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Left Bi-quasi and Minimal Left Bi-quasi Ideals of Ternary Semiring

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Abstract. In this paper, the notions of left bi-quasi ideals and bi-quasi ideals of a ternary semiring have been introduced, which are extensions of the concept of bi-ideals in a ternary semiring. The concept of minimal left bi-quasi ideals, left bi-quasi simple ternary semiring have also been studied. Characterization of regular ternary semiring in terms of left bi-quasi ideals has been obtained.

Keywords: Ternary semiring; Bi-quasi ideal; Minimal left bi-quasi ideal; Left bi-quasi simple ternary semiring; Regular ternary semiring.

1. Introduction

The concept of ternary algebraic system was first introduced by D.H. Lehmer [11] in 1932 which is a generalization of abelian groups. In 1971, Lister [12] introduced ternary rings. To generalize the ternary rings introduced by Lister, in 2003 T.K. Dutta and S. Kar [6] introduced the notion of ternary semirings. Series of studies on ternary semirings have been done by T.K. Dutta et al. [3, 4, 5, 7, 8, 10]. Dixit and Dewan [1] studied about the quasi-ideals and bi-ideals in ternary semigroups. S. Kar [9] generalized the concept of quasi-ideals and bi-ideals in ternary semirings. M.M.K. Rao [13] initiated the notion of bi-quasi ideals and fuzzy bi-quasi ideals of $\Gamma-$semigroup, he also introduced the notion of bi-quasi ideals in semirings [14]. The purpose of the paper is as stated in the abstract.

The structure of the paper is organized as follows: In Section 2, we recall some preliminaries of ternary semiring for their use in the sequel. In Section 3, we introduce the notion of bi-quasi ideals and left (right, lateral) bi-quasi ideals in ternary semirings. We use these concepts to characterize regular ternary semirings. We also obtain some properties of left bi-quasi ideals in ternary semirings. In Section 4, we introduce the concept of minimal bi-quasi ideals and bi-quasi simple ternary semirings. We obtain some characterizations (Theorems 4.3 and 4.4) of left bi-quasi simple ternary semirings. Theorem 4.11, characterizes regular ternary semirings in terms of the left bi-quasi ideals.

2. Preliminaries

In this section, we review some definitions and results which will be used in later sections.

Definition 2.1. [6, Definition 2.1] *A non empty set S together with a binary operation called addition and ternary multiplication, denoted by juxtaposition is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following condition:*

(1) $(abc)de = a(bcd)e = ab(cde)$,
(2) $(a+b)cd = acd + bcd$,
(3) $a(b+c)d = abd + acd$,
(4) $ab(c+d) = abc + abd$, *for all* $a, b, c, d, e \in S$.

Example 2.2. [3, Example 2.2] Let S be a set of continuous functions $f : X \to \mathcal{R}$ where X is a topological space and R is the set of all negative real numbers. Now we define a binary addition and ternary multiplication on S as follows:

(1) $(f+g)(x) = f(x) + g(x)$,
(2) $(fgh)(x) = f(x)g(x)h(x)$, for all $f, g, h \in S$ and $x \in X$.

Then with respect to the binary addition and ternary multiplication S forms a ternary semiring.

Definition 2.3. [6, Definition 2.5] *Let S be a ternary semiring. If there exists an element $a \in S$ such that $0 + x = x$ and $0xy = x0y = xy0 = 0$ for $x, y \in S$, then "0" is called zero element or simply the zero of the ternary semiring S. In this case we say that S is ternary semiring with zero.*

Throughout this paper, S will always denote a ternary semiring with zero and unless, stated otherwise a ternary semiring means a ternary semiring with zero. Let A, B, C be three subsets of S. Then by ABC, we mean the set of all finite sums of the form $\sum a_i b_i c_i$ with $a_i \in A, b_i \in B, c_i \in C$.

