# Relations on Some Varieties of Completely Regular Semigroups 

Mario Petrich<br>Uz garmu 11, 21420 Bol, Brač, Croatia

Received 22 December 2020
Accepted 8 April 2021

Communicated by Victoria Gould
Dedicated to the memory of Professor Yuqi Guo (1940-2019)

AMS Mathematics Subject Classification(2000): 20M07


#### Abstract

The class of completely regular semigroups $\mathcal{C R}$ endowed with the unary operation of inversion within maximal subgroups forms a variety $\mathcal{C R}$ under inclusion. The lattice of its subvarieties is denoted by $\mathcal{L}(C \mathcal{R})$. In previous publications, we constructed a $\cap$-subsemilattice $\Gamma$ of $\mathcal{L}(\mathcal{C})$ and provided each of its members with a basis of identities.

For each of these varieties, we construct the classes of the following relations: trace, kernel, $\mathbf{B}^{\wedge}, \mathbf{B}^{\vee}$ as well as restrictions of these relations to $\Gamma$. A few cases of the kernel relation escape our scrutiny.


Keywords: Semigroups; Completely regular; Variety; Lattices; Kernel; Trace; Local; Core.

## 1. Preliminaries

Completely regular semigroups are provided here with the unary operation of inversion within the maximal subgroups. As such they form a variety $\mathcal{C R}$. We denote the lattice of its subvarieties by $\mathcal{L}(\mathcal{C R})$ ordered by inclusion. In the effort of studying the structure of $\mathcal{L}(\mathcal{C})$, a number of relations have proved quite useful. Moreover, they induce operators on $\mathcal{L}(C \mathcal{R})$ and some of its interesting sublattices.

For each of the following relations on $\mathcal{L}(\mathcal{C R})$ : trace, kernel, $\mathbf{B}^{\wedge}, \mathbf{B}^{\vee}$ for a $\cap$-subsemilattice $\Gamma$ of $\mathcal{L}(\varrho \mathcal{R})$ constructed in [10], we determine
(a) the classes of each variety in $\Gamma$,
(b) the restrictions of these relations to $\Gamma$,
with very few exceptions (concerning the kernel relation).
For each of these relations, we illustrate the situation on $\Gamma$ by a figure. The set $\Gamma$ has 60 elements, so there is abundance of "variety" to contend with. This type of analysis was executed in [11] for the local relation where it depends on a basis of identities of a variety. This is no longer the case for the kernel and trace relations, since lower and upper ends are expressed by formulae.


Figure 1: $\cap$-subsemilattice $\Gamma$ with enclosed sublattice $\Delta$ of $\mathcal{L}(\mathcal{C R})$

In the main body of the paper, for several important relations on $\mathcal{L}(\mathcal{C} \mathcal{R})$, which are all equivalence relations all of whose classes are intervals, we characterize all classes containing varieties in $\Gamma$, with few exceptions.

We follow symbolism and concepts of the book [15]; moreover most of our references stem from this text. We now state a minimum of definitions used throughout.

The set of all idempotents in $S \in \mathcal{C} \mathcal{R}$ is denoted by $E(S)$.
All the relations $\mathbf{R}$ we study in this work are equivalences all of whose classes are intervals. Hence for all $\mathcal{V} \in \mathcal{L}(\varrho \mathcal{R})$, we write its $\mathbf{R}$-class as $\mathcal{V} \mathbf{R}=\left[\mathcal{V}_{R}, \mathcal{V}^{R}\right]$. For example $L \mathcal{O}$ or $C H \mathcal{A}$, where $L$ stands for local, $\mathcal{O}$ for orthodox, $C$ for core, and $H \mathcal{A}$ for the class of all $S \in \mathcal{C} \mathcal{R}$ all of whose subgroups are abelian, called overabelian.

We often write the meet of a finite number of varieties by juxtaposition of their acronyms with a minimum of parentheses for easy and unambiguous identification of the variety. For any $\mathcal{V} \in \mathcal{L}(\varrho \mathcal{R})$, we write

$$
[\mathcal{V})=\{\mathcal{U} \in \mathcal{L}(\mathcal{C R}) \mid \mathcal{V} \subseteq \mathcal{U}\}, \quad(\mathcal{V}]=\{\mathcal{U} \in \mathcal{L}(\mathcal{C R}) \mid \mathcal{U} \subseteq \mathcal{V}\}
$$

In [10], we introduced the $\cap$-subsemilattice $\Gamma$ of $\mathcal{L}(\mathcal{C} \mathcal{R})$, see Figure 1. For each variety in $\Gamma$, in [11] we gave at least one basis of identities, and characterized its (local) L-classes. For example, when we consider an interval $[\mathcal{U}, \mathcal{V}]$ with $\mathcal{U}, \mathcal{V} \in \Gamma$, it may mean within $\mathcal{L}(\mathcal{R})$ or within $\Gamma$.

