + \int_a^b \left(\sum_{r=1}^{m} p_r G(x_r, s) \right) f''(s)\, ds. \end{aligned} \tag{33}$$

Applying the Able-Gontscharoff (10) identity with $f \to f''$ and $n \to n-2$, it is easy to see that

$$f''(s) = \sum_{i=0}^{k} \frac{(s-a)^i}{i!} f^{(i+2)}(a) + \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(s-a)^{k+1+i}(a-b)^{j-i}}{(k+1+i)!\,(j-i)!} f^{(k+3+j)}(b)$$
$$+ \int_a^b G_{n-2}(s,t) f^{(n)}(t)dt. \qquad \blacksquare$$

In similar manner we can state further results related to other Green functions $G_1 - G_4$ as follows:

Theorem 3.2. *Let $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $f \in C^n[\alpha, \beta]$ and $\boldsymbol{x} \in [a,b]^m$, $\boldsymbol{p} \in \mathbb{R}^m$. Also let $G_1 - G_4$ and G_n be defined by $(25)-(28)$ and (2) respectively. Then for $l \in \{4,5,6,7\}$*

$$\sum_{r=1}^{m} p_r f(x_r) = \theta_l(f, G_{l-3}) + \int_a^b \int_a^b \left(\sum_{r=1}^{m} p_r G_{l-3}(x_r, s) \right) G_{n-2}(s,t) f^{(n)}(t)dtds,$$

where

$$\begin{aligned} \theta_4(f,G_1) = {} & (f(a) - af'(b)) \sum_{r=1}^{m} p_r + f'(b) \sum_{r=1}^{m} p_r x_r \\ & + \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \sum_{r=1}^{m} p_r G_1(x_r,s)(s-a)^i ds \\ & + \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i} f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\ & \times \int_a^b \sum_{r=1}^{m} p_r G_1(x_r,s)(s-a)^{k+1+i} ds, \end{aligned} \tag{34}$$

$$\begin{aligned} \theta_5(f,G_2) = {} & (f(b) - bf'(a)) \sum_{r=1}^{m} p_r - f'(a) \sum_{r=1}^{m} p_r x_r \\ & + \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \sum_{r=1}^{m} p_r G_2(x_r,s)(s-a)^i ds \\ & + \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i} f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\ & \times \int_a^b \sum_{r=1}^{m} p_r G_2(x_r,s)(s-a)^{k+1+i} ds, \end{aligned} \tag{35}$$

$$\theta_6(f,G_3) = (f(b) - bf'(b) + (f'(b) - f'(a))a) \sum_{r=1}^{m} p_r + f'(a) \sum_{r=1}^{m} p_r x_r \tag{36}$$

$$
\begin{aligned}
&+\sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \sum_{r=1}^{m} p_r G_3(x_r,s)\,(s-a)^i\,ds \\
&+\sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}\,(b-a)^{j-i}\,f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\
&\times \int_a^b \sum_{r=1}^{m} p_r G_3(x_r,s)\,(s-a)^{k+1+i}\,ds,
\end{aligned}
$$

$$
\begin{aligned}
\theta_7(f,G_4) \;=\;& (f(a)-af'(a)-(f'(b)-f'(a))b)\sum_{r=1}^{m} p_r + f'(b)\sum_{r=1}^{m} p_r x_r \qquad (37)\\
&+\sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \sum_{r=1}^{m} p_r G_4(x_r,s)\,(s-a)^i\,ds \\
&+\sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}\,(b-a)^{j-i}\,f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\
&\times \int_a^b \sum_{r=1}^{m} p_r G_4(x_r,s)\,(s-a)^{k+1+i}\,ds.
\end{aligned}
$$

Proof. The proof is followed by using technique of proof of Theorem 3.1 by using respective identities (24), (29) – (31) for Green functions $G_1 - G_4$. ■