Definition 2.4. [6, Definition 2.6] *An additive subsemigroup T of S is called a ternary subsemiring if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.*

Definition 2.5. [6, Definition 2.7] *An additive subsemigroup I of S is called a left (resp. right, lateral) ideal of S if s_1s_2i (resp. is_1s_2, s_1is_2) $\in I$, for all $s_1, s_2 \in S$ and $i \in I$. If I is both left and right ideal of S, then I is called a two-sided ideal of S. If I is a left, a right, a lateral ideal of S, then I is called an ideal of S.*

Definition 2.6. [9, Definition 3.1] *An additive subsemigroup Q of a ternary semiring S is called a quasi-ideal of S if $QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q$.*

Definition 2.7. [9, Definition 3.14] *A ternary subsemiring B of a ternary semiring S is called a bi-ideal of S if $BSBSB \subseteq B$.*

Definition 2.8. [6, Definition 4.1] *An element a in a ternary semiring S is called regular if there exists an element $x \in S$ such that $axa = a$. A ternary semiring S is called regular if all of its elements are regular.*

Definition 2.9. [2, Definition 1.7] *Let P be an ideal of S. Then P is said to be strongly nilpotent if there exists a positive integer n such that $(PS)^{n-1}P = 0$.*

3. Left Bi-quasi Ideals

Definition 3.1. *A ternary subsemiring L of a ternary semiring S is called a left bi-quasi ideal of S if $SSL \cap LSLSL \subseteq L$.*

Definition 3.2. *A ternary subsemiring R of a ternary semiring S is called a right bi-quasi ideal of S if $RSS \cap RSRSR \subseteq R$.*

Definition 3.3. *A ternary subsemiring M of a ternary semiring S is called a lateral bi-quasi ideal of S if $(SMS + SSMSS) \cap MSMSM \subseteq M$.*

Definition 3.4. *Let S be a ternary semiring. A ternary subsemiring B of S is said to be bi-quasi ideal of S if B satisfies all three conditions $SSB \cap BSBSB \subseteq B$, $(SBS + SSBSS) \cap BSBSB \subseteq B$ and $BSS \cap BSBSB \subseteq B$.*

Example 3.5. Let $S = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} : a, b, c, d, e, f \in Q_0^-, \ f \neq 0 \right\}$ be the ternary semiring with respect to matrix multiplication of the set of all 3×3 upper triangular matrices over Q_0^-, where Q_0^- be the set of all non positive rational numbers. Consider $L = \left\{ \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n \end{pmatrix} : m, n \in Q_0^-, n \neq 0 \right\}$. Then L is not a bi-ideal of the ternary semiring S but L is a left bi-quasi ideal of S. Let $x \in SSL \cap LSLSL$. Then

$$\begin{aligned} x &= \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & e_1 \\ 0 & 0 & f_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & e_2 \\ 0 & 0 & f_2 \end{pmatrix} \begin{pmatrix} 0 & m_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & m_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_2 \end{pmatrix} \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & d_3 & e_3 \\ 0 & 0 & f_3 \end{pmatrix} \begin{pmatrix} 0 & m_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_3 \end{pmatrix} \begin{pmatrix} a_4 & b_4 & c_4 \\ 0 & d_4 & e_4 \\ 0 & 0 & f_4 \end{pmatrix} \begin{pmatrix} 0 & m_4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_4 \end{pmatrix} \end{aligned}$$

This implies that

$$\begin{aligned} \begin{pmatrix} 0 & a_1a_2m_1 & a_1c_2n_1 + b_1e_2n_1 + c_1f_2n_1 \\ 0 & 0 & d_1e_2n_1 + e_1f_2n_1 \\ 0 & 0 & f_1f_2n_1 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & e_3f_4m_2n_3n_4 \\ 0 & 0 & 0 \\ 0 & 0 & f_3f_4n_2n_3n_4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix} (say) \end{aligned}$$

Then the equations

$$\begin{aligned} a_1a_2m_1 &= 0 = u, \\ a_1c_2n_1 + b_1e_2n_1 + c_1f_2n_1 &= e_3f_4m_2n_3n_4 = 0, \\ d_1e_2n_1 + e_1f_2n_1 &= 0, \\ f_1f_2n_1 &= f_3f_4n_2n_3n_4 = v(\neq 0) \end{aligned}$$

have non-trivial solutions. Hence, $x \in L$. Therefore $SSL \cap LSLSL \subseteq L$ which implies L is left bi-quasi ideal of S but $LSLSL \nsubseteq L$ which is evident from the above computation. Hence L is not bi-ideal of S.