By the varietal version of a statement concerning certain fully invariant congruences on a free completely regular semigroup of countably infinite rank, we mean its translation in terms of varieties via the standard antiisomorphism.

Our results are mostly complete with some lacunae in the context of the kernel relation. The set $\Gamma$ can be naturally partitioned as $[\mathcal{T}, L \mathcal{O}]$ and $[\mathcal{R B} \mathcal{A}, \mathcal{C R}]$. The former interval is situated low in $\mathcal{L}(\varrho \mathcal{R})$ while the latter takes up the highest part of $\mathcal{L}(\varrho \mathcal{R})$. As a consequence, the former presents no difficulty, while the latter contains lacunae in our description.

For the $\mathbf{T}$ - and $\mathbf{K}$-relations, we follow [14], and for $\mathbf{B}^{\wedge}$ - and $\mathbf{B}^{\vee}$-relations [7]. In addition, our references include a copious collection of papers and results, generally adapted for the needs of the specific reference.

We first fix some notation that will be used several times.

## 2. Canonical Varieties

Let $X=\left\{x_{1}, x_{2}, \ldots\right\}$ be a countably infinite alphabet. In the free completely regular semigroup on $X$, we single out the following words

$$
G_{2}=x_{2} x_{1}, \quad H_{2}=x_{2}, \quad I_{2}=x_{2} x_{1} x_{2}^{0}
$$

where $x^{0}=x x^{-1}\left(=x^{-1} x\right)$. Overline on a word denotes the reverse (mirror image). For $n>2$, define inductively, for $P \in\{H, I\}$,

$$
G_{n}=x_{n} \bar{G}_{n-1}, \quad P_{n}=G_{n}\left(x_{n} \bar{P}_{n-1}\right)^{0}
$$

We call the varieties $\mathcal{H}_{2}=\mathcal{L} \mathcal{N} O, \overline{\mathcal{H}}_{2}=\mathcal{R} \mathcal{N} \mathcal{O}$, and

$$
\begin{array}{rlrl}
\mathcal{H}_{n} & =\left[G_{n}=H_{n}\right], \quad \overline{\mathcal{H}}_{n}=\left[\bar{G}_{n}=\bar{H}_{n}\right], \\
\mathcal{J}_{n} & =\left[G_{n}=I_{n}\right], & \overline{\mathcal{J}}_{n}=\left[\bar{G}_{n}=\bar{I}_{n}\right],
\end{array}
$$

canonical. See [4].

Notation 2.1. $\mathcal{R}=\mathcal{J}_{3} \cap \overline{\mathcal{J}}_{3}, \mathcal{R} e \mathcal{B}=\mathcal{R} \cap \mathcal{B}$.

It will be convenient to list its properties as follows.

Fact 2.2. We have
(i) $\mathcal{R}=\left[(a x y a)^{0}=\left(a x a^{0} y a\right)^{0}\right]$.
(ii) $\mathcal{R} \cap \mathcal{O}=\mathcal{R} \mathcal{O}=\left[a x y a=a x a^{0} y a\right]$.
(iii) $\mathcal{R} \cap \mathcal{B G}=\mathcal{R B G}=\left[(\text { axya })^{0}=(\text { axaya })^{0}\right]$.
(iv) $\mathcal{R B G} \cap H \mathcal{A}=\mathcal{R B} \mathcal{A}=\mathcal{R} \cap \mathcal{B G} \cap H \mathcal{A}$.

Proof. (i) See [5, Theorem 5.1(iv)].
(ii) For the equality of the first and third varieties, see [5, Theorem 5.4(iii)]. For the second equality, consult [15, Theorem V.3.3].
(iii) This can be derived from part (i) and [15, Proposition V.4.4].
(iv) This requires a straightforward argument.

## 3. T-relation

The relation $\mathbf{T}$ on $\mathcal{L}(\mathcal{C R})$ can be thought of as the varietal version of the $\mathbf{T}$ relation on completely regular semigroups. Indeed, on the lattice of fully invariant congruences on a free completely regular semigroup of countably infinite rank, the trace relation induces a relation which we also denote by $\mathbf{T}$. Its classes are intervals, and we may use the notation: for any $\mathcal{V} \in \mathcal{L}(\mathcal{C})$, its $\mathbf{T}$-class is denoted $\mathcal{V} \mathbf{T}=\left[\mathcal{V}_{T}, \mathcal{V}^{T}\right]$.