Theorem 3.3. *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $f \in C^n[a,b]$, and let $x : [\alpha,\beta] \to [a,b]$, $p : [\alpha,\beta] \to \mathbb{R}$ be continuous functions. Also let $G_1 - G_4$ and G_n be defined by* (25) – (28) *and* (2) *respectively. Then*

$$
\begin{aligned}
&\int_\alpha^\beta p(\tau) f(x(\tau))d\tau \\
&= \theta_8(f,G_0) + \int_a^b \int_a^b \int_\alpha^\beta p(\tau) G(x(\tau),s) G_{n-2}(s,t) f^{(n)}(t) d\tau ds dt,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_8(f,G_0) \;=\;& \frac{f(b)-f(a)}{b-a} \int_\alpha^\beta p(\tau)x(\tau)d\tau + \frac{bf(a)-af(b)}{b-a}\int_\alpha^\beta p(\tau)d\tau \\
&+\sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \int_\alpha^\beta p(\tau)G(x(\tau),s)d\tau(s-a)^i ds \qquad (38)\\
&+\sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}\,(b-a)^{j-i}\,f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\
&\times \int_a^b \int_\alpha^\beta p(\tau)G(x(\tau),s)\,(s-a)^{k+1+i}\,d\tau\,ds.
\end{aligned}
$$

Theorem 3.4. *Let $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $f \in C^n[a,b]$, and let $x : [\alpha, \beta] \to [a,b]$, $p : [\alpha, \beta] \to \mathbb{R}$ be continuous functions and $G_0 - G_4$, G_n be defined by* (22), (25) − (28) *and* (2) *respectively. Then for $l \in \{9, 10, 11, 12\}$*

$$\begin{aligned}
&\int_\alpha^\beta p(\tau) f(x(\tau))\, d\tau \\
&= \theta_l(f, G_{l-8}) + \int_a^b \int_a^b \int_\alpha^\beta p(\tau) G_{l-8}(x(\tau), s) G_{n-2}(s,t) f^{(n)}(t) d\tau\, ds\, dt,
\end{aligned}$$

$$\begin{aligned}
\theta_9(f, G_1) &= (f(a) - af'(b)) \int_\alpha^\beta p(x)\, dx + f'(b) \int_\alpha^\beta p(x) g(x)\, dx \\
&+ \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \int_\alpha^\beta p(\tau) G_1(x(\tau), s) d\tau (s-a)^i ds \\
&+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i} (b-a)^{j-i} f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\
&\times \int_a^b \int_\alpha^\beta p(\tau) G_1(x(\tau), s)\, (s-a)^{k+1+i}\, d\tau\, ds, \\
\theta_{10}(f, G_2) &= (f(b) - bf'(a)) \int_\alpha^\beta p(x)\, dx - f'(a) \int_\alpha^\beta p(x) g(x)\, dx \\
&+ \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \int_\alpha^\beta p(\tau) G_2(x(\tau), s) d\tau (s-a)^i ds \\
&+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i} (b-a)^{j-i} f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\
&\times \int_a^b \int_\alpha^\beta p(\tau) G_2(x(\tau), s)\, (s-a)^{k+1+i}\, d\tau\, ds, \\
\theta_{11}(f, G_3) &= (f(b) - bf'(b) + (f'(b) - f'(a))a) \int_\alpha^\beta p(x) dx \\
&+ f'(a) \int_\alpha^\beta p(x) g(x) dx \\
&+ \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \int_\alpha^\beta p(\tau) G_3(x(\tau), s) d\tau (s-a)^i ds \\
&+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i} (b-a)^{j-i} f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\
&\times \int_a^b \int_\alpha^\beta p(\tau) G_3(x(\tau), s)\, (s-a)^{k+1+i}\, d\tau\, ds, \\
\theta_{12}(f, G_4) &= (f(a) - af'(a) - (f'(b) - f'(a))b)
\end{aligned}$$

$$+\int_{\alpha}^{\beta} p(x)dx + f'(b)\int_{\alpha}^{\beta} p(x)g(x)\,dx$$
$$+\sum_{i=0}^{k}\frac{f^{(i+2)}(a)}{i!}\int_a^b\int_{\alpha}^{\beta} p(\tau)G(x_4(\tau),s)d\tau(s-a)^i ds$$
$$+\sum_{j=0}^{n-k-4}\sum_{i=0}^{j}\frac{(-1)^{j-i}(b-a)^{j-i}f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!}$$
$$\times\int_a^b\int_{\alpha}^{\beta} p(\tau)G_4(x(\tau),s)\,(s-a)^{k+1+i}\,d\tau\,ds.$$