Theorem 3.6. *Let S be a ternary semiring. Then every left (right) ideal L of S is bi-quasi ideal of S.*

Proof. Let L be a left ideal of a ternary semiring S. Then $SSL \subseteq L$. Now $SSL \cap LSLSL \subseteq SSL \subseteq L$. So L is left bi-quasi ideal. Also $(SLS + SSLSS) \cap LSLSL \subseteq LSLSL \subseteq SSSSL \subseteq SSL \subseteq L$. So L is lateral bi-quasi ideal of S. Similarly we can prove L is right bi-quasi ideal of S and consequently L is bi-quasi ideal of S. ■

Theorem 3.7. *Every lateral ideal M of a ternary semiring S is bi-quasi ideal of S.*

Proof. Let S be a ternary semiring and M a lateral ideal of S. Then $(SMS + SSMSS) \subseteq M$. Now $SSM \cap MSMSM \subseteq MSMSM \subseteq SSMSS \subseteq (SMS + SSMSS)$ (Since $0 \in SMS$) $\subseteq M$. Similarly we can prove M is right bi-quasi ideal of S. Again $(SMS + SSMSS) \cap MSMSM \subseteq SMS + SSMSS \subseteq M$. So M is lateral bi-quasi ideal of S. Consequently M is bi-quasi ideal of S. ∎

Corollary 3.8. *Every ideal of ternary semiring is a bi-quasi ideal.*

Theorem 3.9. *Every quasi ideal Q of a ternary semiring S is bi-quasi ideal of S.*

Proof. Let Q be a quasi ideal of ternary semiring S. Then $SSQ \cap (SQS + SSQSS) \cap QSS \subseteq Q$. We have $QSQSQ \subseteq SQS + SSQSS$ (Since $QSQSQ \subseteq SSQSS$ and $0 \in SQS$). Also $QSQSQ \subseteq Q(SSS)S \subseteq QSS$. Thus $QSQSQ \subseteq QSS \cap (SQS + SSQSS)$. So $SSQ \cap QSQSQ \subseteq SSQ \cap (SQS + SSQSS) \cap QSS \subseteq Q$. Hence Q is left bi-quasi ideal of S. Similarly we can prove Q is right bi-quasi ideal of S. Further $QSQSQ \subseteq SSQ$ and $QSQSQ \subseteq QSS$ implies that $QSQSQ \subseteq SSQ \cap QSS$. Thus $(SQS + SSQSS) \cap QSQSQ \subseteq SSQ \cap (SQS + SSQSS) \cap QSS \subseteq Q$ that proves Q is lateral bi-quasi ideal of S. Consequently Q is bi-quasi ideal of ternary semiring S. ∎

Theorem 3.10. *Every bi-ideal of a ternary semiring S is a bi-quasi ideal of S.*

Proof. Let B be a bi-ideal of ternary semiring S. Then $BSBSB \subseteq B$. Therefore $SSB \cap BSBSB \subseteq BSBSB \subseteq B$, $BSS \cap BSBSB \subseteq BSBSB \subseteq B$ and $(SBS + SSBSS) \cap BSBSB \subseteq BSBSB \subseteq B$. Hence B is a bi-quasi ideal of S. ∎

Theorem 3.11. [3, Theorem 3.4] *Let S be a ternary semiring. S is regular if and only if $RML = R \cap M \cap L$, for any right ideal R, lateral ideal M and left ideal L.*