Our purpose in this section is the determination of all $\mathbf{T}$-classes of varieties in $\Gamma$. The prime reference here is [14].

We start with citation of the relevant literature needed in our discussion.

Fact 3.1. Let $\mathcal{V}=\left[u_{\alpha}=v_{\alpha}\right]_{\alpha \in A} \in \mathcal{L}(\mathcal{C R})$.
(i) $\mathcal{V}^{T}$ equals the Malcev product $\mathcal{G} \circ \mathcal{V}$.
(ii) If $\mathcal{V} \in \mathcal{B} \mathcal{G}$, then $\mathcal{V}_{T}=\mathcal{V} \cap \mathcal{B}$.
(iii) The mapping $\mathcal{V} \mapsto \mathcal{V}^{T}$ is an $\cap$-endomorphism,
$\mathcal{V}^{T}=\left[u_{\alpha}^{0}=v_{\alpha}^{0},\left(x u_{\alpha} y\right)^{0}=\left(x v_{\alpha} y\right)^{0}\right]$ where $x, y \notin \bigcup_{\alpha \in A} c\left(u_{\alpha}, v_{\alpha}\right)$.
Proof. (i) See [14, Theorem 6.2].
(ii) By [14, Theorem 6.2], we have $\mathcal{V}_{T}=\langle\mathcal{V} \cap \mathcal{F}\rangle$ where $\mathcal{F}=\left\{S \in \mathcal{C R} \mid \mu_{S}=\right.$ $\varepsilon\}$. If $S \in \mathcal{B} \mathcal{G}$, we have $\mu=\mathcal{H}$ and thus $\mathcal{B G} \cap \mathcal{F}=\mathcal{B}$. Hence for $\mathcal{V} \in \mathcal{L}(\mathcal{B} \mathcal{G})$, we get

$$
\mathcal{V}_{T}=\langle\mathcal{V} \cap \mathcal{F}\rangle=\langle(\mathcal{V} \cap \mathcal{B G}) \cap \mathcal{F}\rangle=\langle\mathcal{V} \cap(\mathcal{B G} \cap \mathcal{F})\rangle=\langle\mathcal{V} \cap \mathcal{B}\rangle=\mathcal{V} \cap \mathcal{B}
$$

(iii) See [14, Proposition 7.10] and [17, Theorem 3.9].

The completeness assertion can be easily checked by inspection.
The subsets $(T \mathcal{O}]$ and $[(L \mathcal{O}) H \mathcal{A}]$ of $\Gamma$ form a partition of $\Gamma$. It is clear that no variety in the former is $\mathbf{T}$-related to a variety in the latter.

Theorem 3.2. We classify T-classes of varieties in $\{(T \mathcal{O}],[(L \mathcal{O}) H \mathcal{A})\} \cap \Gamma$ as follows.
(i) The following intervals form the complete set of $\mathbf{T}$-classes of the varieties in $(T \mathcal{O}] \cap \Gamma$.

$$
\begin{gathered}
{[\mathcal{T}, \mathcal{G}],[\mathcal{R B}, \mathcal{C S}],[\mathcal{S}, \mathcal{S G}],[\mathcal{N B}, \mathcal{N B} \mathcal{G}],[\mathcal{R e} \mathcal{B}, \mathcal{R B \mathcal { B }}],} \\
{[\mathcal{B}, \mathcal{B G}],[\mathcal{O}, T \mathcal{O}],[\mathcal{O}(H \mathcal{A}),(T \mathcal{O}) T H \mathcal{A}] .}
\end{gathered}
$$

(ii) For every $\mathcal{V} \in[(L \mathcal{O}) H \mathcal{A})$, we have $\mathcal{V} \mathbf{T} \cap \Gamma=\left\{\mathcal{V}_{T}\right\}$ and thus $\left[\mathcal{V}, \mathcal{V}^{T}\right]$ is a $\mathbf{T}$-class and $\left[\mathcal{V}, \mathcal{V}^{T}\right] \cap \Gamma=\{\mathcal{V}\}$.