Theorem 3.5. *Let $n,k\in\mathbb{N}$, $n\geq 4$, $0\leq k\leq n-1$, $\boldsymbol{x}\in[a,b]^m$ and $\boldsymbol{p}\in\mathbb{R}^m$. Also let G_0,G_1-G_4, G_n be defined by* (22), (25) − (28) *and* (2) *respectively.*

If $f:[a,b]\to\mathbb{R}$ is $n-$convex, and

$$\int_a^b\left(\sum_{r=1}^{m}p_rG_{l-3}(x_r,s)\right)G_{n-2}(s,t)ds\geq 0,\qquad t\in[a,b],\tag{39}$$

then for $l\in\{3,4,5,6,7\}$

$$\sum_{r=1}^{m}p_r\,f\,(x_r)\;\geq\;\theta_l(f,G_{l-3}).\tag{40}$$

If the reverse inequality in (39) *holds, then also the reverse inequality in* (40) *holds.*

Proof. It follows from n-convexity of a function f and from Theorem 3.1. ∎

As from (3) we have $(-1)^{n-k-3}G_{n-2}(s,t)\geq 0$, therefore for the case when n is even and k is odd or n is odd and k is even, it is enough to assume that $\sum_{r=1}^{m}p_rG(x_r,s)\geq 0, s\in[\alpha,\beta]$, instead of the assumption (39) in Theorem 3.5. Similarly we can discuss for the reverse inequality in (40).

Integral version of the above theorem can be stated as:

Theorem 3.6. *Let $n,k\in\mathbb{N}$, $n\geq 4$, $0\leq k\leq n-1$, $x:[\alpha,\beta]\to[a,b]$, $p:[\alpha,\beta]\to\mathbb{R}$ be continuous functions and G_0,G_1-G_4, G_n be defined by* (22), (25) − (28) *and* (2) *respectively. If $f:[a,b]\to\mathbb{R}$ is $n-$convex, and*

$$\int_a^b\int_{\alpha}^{\beta} p(\tau)G_{l-8}(x(\tau),s)G_{n-2}(s,t)d\tau\,ds\geq 0,\tag{41}$$

then $l\in\{8,9,10,11,12\}$

$$\int_{\alpha}^{\beta} p(\tau)f(x(\tau))d\tau\geq\theta_l(f,G_{l-8}).\tag{42}$$

If the reverse inequality in (41) *holds, then also the reverse inequality in* (42) *holds.*

As from (3) we have $(-1)^{n-k-3}G_{n-2}(s,t) \geq 0$, therefore for the case when n is even and k is odd or n is odd and k is even, it is enough to assume that $\int_a^b p(\tau)G_{l-8}(x(\tau),s)d\tau \geq 0, s \in [\alpha,\beta],\ l \in \{8,9,10,11,12\}$, instead of the assumption (41) in Theorem 3.6. Similarly we can discuss for the reverse inequality in (42).

If we deal with assumptions from Remark 1.1, which are equivalent to Popoviciu's conditions for positivity of sum involving convex function f, then for some combinations of n and k we get result for n-convex function f. Precisely, we get the following theorem.

Theorem 3.7. *Let $n,k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$. Let G_0 and $G_1 - G_4$ be defined by* (22), (25) – (28) *and let $f : [a,b] \to \mathbb{R}$ be $n-$convex. Let $\mathbf{x} \in [a,b]^m$ and $\mathbf{p} \in \mathbb{R}$ satisfy*

$$\sum_{r=1}^{m} p_r = 0, \qquad \sum_{r=1}^{m} p_r|x_r - x_s| \geq 0, \ \textit{for } s \in \{1,2,\ldots,m\}.$$

(i) *If n is even and k is odd or n is odd and k is even, then for $l \in \{0,1,2,3,4\}$*

$$\begin{aligned}\sum_{r=1}^{m} p_l f(x_r) \geq{}& \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \sum_{r=1}^{m} p_r G_l(x_r,s)(s-a)^i\, ds \\ &+ \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}(b-a)^{j-i} f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\ &\times \int_a^b \sum_{r=1}^{m} p_r G_l(x_r,s)(s-a)^{k+1+i}\, ds. \end{aligned} \tag{43}$$

Moreover if $f^{(i+2)}(a) \geq 0$ for $i \in \{0,\ldots,k\}$ and $f^{(k+3+j)}(b) \geq 0$ if $j-i$ is even and $f^{(k+3+j)}(b) \leq 0$ if $j-i$ is odd for $i \in \{0,\ldots,j\}$ and $j \in \{0,\ldots,n-k-4\}$, then $\sum_{r=1}^{m} p_r f(x_r) \geq 0$.