Theorem 3.12. *Let S be a regular ternary semiring. Then a ternary subsemiring L is a left bi-quasi ideal of S if and only if $L = LSL$.*

Proof. Suppose L is a left bi-quasi ideal of a regular ternary semiring S. If $a \in L$, then there exists $s \in S$ such that $a = asa \in LSL$ that implies $L \subseteq LSL$. Again $LSL \subseteq SSL$ and by regularity, $LSL \subseteq LSLSL$. Thus $LSL \subseteq SSL \cap LSLSL \subseteq L$. Hence $L = LSL$.

Conversely, suppose $L = LSL$. Since S is regular, $L \subseteq LSL \subseteq SSL$. Thus $SSL \cap LSLSL = SSL \cap L \subseteq L$. Consequently L is a left bi-quasi ideal of S. ∎

Theorem 3.13. *Let S be a regular ternary semiring. Then every left bi-quasi ideal of S is an ideal of S.*

Proof. Let S be a regular ternary semiring and L a left bi-quasi ideal of S. By Theorem 3.11, we have $LSS \cap (SLS + SSLSS) \cap SSL = LSSSLSSSL + LSSSSLSSSSL \subseteq LSLSL + LSSLSSL \subseteq LSLSL + LSLSL$ (cf. Theorem 3.12) $\subseteq LSLSL$. Also $LSS \cap (SSL + SSLSS) \cap SSL = LSSSLSSSL + LSSSSLSSSSL \subseteq LSLSL + LSSLSSL \subseteq SSSSL + SSSSSSL \subseteq SSL + SSL \subseteq SSL$. Therefore $LSS \cap (SSL + SSLSS) \cap SSL \subseteq SSL \cap LSLSL \subseteq L$.

Hence L is quasi ideal of S. Since in regular ternary semiring every bi-ideal is quasi ideal and every quasi ideal is an ideal, therefore in regular ternary semiring every bi-quasi ideal is an ideal. ■

Theorem 3.14. *Let S be a regular ternary semiring. Then every left bi-quasi ideal is a bi-ideal of S.*

Proof. Suppose L is a left bi-quasi ideal of S. Clearly by Theorem 3.12, $LSLSL \subseteq L$. Consequently L is a bi-ideal of S. ■

Theorem 3.15. *Let S be a ternary semiring and $\{L_i\}_{i \in I}$ a family of left bi-quasi ideals of S. Then $\bigcap_{i \in I} L_i$ is a left bi-quasi ideal of S, provided $\bigcap_{i \in I} L_i \neq \phi$.*

Proof. Let $L = \bigcap_{i \in I} L_i$. Then $SSL_i \cap L_iSL_iSL_i \subseteq L_i$, for all $i \in I$. We have $SSL \cap LSLSL \subseteq SSL_i \cap L_iSL_iSL_i \subseteq L_i$, for all $i \in I$. Hence L is left bi-quasi ideal of ternary semiring S. ■

We obtain the following proposition by routine verification.

Proposition 3.16. *Let S be a ternary semiring, L a left ideal of S, M a lateral ideal of S and R a right ideal of S. Then $L \cap M \cap R$ is a left bi-quasi ideal of S.*

Theorem 3.17. *If B is a bi-quasi ideal and T is a ternary subsemiring of a ternary semiring S, then $B \cap T$ is bi-quasi ideal of T.*

Proof. Clearly $B \cap T$ is a subsemiring of T. Now $TT(B \cap T) \cap ((B \cap T)T(B \cap T)T(B \cap T)) \subseteq TTT \cap TTTTT \subseteq T$. Also $TT(B \cap T) \cap ((B \cap T)T(B \cap T)T(B \cap T)) \subseteq TTB \cap BTBTB \subseteq B$. Which implies that $TT(B \cap T) \cap ((B \cap T)T(B \cap T)T(B \cap T)) \subseteq B \cap T$. Consequently $B \cap T$ is left bi-quasi ideal of T. Similarly we can prove that $B \cap T$ is right and lateral bi-quasi ideal of T. Hence $B \cap T$ is bi-quasi ideal of T. ■