Proof. (i) We verify first that these intervals are T-classes. We will freely use Facts 3.1 and 4.1 without specific reference and will refer to each interval by its row and the order in its row.
$(1,1): \mathcal{T}^{T}=\mathcal{G} \circ \mathfrak{T}=\mathcal{C S}, \mathcal{G}_{T}=\mathcal{G} \cap \mathcal{B}=\mathcal{T}$.
$(1,2):(\mathcal{R B})^{T}=\mathcal{G} \circ \mathcal{R B}=\mathcal{C S},(\mathcal{C S})_{T}=\mathcal{C S} \cap \mathcal{B}=\mathcal{R B}$.
$(1,3): \mathcal{S}^{T}=\mathcal{G} \circ \mathcal{S}=\mathcal{S G},(\mathcal{S G})_{T}=\mathcal{S G} \cap \mathcal{B}=\mathcal{S}$.
$(1,4):(\mathcal{N B})^{T}=\mathcal{G} \circ \mathcal{N B}=\mathcal{N B G},(\mathcal{N B G})_{T}=\mathcal{R B G} \cap \mathcal{B}=\mathcal{N B}$.
$(1,5):(\mathcal{R e} e)^{T}=\mathcal{G} \circ \mathcal{R} e \mathcal{B}=\mathcal{R B G},(\mathcal{R B G})_{T}=\mathcal{R B G} \cap \mathcal{B}=\mathcal{R} e \mathcal{B}$.
$(2,1): \mathcal{B}^{T}=\mathcal{G} \circ \mathcal{B}=\mathcal{B G},(\mathcal{B G})_{T}=\mathcal{B G} \cap \mathcal{B}=\mathcal{B}$.
$(2,2): \mathcal{O}^{T}=T \mathcal{O},(T \mathcal{O})_{T}=\mathcal{O}_{T}=\left(\mathcal{G}^{K}\right)_{T}=\mathcal{G}^{K} \mathcal{O}$.
$(2,3):(\mathcal{O}(H \mathcal{A}))^{T}=\mathcal{O}^{T}(H \mathcal{A})^{T}=(T \mathcal{O}) T H \mathcal{A}$,

$$
(\mathcal{O}(H \mathcal{A}))_{T}=\left(\mathcal{O}^{K}(H \mathcal{A})^{K}\right)_{T}=\left(\left(\mathcal{O}(H \mathcal{A})^{K}\right)_{T}=(\mathcal{O}(H \mathcal{A}))^{K}=\mathcal{O}(H \mathcal{A})\right.
$$

(ii) From Fig. 2, we note that the varieties $\mathcal{V}$ in $\Gamma$ not in part (i) are

$$
\begin{equation*}
\mathcal{C R}, C H \mathcal{A}, \mathcal{C}, H \mathcal{A}, L \mathcal{O},(L \mathcal{O}) C H \mathcal{A},(L \mathcal{O}) \mathcal{C},(L \mathcal{O}) H \mathcal{A} . \tag{1}
\end{equation*}
$$

We postpone the proof that they satisfy $\mathcal{V}=\mathcal{V}_{T}$ until the end of the next section. It follows that their T-classes are $\left[\mathcal{V}, \mathcal{V}^{T}\right]$ and satisfy $\left[\mathcal{V}, \mathcal{V}^{T}\right] \cap \Gamma=\{\mathcal{V}\}$.

For all the varieties in $\Gamma$, we characterize the lower ends of their $\mathbf{T}$-classes, and upper ends as well except those in the list (1). The following proposition


Figure 2: T-relation restricted to $\Gamma$ (Theorem 3.2)
provides a basis of identities for upper ends in a very general setting. Recall that if $w$ is either a word or an element of the free completely regular semigroup, $h(w)$ - the head of $w$ and $t(w)$ - the tail of $w$ are the first and the last letters of $w$, respectively.

Proposition 3.3. Let $\mathcal{V}=\left[u_{\alpha}, v_{\alpha}\right]_{\alpha \in A} \in \mathcal{L}(\mathcal{R})$ have the property that for every $\alpha \in A, h\left(u_{\alpha}\right)=h\left(v_{\alpha}\right)$ and $t\left(u_{\alpha}\right)=t\left(v_{\alpha}\right)$ for all $\alpha \in A$. Then $\mathcal{V}^{T}=\left[\left(x u_{\alpha} y\right)^{0}=\right.$
$\left.\left(x v_{\alpha} y\right)^{0}\right]$.
Proof. This follows easily from Fact 3.1(iii).

In view of this proposition, we can say that we have characterized the $\mathbf{T}$ classes of all varieties in $\Gamma$.

## 4. K-Relation

The preamble to Section 3 remains literally valid also in this case. We will witness the rich interplay of the $\mathbf{K}$ - and T-relations. As usual, we start with citations from the literature.