(ii) *If n and k both are even or both are odd, then reverse inequality holds in* (43).
Moreover if $f^{(i+2)}(a) \leq 0$ for $i \in \{0,\ldots,k\}$ and $f^{(k+3+j)}(b) \leq 0$ if $j-i$ is even and $f^{(k+3+j)}(b) \geq 0$ if $j-i$ is odd for $i \in \{0,\ldots,j\}$ and $j \in \{0,\ldots,n-k-4\}$, then $\sum_{r=1}^{m} p_r f(x_r) \leq 0$.

Proof. (i) By using (3) we have $(-1)^{n-k-3}G_{n-2}(s,t) \geq 0,\ a \leq s,t \leq b$, therefore if n is even and k is odd or n is odd and k is even then $G_{n-2}(s,t) \geq 0$. Since G is convex and G_{n-2} is nonnegative, the inequality (39) holds. Hence by Theorem 3.5 the inequality (43) holds. By using the other conditions the nonnegativity of the right-hand side of (43) is obvious.

Similarly we prove (ii). ■

The integral version of Theorem 3.7 can be stated as:

Theorem 3.8. *Let $n, k \in \mathbb{N}$, $n \geq 4$, $0 \leq k \leq n-1$, $x : [\alpha, \beta] \to [a, b]$ and $p : [\alpha, \beta] \to \mathbb{R}$ be any continuous functions. Also let G, $G_1 - G_4$ be defined by* (22) *and* (25) − (28)*. Consider $f : [a, b] \to \mathbb{R}$ is $n-$convex and*

$$\int_\alpha^\beta p(\tau)\, d\tau \geq 0, \qquad \int_\alpha^\beta p(\tau)(x(\tau)-t)_+\, d\tau \geq 0 \ \text{ for } \ t \in [a,b], \tag{44}$$

(i) *If n is even and k is odd or n is odd and k is even, then for $l \in \{0, 1, 2, 3, 4\}$*

$$\begin{aligned}\int_\alpha^\beta p(\tau) f(x(\tau)) d\tau \geq & \sum_{i=0}^{k} \frac{f^{(i+2)}(a)}{i!} \int_a^b \int_\alpha^\beta p(\tau) G_l(x(\tau), s)(s-a)^i d\tau\, ds \\ & + \sum_{j=0}^{n-k-4} \sum_{i=0}^{j} \frac{(-1)^{j-i}\,(b-a)^{j-i}\, f^{(k+3+j)}(b)}{(k+1+i)!\,(j-i)!} \\ & \times \int_a^b \int_\alpha^\beta p(\tau) G_l(x(\tau), s)\,(s-a)^{k+1+i}\, d\tau\, ds.\end{aligned} \tag{45}$$

Moreover if $f^{(i+2)}(a) \geq 0$ for $i \in \{0, \ldots, k\}$ and $f^{(k+3+j)}(b) \geq 0$ if $j-i$ is even and $f^{(k+3+j)}(b) \leq 0$ if $j-i$ is odd for $i \in \{0, \ldots, j\}$ and $j \in \{0, \ldots, n-k-4\}$, then the right-hand side of (45) *is nonnegative, that is integral version of* (5) *holds.*

(ii) *If n and k both are even or both are odd, then reverse inequality holds in* (45)*.*
Moreover if $f^{(i+2)}(a) \leq 0$ for $i \in \{0, \ldots, k\}$ and $f^{(k+3+j)}(b) \leq 0$ if $j-i$ is even and $f^{(k+3+j)}(b) \geq 0$ if $j-i$ is odd for $i \in \{0, \ldots, j\}$ and $j \in \{0, \ldots, n-k-4\}$, then the right hand side of the reverse inequality in (45) *is nonpositive, that is the reverse inequality in the integral version of* (5) *holds.*