Theorem 3.18. *Let L be a left ideal, R a right ideal of a ternary semiring S and e a multiplicative idempotent element of S. Then RSe, eSL are left bi-quasi ideals of S.*

Proof. To show RSe is a left bi-quasi ideal of S, it is enough if we prove $RSe = R \cap (SeS + SSeSS) \cap SSe$ (cf. Proposition 3.16). Clearly $RSe \subseteq R \cap SSe$. Let $a \in R \cap SSe$. Then $a \in R$ and $a \in SSe$. Now $a \in SSe$ implies that $a = \sum_{i=1}^{m} s_i t_i e$ for some $s_i, t_i \in S$. Therefore $aee = (\sum_{i=1}^{m} s_i t_i e)ee = \sum_{i=1}^{m} s_i t_i (eee) = \sum_{i=1}^{m} s_i t_i e = a$. implies that $a \in Ree \subseteq RSe$ and hence $RSe = R \cap SSe$.

Again $a = aee \in SeS$ and $0 \in SSeSS$ implies that $a + 0 = a \in SeS + SSeSS$. Thus $R \cap SSe \subseteq SeS + SSeSS$. Consequently $RSe = R \cap (SeS + SSeSS) \cap SSe$.

Similarly we can show that eSL is left bi-quasi ideal of S. ■

Theorem 3.19. *Let e and f be two multiplicative idempotent elements of ternary semiring S and M a lateral ideal of S. Then $eSMSf$ is a left bi-quasi ideal of S.*

Proof. Let M be a lateral ideal of ternary semiring S. Then clearly $eSMSf \subseteq eSS \cap M \cap SSf$. Let $a \in eSS \cap M \cap SSf$. Then $a \in eSS$, $a \in M$ and $a \in SSf$. Now $a \in eSS$ and $a \in SSf$ imply that $a = \sum_{i=1}^{m} es_it_i = \sum_{j=1}^{n} u_jv_jf$ for some $s_i, t_i, u_j, v_j \in S$. Therefore $eeaff = ee(\sum_{i=1}^{m} es_it_i)ff = (\sum_{i=1}^{m} es_it_i)ff = (\sum_{j=1}^{n} u_jv_jf)ff = \sum_{j=1}^{n} u_jv_jf = a$. So $a \in eeMff \subseteq eSMSf$. Thus $eSMSf = eSS \cap M \cap SSf$. Consequently, $eSMSf$ is a left bi-quasi ideal of S. ■

4. Minimal Left Bi-quasi Ideals

Definition 4.1. *Let S be a ternary semiring. A left bi-quasi ideal L of S is called a minimal left bi-quasi ideal of S if it does not contain any left bi-quasi ideal of S.*

Definition 4.2. *A ternary semiring S is called left bi-quasi simple if S is the unique left bi-quasi ideal of S.*

A characterization of left bi-quasi simple ternary semiring is obtained in the following theorem:

Theorem 4.3. *A ternary semiring S is left bi-quasi simple if and only if $SSa \cap aSaSa = S$ for all $a \in S$.*

Proof. Let S be a left bi-quasi simple and let $a \in S$. Since $SSa \cap aSaSa$ is the intersection of two left bi-quasi ideal of S. Thus $SSa \cap aSaSa$ is a left bi-quasi ideal of S. Therefore $SSa \cap aSaSa = S$ for all $a \in S$.