Fact 4.1. Let $\mathcal{V}=\left[u_{\alpha}=v_{\alpha}\right]_{\alpha \in A} \in \mathcal{L}(\mathcal{C R})$.
(i) $\mathbf{K}$ contain a complete congruence.
(ii) The mapping $\mathcal{V} \mapsto \mathcal{V}_{K}$ is a complete $\vee$-endomorphism.
(iii) The mapping $\mathcal{V} \mapsto \mathcal{V}^{K}$ is a complete endomorphism.
(iv) If $\mathcal{V} \subseteq L \mathcal{O}$, then $\mathcal{V}_{K}=\mathcal{V} \cap \mathcal{G}$, if $\mathcal{V} \subseteq \mathcal{O}$, and $\mathcal{V}_{K}=\mathcal{V} \cap \mathcal{C S}$ otherwise.
(v) If $\mathcal{V} \in\{H \mathcal{A}, \mathcal{C}, C H \mathcal{A}, \mathcal{C} \mathcal{R}\}$, then $\mathcal{V}^{K}=\mathcal{V}$.
(vi) If $\mathcal{V}^{K}=\mathcal{V}^{T}$, then $\mathcal{V}=\mathcal{C} \mathcal{R}$.
(vii) $\mathcal{V}^{K}=\left(\mathcal{V}^{K}\right)_{T}$.
(viii) If $\mathcal{V}=\mathcal{V}^{T} \supset \mathcal{R e} \mathcal{B}$, then $\mathcal{V}=\mathcal{V}_{K}$.
(ix) $\nu^{K C}=\nu^{C K}, \nu^{T C}=\mathcal{V}^{C T}$.
(x) If $\mathcal{V} \supseteq \mathcal{S}$, then $\mathcal{V}^{K}=\left[x u_{\alpha} y\left(x v_{\alpha} y\right)^{-1} \in E\right]_{\alpha \in A}$ where $w \in E$ means $w^{2}=w$.

Proof. (i) See [3, Theorem 11].
(ii) This follows from [15, Lemma I.2.2].
(iii) This follows from [16, Theorem 1(3)] via [3, Theorem 14].
(iv) See [16, Theorem 2] and [14, Theorem 5.8].
(v) See [10, Lemma 3.5].
(vi) This follows directly from [12, Theorem 4.6(i)].
(vii) This is the varietal version of [2, Proposition 8.1].
(viii) This is the varietal version of [2, Proposition 8.2].
(ix) See [12, Lemmas 5.5 and 5.3 , respectively].
(x) See [1, Proposition 7.2(ii) and Corollary 6.5].

It follows from Fact $4.1(\mathrm{iv})$ that for any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{C})$, if $\mathcal{U} \subseteq L \mathcal{O}$, then also $\mathcal{V} \subseteq L \mathcal{O}$. Hence no variety $\mathcal{U} \in(L \mathcal{O}) \cap \Gamma$ is K-related to a variety in $[\mathcal{R B} \mathcal{A}) \cap \Gamma$. Clearly the sets $(L \mathcal{O}] \cap \Gamma$ and $[\mathcal{R B} \mathcal{A}) \cap \Gamma$ form a partition of $\Gamma$.

Theorem 4.2. The following intervals form the complete set of $\mathbf{K}$-classes of varieties in $(L \mathcal{O}) \cap \Gamma$.

$$
[\mathcal{T}, \mathcal{B}],[\mathcal{A}, \mathcal{O}(H \mathcal{A})],[\mathcal{G}, \mathcal{O}],[(\mathcal{C S}) H \mathcal{A},(L \mathcal{O}) H \mathcal{A}]
$$



Figure 3: K-relation restricted to $((L \mathcal{O}] \cup[H \mathcal{A}) \cup[\mathcal{R B G})) \cap \Gamma$ (Thms. 4.2, 4.3)
$[(\mathcal{C S}) \mathcal{C},(L \mathcal{O}) \mathcal{C}],[(\mathcal{C S}) C H \mathcal{A},(L \mathcal{O}) C H \mathcal{A}],[\mathcal{C S}, L \mathcal{O}]$.

Proof. That the first, the third, and the last intervals are K-classes is folklore and can be deduced from [16, Theorem 2]. That the second and the fourth intervals are K-classes was proved in [9, Lemma 5.3], the argument of whose proof may be readily adapted to the proofs for the fifth and sixth intervals.

It is clear from Figure 3 that this covers all the cases.

The next result has a different flavor.

## Theorem 4.3.

(i) Each variety $\mathcal{V} \in[H \mathcal{A}) \cap \Gamma$ satisfies $\mathcal{V}=\mathcal{V}^{K}=\mathcal{V}_{T}$.
(ii) Each variety $\mathcal{V} \in[\mathcal{R B G}) \cap \Gamma$ satisfies $\mathcal{V}=\mathcal{V}_{K}=\mathcal{V}^{T}$.