4. Related Inequalities for n-Convex Functions at a Point

Under the assumptions of Theorems 2.1, 3.1 and 3.2 here we define some linear functional as follows:

$$A_1(m, f, \mathbf{x}, \mathbf{p}, [a,b]) = \sum_{r=1}^{m} p_r f(x_r) - \theta_1(f) \tag{46}$$

For $l \in \{3, 4, 5, 6, 7\}$

$$A_l(m, f, \mathbf{x}, \mathbf{p}, [a,b]) = \sum_{r=1}^{m} p_r f(x_r) - \theta_l(f, G_{l-3}) \tag{47}$$

Similarly, under the assumptions of Theorem 2.2, 3.3 and 3.4 here we introduce some further linear functional as follows:

$$A_2([\alpha,\beta],f,x,p,[a,b]) = \int_{\alpha}^{\beta} p(\tau)f(x(\tau))\,d\tau - \theta_2(f) \tag{48}$$

For $l \in \{8,9,10,11,12\}$ we have

$$A_l([\alpha,\beta],f,x,p,[a,b]) = \int_{\alpha}^{\beta} p(\tau)f(x(\tau))\,d\tau - \theta_l(f,G_{l-8}) \tag{49}$$

Here we also define some new functionals B_1 and B_l for $l \in \{3,4,5,6,7\}$ as follows:

$$B_1(m,t,\mathbf{x},\mathbf{p},[a,b]) = \sum_{r=1}^{m} p_r G_n\left(x_r,t\right), \tag{50}$$

$$B_l(m,t,\mathbf{x},\mathbf{p},[a,b]) = \int_a^b \sum_{r=1}^{m} p_r G(x_r,s)G_{n-2}(s,t)ds \geq 0 \tag{51}$$

for all $t \in [a,b]$. And B_2 and B_l for $l \in \{8,9,10,11,12\}$ are defined as

$$B_2([\alpha,\beta],t,x,p,[a,b]) = \int_{\alpha}^{\beta} p(\tau)G_n\left(x(\tau),t\right)d\tau\,dx \geq 0, \tag{52}$$

$$B_l([\alpha,\beta],t,x,p,[a,b]) = \int_a^b \int_{\alpha}^{\beta} p(\tau)G_{l-8}(x(\tau),s)G_{n-2}(s,t)d\tau\,ds \geq 0 \tag{53}$$

for all $t \in [a,b]$. For the sake of brevity we consider $A_l(\cdot,f,\cdot,\cdot,\cdot) = A_l(f)$ and $B_l(\cdot,t,\cdot,\cdot,\cdot) = B_l(t)$. We state our next result.

In this section we will give related results for the class of n-convex functions at a point introduced in [12].

Definition 4.1. *Let I be an interval in $\mathbb{R}$, c a point in the interior of I and $n \in \mathbb{N}$. A function $f : I \to \mathbb{R}$ is said to be n-convex at point c if there exists a constant K such that the function*

$$F(x) = f(x) - \frac{K}{(n-1)!}x^{n-1} \tag{54}$$

is $(n-1)$-concave on $I \cap (-\infty,c]$ and $(n-1)$-convex on $I \cap [c,\infty)$. A function f is said to be n-concave at point c if the function $-f$ is n-convex at point c.

A property that explains the name of the class is the fact that a function is n-convex on an interval if and only if it is n-convex at every point of the interval (see [12]). For further details on the topic kindly see [12].

Let e_i denote the monomials $e_i(x) = x^i$, $i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout this section we let $c \in \langle a,b\rangle$.