Conversely, suppose $SSa \cap aSaSa = S$ for all $a \in S$ and T is a left bi-quasi ideal of S. Let $b \in T$. Then $S = SSb \cap bSbSb \subseteq SST \cap TSTST \subseteq T \subseteq S$. So $T = S$ and hence S is left bi-quasi simple. ■

Theorem 4.4. *Let S be a ternary semiring. Then following are equivalent:*

(1) *S is left bi-quasi simple,*
(2) *$SSa = S$ for all $a \in S$,*
(3) *$\langle a\rangle_{lbq} = S$ for all $a \in S$, where $\langle a\rangle_{lbq}$ be the smallest left bi-quasi ideal of S containing a.*

Proof. (1) $\Rightarrow$ (2) Let S be left bi-quasi simple. For $a \in S$, SSa is left ideal of S. So SSa is left bi-quasi ideal of S. Therefore $SSa = S$ for all $a \in S$.

(2) $\Rightarrow$ (3) Let $\langle a\rangle_{lbq}$ be the smallest left bi-quasi ideal of S containing a. Then $SSa \subseteq \langle a\rangle_{lbq} \Rightarrow S \subseteq \langle a\rangle_{lbq}$. So $S = \langle a\rangle_{lbq}$.

(3) ⇒ (1) Suppose A is a left bi-quasi ideal of S and $a \in A$. Now, $\langle a \rangle_{lbq} \subseteq A$, (by (3)) $S \subseteq A$ implies $A = S$. Consequently S is left bi-quasi simple. ■

Theorem 4.5. *Let S be a ternary semiring and B be a left bi-quasi ideal of S. Then B is minimal if and only if B is the intersection of minimal left ideal and minimal bi-ideal of S.*

Proof. Let B be a minimal left bi-quasi ideal of S. Then $SSB \cap BSBSB \subseteq B$. Let $b \in B$. Then SSb is a left ideal and $bSbSb$ is a bi-ideal of S. So, their intersection $SSb \cap bSbSb$ is a left bi-quasi ideal of S. Further $SSb \cap bSbSb \subseteq SSB \cap BSBSB \subseteq B$. Since B is minimal so $SSb \cap bSbSb = B$. Now it remains to show that SSb and $bSbSb$ are minimal left ideal and minimal bi-ideal of S respectively. If possible, let L be a left ideal of S such that $L \subseteq SSb$. Then $SSL \subseteq L \subseteq SSb$. Thus $SSL \cap bSbSb \subseteq SSb \cap bSbSb = B$. By minimality of B, $SSL \cap bSbSb = B$. This implies that $B \subseteq SSL$. Also $SSb \subseteq SSB \subseteq SS(SSL) \subseteq SSL \subseteq L$. Thus $L = SSb$. Hence SSb is minimal left ideal of S. If possible, let K be a left bi-quasi ideal of S such that $K \subseteq bSbSb$. Then $KSKSK \subseteq K \subseteq bSbSb$. Now $SSb \cap KSKSK \subseteq SSb \cap bSbSb = B$. By minimality of B, $SSL \cap KSKSK = B$. So $B \subseteq KSKSK$. Again $bSbSb \subseteq BSBSB \subseteq (KSKSK)S(KSKSK)S(KSKSK) \subseteq KSSSKSSSKSSSKSSSK \subseteq KSKSKSKSK \subseteq KSSSKSSSK \subseteq KSKSK \subseteq K$. Thus $K = bSbSb$. Hence $bSbSb$ is minimal bi-ideal of S.

Conversely, let L be a minimal left ideal and K a minimal bi-ideal of S. Then $B = L \cap K$ is a left bi-quasi ideal of S. Let B_1 be a left bi-quasi ideal of S such that $B_1 \subseteq B$. Then $SSB_1 \subseteq SSB \subseteq SSL \subseteq L$. L is minimal left ideal of S implies $L = SSB_1$. Also $B_1SB_1SB_1 \subseteq BSBSB \subseteq KSKSK \subseteq K$. By minimality of K, $K = B_1SB_1SB_1$. Further $B = L \cap K = SSB_1 \cap B_1SB_1SB_1 \subseteq B_1$. Thus $B = B_1$. Hence B is minimal left bi-quasi ideal of S. ■