In both cases, no two distinct varieties are either $\mathbf{K}$ - or $\mathbf{T}$-related.
Proof. (i) First $\mathcal{V}=\mathcal{V}^{K}$ is the content of Fact 4.1(v); now apply Fact 4.1(vii).
(ii) By Theorem 4.2, we get $\mathcal{V}=\mathcal{V}^{T}$, and hence, since clearly $\mathcal{V} \supseteq \operatorname{Re} \mathcal{B}$, Fact 4.1(viii), we obtain $\mathcal{V}=\mathcal{V}_{K}$.

The final statement of the theorem obviously holds in both cases.

Next we complete the proof of Theorem 3.2(ii). We are interested here in $\mathcal{V}_{T}$. For the varieties $\mathcal{V} \in[H \mathcal{A}) \cap \Gamma$, we have $\mathcal{V}=\mathcal{V}_{T}$ by Theorem 4.3(i). For the varieties $\mathcal{V} \in[(L \mathcal{O}) H \mathcal{A}, L \mathcal{O}] \cap \Gamma$, in view of Fact 4.1 (vii), it suffices to prove that $\mathcal{V}=\mathcal{V}^{K}$. But this follows directly from Theorem 4.2.

This takes care of Theorem 3.2.
For the subject of $\mathbf{K}$-classes, we are still missing
(a) lower ends of K-classes of the varieties $H \mathcal{A}, \mathcal{C}, C H \mathcal{A}$ since it is well known that $\mathcal{C} \mathcal{R}_{K}=\mathcal{C} \mathcal{R}$,
(b) upper ends of $\mathbf{K}$-classes of the varieties $\mathcal{R B} \mathcal{G}, \mathcal{B G}, T \mathcal{O}$; they can be computed using Fact 4.1(x),
(c) K-classes of varieties in $[\mathcal{R B} \mathcal{A},(T \mathcal{O}) C H \mathcal{A}] \cap \Gamma$.

The main difficulty here is that we have no basis of identities of the lower ends of $\mathbf{K}$-classes in general. For basis of identities of the upper ends of $\mathbf{K}$-classes, see Fact 4.1(x). In [11], we constructed bases for all varieties in $\Gamma$. We now dedicate the remainder of this section to studying the K-relation in the interval $[\mathcal{R B} \mathcal{A}, C H \mathcal{A}] \cap \Gamma$.

Theorem 4.4. In each of the following sets, no two varieties are $\mathbf{K}$-related.

$$
\begin{gathered}
\{(T \mathcal{O}) H \mathcal{A},(T \mathcal{O}) \mathcal{C},(T \mathcal{O}) C H \mathcal{A}, T \mathcal{O}\} \\
\{\mathcal{B} \mathcal{A},(\mathcal{B G}) \mathcal{C},(\mathcal{B G}) C H \mathcal{A}, \mathcal{B G}\} \\
\{\mathcal{R B} \mathcal{A},(\mathcal{R B} \mathcal{G}) \mathcal{C},(\mathcal{R B G}) C H \mathcal{A}, \mathcal{R B}\}
\end{gathered}
$$

Proof. By Fact $4.1(\mathrm{vi})$, if $\mathcal{V}^{K}=\mathcal{V}^{T}$, then $\mathcal{V}=\mathcal{C} \mathcal{R}$. In Theorem 3.2, we have seen that each set in the statement of the theorem is contained in some T-class. The assertion follows.

We have seen in Theorem 4.3 that no two varieties in the set $\{H \mathcal{A}, \mathcal{C}, C H \mathcal{A}$, $\mathcal{C R}\}$ are K-related. This takes care of the K-relation on "almost vertical lines"
in the interval $[\mathcal{R B} \mathcal{A}, \mathcal{C}] \cap \Gamma$. For the "almost horizontal" lines in Figure 2 of this interval, we now propose

Conjecture 4.5. In each of the following sets
(4.1) $\{(\mathcal{R B G}) C H \mathcal{A},(\mathcal{B G}) C H \mathcal{A},(T \mathcal{O}) C H \mathcal{A}, C H \mathcal{A}\}$,
(4.2) $\{(\mathcal{R B G}) \mathcal{C},(\mathcal{B G}) \mathcal{C},(T \mathcal{O}) \mathcal{C}, \mathcal{C}\}$,
(4.3) $\{\mathcal{R B} \mathcal{A}, \mathcal{B} \mathcal{A},(T \mathcal{O}) H \mathcal{A}, H \mathcal{A}\}$
no two distinct varieties are K-related.

No two varieties in the $\operatorname{set}\{\mathcal{R B} \mathcal{B}, \mathcal{B} \mathcal{G}, T \mathcal{O}, \mathcal{C R}\}$ are $\mathbf{K}$-related, as we have seen in Theorem 4.3. But now, unfortunately, the varieties within (4.1), (4.2), (4.3) are not T-related. In the next two propositions, we characterize the possible K-relationship between some of these varieties.