Theorem 4.2. *Let* $\mathbf{x} \in [a,c]^{m_1}$*,* $\mathbf{p} \in \mathbb{R}^{m_1}$*,* $\mathbf{y} \in [c,b]^{m_2}$ *and* $\mathbf{q} \in \mathbb{R}^{m_2}$ *be such that for each* $l \in \{1,3,4,5,6,7\}$

$$B_l(m_1, t, \mathbf{x}, \mathbf{p}, [a,c]) \geq 0 \quad \text{for all } t \in [a,c], \tag{55}$$

$$B_l(m_2, t, \mathbf{y}, \mathbf{q}, [c,b]) \geq 0 \quad \text{for all } t \in [c,b], \tag{56}$$

and

$$A_l(m_1, e_n, \mathbf{x}, \mathbf{p}, [a,c]) = A_l(m_2, e_n, \mathbf{y}, \mathbf{q}, [c,b]) \tag{57}$$

where G_l *are Green functions given by* (22), (24), (29), (30) *and* (31) *respectively for* $l \in \{0,1,2,3,4\}$*, and* A_l *and* B_l *be the linear functionals given by* (46) – (49) *and* (50)–(54)*. If* $f : [a,b] \to \mathbb{R}$ *is* $(n+1)$*-convex at point* c*, then*

$$A_l(m_1, f, \mathbf{x}, \mathbf{p}, [a,c]) \leq A_l(m_2, f, \mathbf{y}, \mathbf{q}, [c,b]). \tag{58}$$

If the inequalities in (55) *and* (56) *are reversed, then* (58) *holds with the reversed sign of inequality.*

Proof. Fix $l \in \{1,3,4,5,6,7\}$. Let $F = f - \frac{K}{n!}e_n$ be as in Definition 4.1, i.e., the function F is n-concave on $[a,c]$ and n-convex on $[c,b]$. Applying Theorems 2.3 and 3.5 to F on the interval $[a,c]$ we have

$$0 \geq A_l(m_1, F, \mathbf{x}, \mathbf{p}, [a,c]) = A_l(m_1, f, \mathbf{x}, \mathbf{p}, [a,c]) - \frac{K}{n!}A_l(m_1, e_n, \mathbf{x}, \mathbf{p}, [a,c]) \tag{59}$$

and again applying Theorems 2.3 and 3.5 to F on the interval $[c,b]$ we have

$$0 \leq A_l(m_2, F, \mathbf{y}, \mathbf{q}, [c,b]) = A_l(m_2, f, \mathbf{y}, \mathbf{q}, [c,b]) - \frac{K}{n!}A_l(m_2, e_n, \mathbf{y}, \mathbf{q}, [c,b]). \tag{60}$$

So we do have

$$\begin{aligned} &A_l(m_1, f, \mathbf{x}, \mathbf{p}, [a,c]) - \frac{K}{n!}A_l(m_1, e_n, \mathbf{x}, \mathbf{p}, [a,c]) \\ \leq &A_l(m_2, f, \mathbf{y}, \mathbf{q}, [c,b]) - \frac{K}{n!}A_l(m_2, e_n, \mathbf{y}, \mathbf{q}, [c,b]). \end{aligned} \tag{61}$$

Hence, we finally get our required result by using assumption (57). ■

Theorem 4.3. *Let* $x : [\alpha,\beta] \to [a,c]$*,* $p : [\alpha,\beta] \to \mathbb{R}$*,* $y : [\gamma,\delta] \to [c,b]$*,* $q : [\gamma,\delta] \to \mathbb{R}$ *be such that for each* $l \in \{5,6,7,8\}$

$$B_l([\alpha,\beta], t, x, p, [a,c]) \geq 0 \quad \text{for all } t \in [a,c], \tag{62}$$

$$B_l([\gamma,\delta], t, y, q, [c,b]) \geq 0 \quad \text{for all } t \in [c,b] \tag{63}$$

and

$$A_l([\alpha,\beta], e_n, x, p, [a,c]) = A_l([\gamma,\delta], e_n, y, p, [c,b]) \tag{64}$$

where G_l are Green functions given by (22), (24), (29), (30) *and* (31) *respectively for $l \in \{0, 1, 2, 3, 4\}$, and A_l and B_l be the linear functionals given by* (46) − (49) *and* (50)–(54).

$$A_l([\alpha, \beta], f, x, p, [a, c]) \leq A_l([\gamma, \delta], f, y, p, [c, b]). \tag{65}$$

If the inequalities in (62) *and* (63) *are reversed, then* (65) *holds with the reversed sign of inequality.*

5. Bounds for Remainders and Functionals

Let $\digamma, h : [a, b] \to \mathbb{R}$ be two Lebesgue integrable functions. We consider the Čebyšev functional

$$T(\digamma, h) = \frac{1}{b-a} \int_a^b \digamma(\xi) h(\xi) d\xi - \left(\frac{1}{b-a} \int_a^b \digamma(\xi) d\xi\right) \left(\frac{1}{b-a} \int_a^b h(\xi) d\xi\right). \tag{66}$$

A bound for Čebyšev functional is given in following proposition in which pre-Grüss inequality is given (see [10]).