Theorem 4.6. [2, Theorem 2.6] *Let B be a minimal bi-ideal of a ternary semiring S with no nonzero strongly nilpotent ideals. Then B can be represented in the form of RML with minimal right ideal R, minimal lateral ideal M and minimal left ideal L of S.*

Corollary 4.7. *If B is minimal left bi-quasi ideal of ternary semiring S with no non-zero strongly nilpotent ideals then by Theorem 4.6, B can be represented in the form $L_1 \cap RML$ with minimal right ideal R, minimal left ideals L, L_1 and minimal lateral ideal M.*

Theorem 4.8. *Let B be a left bi-quasi ideal of S. If B is minimal then any two non zero element of B generate the same ideal (left, lateral, right) of S.*

Proof. Let B be a minimal left bi-quasi ideal of S and $0 \neq a \in B$. Then the left ideal $\langle a \rangle_l$ generated by a, is a left bi-quasi ideal of S. So $B \cap \langle a \rangle_l$ is a left bi-quasi ideal of S. By minimality of B, $B = B \cap < a >_l$. Thus $B \subseteq \langle a \rangle_l$. Now for any non zero element $b \in B$, $b \in B \subseteq \langle a \rangle_l \Rightarrow \langle b \rangle_l \subseteq \langle a \rangle_l$. Similarly, $\langle a \rangle_l \subseteq \langle b \rangle_l$. Hence $\langle a \rangle_l = \langle b \rangle_l$. ■

Lemma 4.9. *Let B be a left bi-quasi ideal of a ternary semiring S and T a ternary subsemiring of S. If T is left bi-quasi simple such that $T \cap B \neq \phi$, then $T \subseteq B$.*

Proof. Let $a \in T \cap B$. Since $TTa \cap aTaTa$ is left bi-quasi ideal of T and T is left bi-quasi simple, so $TTa \cap aTaTa = T$ (cf. Theorem 4.3). Now $T = TTa \cap aTaTa \subseteq TTB \cap BTBTB \subseteq SSB \cap BSBSB \subseteq B$ (as B is a left bi-quasi ideal of S). Hence $T \subseteq B$. ■

In view of the above lemma we have the following theorem.

Theorem 4.10. *Let S be a ternary semiring and L a left bi-quasi ideal of S. Then the following statements are true:*

(1) *Let L be a left ideal of S and a minimal left bi-quasi ideal of S. Then L is left bi-quasi simple.*
(2) *Let L be left bi-quasi simple. Then L is minimal left bi-quasi ideal of S.*

Proof. (1) Suppose L is left ideal of ternary semiring S and minimal left bi-quasi ideal of S. Then $SSL \cap LSLSL \subseteq L$. To show L is left bi-quasi simple, let A be a left bi-quasi ideal of L. Then $LLA \cap ALALA \subseteq A$.

Now define $H = \{h \in A : h \in LLA \cap ALALA\}$. Then $H \subseteq A \subseteq L$. We want to show that H is left bi-quasi ideal of S. Let $h_1, h_2, h_3 \in H$. Then $h_1 = \sum q_i p_i a_i = \sum b_i r_i c_i s_i d_i$, $h_2 = \sum q_j p_j a_j = \sum b_j r_j c_j s_j d_j$ and $h_3 = \sum q_k p_k a_k = \sum b_k r_k c_k s_k d_k$ for some a_i, a_j, a_k, b_i, b_j, b_k, c_i, c_j, c_k, $d_i, d_j, d_k \in A$ and p_i, p_j, p_k, q_i, q_j, q_k, r_i, r_j, r_k, $s_i, s_j, s_k \in L$. Obviously $h_1 + h_2 \in H$.

Then we have, $h_1 h_2 h_3 = p_i q_i a_i p_j q_j a_j p_k q_k a_k = b_i r_i c_i s_i d_i b_j r_j c_j s_j d_j b_k r_k c_k s_k d_k$. Since L is a left ideal of S, $(p_i q_i a_i p_j q_j)(a_j p_k q_k) a_k \in LLA$ and $b_i (r_i c_i s_i) d_i (b_j r_j c_j s_j d_j b_k r_k c_k s_k) d_k \in ALALA$. Thus $h_1 h_2 h_3 \in H$. Therefore H is a subsemiring of S. To show, H is a left bi-quasi ideal of S.