Of the three sets (4.1)-(4.3), as samples, we have chosen below two pairs of varieties in (4.3) giving some alternatives. For the first pair, we prove a lemma.

Lemma 4.6. We have $\mathcal{R}_{K}=\mathcal{R}$.
Proof. From Notation 2.1, we have by definition that $\mathcal{R}=\overline{\mathcal{J}}_{3} \cap \overline{\mathcal{J}}_{3}$. Now Fact 3.1(iii) and [6, Theorem 10.2(ii)] yield

$$
\mathcal{R}^{T}=\left(\overline{\mathcal{J}}_{3} \cap \overline{\mathcal{J}}_{3}\right)^{T}=\overline{\mathfrak{J}}_{3}^{T} \cap \overline{\mathfrak{J}}_{3}^{T}=\mathcal{J}_{3} \cap \overline{\mathcal{J}}_{3}=\mathcal{R} .
$$

Trivially, $\mathcal{R} e \mathcal{B} \subseteq \mathcal{R}$ which by Fact 4.1 (viii) gives $\mathcal{R}=\mathcal{R}_{K}$.

Proposition 4.7. The following statements are equivalent:
(i) $\mathcal{B} \mathcal{A} \mathbf{K} \mathcal{R B} \mathcal{A}$.
(ii) $(\mathcal{B} \mathcal{A})_{K} \subseteq \mathcal{R}$.
(iii) $\mathcal{B} \mathcal{A} \subseteq \mathcal{R}^{K}$.

Proof. (i) $\Rightarrow$ (ii). By Fact 4.1(iii), we obtain

$$
(\mathcal{B} \mathcal{A})^{K}=(\mathcal{R B} \mathcal{A})^{K}=(\mathcal{R} \cap \mathcal{B} \mathcal{A})^{K}=\mathcal{R}^{K} \cap(\mathcal{B} \mathcal{A})^{K}
$$

whence $(\mathcal{B} \mathcal{A})^{K} \subseteq \mathcal{R}^{K}$ which implies $(\mathcal{B} \mathcal{A})_{K} \subseteq \mathcal{R}_{K}$ and Lemma 4.6 yields $(\mathcal{B} \mathcal{A})_{K} \subseteq$ $\mathcal{R}$.
(ii) $\Rightarrow$ (iii). It suffices to apply the operator $\mathcal{V} \mapsto \mathcal{V}^{K}$.
(iii) $\Rightarrow$ (ii). It suffices to apply the operator $\mathcal{V} \mapsto \mathcal{V}_{K}$.
(ii) $\Rightarrow$ (i). Using the same references, we get $(\mathcal{B} \mathcal{A})^{K} \subseteq \mathcal{R}^{K}$ whence $(\mathcal{B} \mathcal{A})^{K}=$ $\mathcal{R}^{K}(\mathcal{B} \mathcal{A})^{K}=(\mathcal{R}(\mathcal{B} \mathcal{A}))^{K}$ and thus $\mathcal{B} \mathcal{A} \mathbf{K} \mathcal{R}(\mathcal{B} \mathcal{A})$.

In part (ii) above, we have no idea how to compute $(\mathcal{B} \mathcal{A})_{K}$. In part (iii), the variety $\mathcal{R}^{K}$ causes a difficulty. It is even more obscure if we write this in terms of implications of identities using $\mathcal{B} \mathcal{A}=\left[a b a^{0}=a^{0} b a\right]$.

For the remainder of the paper, we will write the operators $T, K, L$ before their argument. (We have done this from the beginning for $H \mathcal{A}$.)

For the second pair, we also need the following proposition.

Proposition 4.8. We have

$$
H \mathcal{A} \mathbf{K}(T \mathcal{O}) H \mathcal{A} \Leftrightarrow H \mathcal{A} \subseteq\{S \in \mathcal{C \mathcal { R }} \mid C(S) \in K \mathcal{B} \mathcal{G}\}
$$

Proof. By (v) and (ii) of Fact 4.1, we get

$$
\begin{aligned}
H \mathcal{A} \mathbf{K}(T \mathcal{O}) H \mathcal{A} & \Leftrightarrow H \mathcal{A} \mathbf{K}(C \mathcal{B G}) H \mathcal{A} \Leftrightarrow(H \mathcal{A})^{K}=(C \mathcal{B G})^{K}(H \mathcal{A})^{K} \\
& \Leftrightarrow H \mathcal{A}=(C \mathcal{B G})^{K} H \mathcal{A} \Leftrightarrow H \mathcal{A} \subseteq(C \mathcal{B G})^{K}=\mathcal{B G}^{K C} \\
& \Leftrightarrow H \mathcal{A} \subseteq\{S \in \mathcal{C \mathcal { R }} \mid C(S) \in K \mathcal{B G}\}
\end{aligned}
$$

It is not clear how to violate this condition since $K \mathcal{B} \mathcal{G}$ is still an enigma.