Proposition 5.1. *Let $\digamma, h : [a, b] \to \mathbb{R}$ be integrable s. t. $\digamma h \in L(a, b)$. If*

$$\gamma_1 \leq h(\eta) \leq \gamma_2 \quad \textit{for} \quad \eta \in [a, b],$$

then

$$|T(\digamma, h)| \leq \frac{1}{2}(\gamma_2 - \gamma_1)\sqrt{T(\digamma, \digamma)}.$$

For the sake of brevity we consider $A_l(\cdot, f, \cdot, \cdot, \cdot) = A_l(f)$ and $B_l(\cdot, t, \cdot, \cdot, \cdot) = B_l(t)$ which was defined in previous section. Now we state our next result.

Theorem 5.2. *Let $l \in \{1, \dots, 12\}$. Let $f \in C^n[a, b]$ such that for real numbers γ_1 and γ_2 we have*

$$\gamma_1 \leq f^{(n)}(\eta) \leq \gamma_2 \quad \textit{for } \eta \in [a, b].$$

Then in representation

$$A_l(f) = \frac{\left[f^{n-1}(b) - f^{n-1}(a)\right]}{b-a} \int_a^b B_l(\xi) d\xi + (b-a) R_n^l, \tag{67}$$

remainder R_n^l satisfies estimation

$$|R_n^l| \leq \frac{1}{2}(\gamma_2 - \gamma_1)\sqrt{T(B_l, B_l)}. \tag{68}$$

Proof. Fix $l \in \{1, \dots, 12\}$.

Starting with Čebyšev functional

$$T(F,h)=\frac{1}{b-a}\int_a^b F(\xi)h(\xi)d\xi-\left(\frac{1}{b-a}\int_a^b F(\xi)d\xi\right)\left(\frac{1}{b-a}\int_a^b h(\xi)d\xi\right).$$

Now replacing F by B_l and h by $f^{(n)}$, we obtain

$$\begin{aligned}T(B_l,f^{(n)}) &= \frac{1}{b-a}\int_a^b B_l(\xi)f^{(n)}(\xi)d\xi \\ &\quad -\left(\frac{1}{b-a}\int_a^b B_l(\xi)d\xi\right)\times\left(\frac{1}{b-a}\int_a^b f^{(n)}(\xi)d\xi\right).\end{aligned}$$

which in turn gives us

$$(b-a)T(B_l,f^{(n)})=\int_a^b B_l(\xi)f^{(n)}(\xi)d\xi-\left(\int_a^b B_l(\xi)d\xi\right)\left(\frac{1}{b-a}\int_a^b f^{(n)}(\xi)d\xi\right).$$

which can be written as

$$\int_a^b B_l(\xi)f^{(n)}(\xi)d\xi=\left(\frac{1}{b-a}\int_a^b f^{(n)}(\xi)d\xi\right)\left(\int_a^b B_l(\xi)d\xi\right)+(b-a)T(B_l,f^{(n)}).$$

and finally we get

$$A_l(f)=\int_a^b B_l(\xi)f^{(n)}(\xi)d\xi=\frac{f^{n-1}(b)-f^{n-1}(a)}{(b-a)}\int_a^b B_l(\xi)d\xi+(b-a)R_n^l,$$

where we used definition of A_l from previous sections and

$$R_n^l=\frac{1}{(b-a)}\left(\int_a^b B_l(\xi)f^{(n)}(\xi)d\xi-\frac{1}{b-a}\int_a^b f^{(n)}(\xi)d\xi\int_a^b B_l(\xi)d\xi\right).$$

satisfying the inequality using Proposition 5.11

$$|R_n^l|=|T(B_l,f^{(n)}|\le\frac{1}{2}(\gamma_2-\gamma_1)\sqrt{T(B_l,B_l)}.$$

■

Now we give some Ostrowski-type inequalities related to the generalized linear inequalities.