Let $h \in SSH \cap HSHSH$. Then $h = \sum q_i^{'} p_i^{'} a_i^{'} = \sum b_i^{'} r_i^{'} c_i^{'} s_i^{'} d_i^{'}$ where $q_i^{'}, p_i^{'}, r_i^{'}, s_i^{'} \in S$ and $a_i^{'}, b_i^{'}, c_i^{'}, d_i^{'} \in H$. Now $h = \sum q_i^{'} p_i^{'} a_i^{'} = \sum \sum q_i^{'} p_i^{'} q_{ij} p_{ij} a_{ij} \in LLA$, (since L is a left ideal of S) where $a_i^{'} = \sum q_{ij} p_{ij} a_{ij} = \sum b_{ij} r_{ij} c_{ij} s_{ij} d_{ij}$.

Also $h = \sum b_i^{'} r_i^{'} c_i^{'} s_i^{'} d_i^{'} = \sum \sum \sum (b_{ki} r_{ki} c_{ki} s_{ki} d_{ki}) r_i^{'} c_i^{'} s_i^{'} (b_{li} r_{li} c_{li} s_{li} d_{li}) = \sum \sum \sum b_{ki} (r_{ki} c_{ki} s_{ki}) d_{ki} (r_i^{'} c_i^{'} s_i^{'} b_{li} r_{li} c_{li} s_{li}) d_{li} \in ALALA$, (since L is a left ideal) where $b_i^{'} = \sum b_{ki} r_{ki} c_{ki} s_{ki} d_{ki}$ and $d_i^{'} = \sum b_{li} r_{li} c_{li} s_{li} d_{li}$. Consequently, $h \in LLA \cap ALALA \subseteq A$ this implies that $h \in H$. So $SSH \cap HSHSH \subseteq H$. By minimality of L, $H = L$. Thus $L = H \subseteq A \subseteq L$ implies $A = L$. Hence L is simple.

(2) Let L be a left bi-quasi simple ideal of ternary semiring S. Let L_1 be a left bi-quasi ideal of S such that $L_1 \subseteq L$. By Lemma 4.9, $L \subseteq L_1$ implies $L = L_1$. Hence L is minimal left bi-quasi ideal of S. ■

To conclude the paper we obtain the following characterization of a regular ternary semiring in terms of left bi-quasi ideals.

Theorem 4.11. *Let S be a ternary semiring. Then S is regular ternary semiring if and only if $B = SSB \cap BSBSB$ for every left bi-quasi ideal B of S.*

Proof. Let S be a regular ternary semiring and B a left bi-quasi ideal of S. Then $SSB \cap BSBSB \subseteq B$. Take $a \in B$, then there exist $x \in S$ such that $a = axa$ (as S is regular). Now $a = axa = axaxa \in BSBSB$ also $a = axa \in SSB$. So $a \in SSB \cap BSBSB$. Thus $B \subseteq SSB \cap BSBSB$. Hence $B = SSB \cap BSBSB$ for every left bi-quasi ideal B of S.

Conversely, suppose $B = SSB \cap BSBSB$ for every left bi-quasi ideal B of S. Let R, M, L be right, lateral and left ideals of S respectively. Then $Q = R \cap M \cap L$ is a left bi-quasi ideal of S (cf. Proposition 3.16). So by the given condition $Q = SSQ \cap QSQSQ$, which implies $R \cap M \cap L = SSQ \cap QSQSQ \subseteq QSQSQ \subseteq RSMSL \subseteq RML$. Also we have $RML \subseteq R \cap M \cap L$. Hence $RML = R \cap M \cap L$. Consequently, by Theorem 3.11, S is regular. ■

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