Theorem 5.3. *Let for $l\in\{1,\dots,12\}$ A_l and B_l be linear functionals as defined in previous section. Furthermore, let (q,r) be a pair of conjugate exponents, i.e., $1\le q,r\le\infty$, $\frac{1}{q}+\frac{1}{r}=1$. Let $f^{(n)}\in L_q[a,b]$ for $n\ge 1$. Then we have for $l\in\{1,2,3,4,5,6,7,8\}$*

$$|A_l(f)|\le\|f^{(n)}\|_q\,\|B_l\|_r\,. \tag{69}$$

The constant on right hand side of (69) *is sharp for* $1 < q \leq \infty$ *and the best possible for* $q = 1$.

Remark 5.4. For idea of the proof kindly see [9].

Using the same method as given in [2] we can state mean value theorems and results connected with exponentially convexity.

References

[1] R.P. Agarwal and P.J.Y. Wong, *Error Inequalities in Polynomial Interpolation and their Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[2] A.A. Aljinović, A.R. Khan, J.E. Pečarić, Weighted majorization theorems via generalization of Taylors formula, *J. Inequal. Appl.* (2015), Art. No. 196, 22 pages.

[3] T.W. Anderson, *An Introduction to Multivariate Statistical Analysis*, Wiley, **3rd** Ed., 2003.

[4] P.J. Davis, *Interpolation and Approximation*, Blaisdell, Boston, 1961.

[5] V.L. Gontscharoff, *Theory of Interpolation and Approximation of Functions*, Gostekhizdat, Moscow, 1954.

[6] A.R. Khan, J.E. Pečarić, Hermite interpolation with Green functions and positivity of general linear inequalities for $n-$convex functions, *J. Math. Anal.* **11** (4) (2020) 1–15.

[7] A.R. Khan, J.E. Pečarić, Positivity of general linear inequalities for $n-$convex functions via the Taylor formula using new Green functions, *Commun. Optim. Theory* (to appear)

[8] A.R. Khan, J. Pečarić, M. Praljak, S. Varošanec, *General Linear Inequalities and Positivity / Higher Order Convexity*, Monographs in inequalities **12**, Element, Zagreb, 2017.

[9] A.R. Khan, J.E. Pečarić, M. Praljak, Popoviciu type inequalities for n-convex functions via extension of Montgomery identity, *An. Ştiinţ. Uni. "Ovidius" Constanţa Ser. Mat.* **24** (3) (2016) 161–188.

[10] M. Matić, J. Pečarić, N. Ujević, Improvement and further generalization of some inequalities of Ostrowski-Grüss type, *Computer Math. Appl.* **39** (3-4) (2000) 161–175.

[11] J. Pečarić and V. Čuljak, Inequality of Petrovi and Giaccardi for convex function of higher order, *Southeast Asian Bull. Math.* **26** (2003) 57–61.

[12] J. Pečarić, M. Praljak, A. Witkowski, Linear operator inequality for n-convex functions at a point, *Math. Ineq. Appl.* **18** (4) (2015) 1201–1217.

[13] J. Pečarić and J. Perić, Improvement of the Giaccardi and the Petrović inequality and related Stolarsky type means, *An. Univ. Craiova Ser. Mat. Inform.* **39** (1) (2012) 65–75.

[14] J.E. Pečarić, F. Proschan, Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.

[15] T. Popoviciu, *Les Fonctions Convexes*, Hermann and Cie, Editeurs, Paris, 1944.

[16] T. Popoviciu, Notes sur les fonctions convexes d'ordre superieur III, *Mathematica (Cluj)* **16** (1940) 74–86.

[17] T. Popoviciu, Notes sur les fonctions convexes d'ordre superieur IV, *Disqusitiones Math.* **1** (1940) 163–171.

[18] J.M. Whittaker, *Interpolation Function Theory*, Cambridge, 1935.

[19] D.V. Widder, *The Laplace Transform*, Princeton Univ. Press, New Jersey, 1941.