## 5. Extension of Figures 2 and 3

By treating the partition $\{(T \mathcal{O}],[(L \mathcal{O}) H \mathcal{A})\}$ in Theorem 3.2, we have seen $\mathbf{T}$-classes in $\Gamma$. In Theorem 4.2, we classified the K-classes in $(L \mathcal{O}] \cap \Gamma$. Theorem 4.4 handles K-relation along three more intervals. The material concerning the K-relation ended with a conjecture. Hence K-relation on $\Gamma$ was not completely determined.

We treat here a different kind of subject concerning both K- and T-relations. In addition to Fact 3.1(v), we will need the following information.

Lemma 5.1. If $\mathcal{V} \in\{T \mathcal{O}, \mathcal{B G}, \mathcal{R B G}\}$, then $\mathcal{V}=\mathcal{V}^{T}$.
Proof. Since $T \mathcal{O}=\mathcal{O}^{T}$ and $\mathcal{B G}=\mathcal{B}^{T}$, we have the contention for the first two varieties. Next using [13, Proposition 7.10], we obtain

$$
T \mathcal{R B G}=\mathcal{R B G}^{T}=(\mathcal{R} \cap \mathcal{B G})^{T}=\mathcal{R}^{T} \cap \mathcal{B} \mathcal{G}^{T}=\mathcal{R B G}
$$

the assertion follows.

Corollary 5.2. If $\mathcal{V} \in\{T \mathcal{O}, \mathcal{B} \mathcal{G}, \mathcal{R B}\}$, then $\mathcal{V}=\mathcal{V}_{K}$.

We need a final preliminary result as follows.

Fact 5.3. Let $\mathcal{V} \in[\mathcal{T}, \mathcal{C R})$. Then

$$
\mathcal{V} \subset \mathcal{V}^{K T} \subset \mathcal{V}^{(K T)^{2}} \subset \cdots \subset \mathcal{C R}
$$

and $\bigvee_{n=1}^{\infty} \mathcal{V}^{(K T)^{n}}=\mathcal{C \mathcal { R }}$
Proof. This is [12, Theorem 4.6(i)]

We are now ready for a heuristic argument. Indeed, we can extend Figs. 2 and 3 by using the idea of Fact 5.3 as follows. First the varieties

$$
\begin{equation*}
T \mathcal{O}, \mathcal{B G}, \mathcal{R B G} \tag{2}
\end{equation*}
$$

are upper ends of their $\mathbf{T}$-classes, and are trivially above $\mathcal{R} e \mathcal{B}$. Hence we can apply the varietal version of [2, Proposition 2] to conclude that all the varieties in (2) are least elements of their K-classes thereby

$$
[T \mathcal{O}, K T \mathcal{O}], \quad[\mathcal{B G}, K \mathcal{B G}], \quad[\mathcal{R B G}, K \mathcal{R B G}]
$$

are K-classes. But now, the varietal version of [2, Proposition 1] yields that $K T \mathcal{O}, K \mathcal{B} \mathcal{G}, K \mathcal{R B G}$ are the least elements in their $\mathbf{T}$-classes. Thus TKTO, $T K \mathcal{B G}, T K \mathcal{R B G}$ are the greatest elements in their $\mathbf{T}$-classes.

Evidently, this procedure can be repeated indefinitely with the join of these varieties equaling $\mathcal{C} \mathcal{R}$ by [12, Theorem 4.4].

We can perform similar analysis with the varieties $H \mathcal{A}, \mathcal{C}, C H \mathcal{A}$ as well since they are upper ends of their T-classes. In particular $T H \mathcal{A}, T \mathcal{C}, T C H \mathcal{A}$ are the greatest elements in their T-classes. Hence they are the least elements of their K-classes. Therefore $K T H \mathcal{A}, K T \mathcal{C}, K T C H \mathcal{A}$ are the greatest elements in their K-classes.

Since $T \mathcal{O}, \mathcal{B G}, \mathcal{R B G}$ are the upper ends of their $\mathbf{T}$-classes, by Fact 4.1(vii), we conclude that

$$
[T \mathcal{O}, K T \mathcal{O}], \quad[\mathcal{B G}, K \mathcal{B} \mathcal{G}], \quad[\mathcal{R B G}, K \mathcal{R B G}]
$$

are K-classes. This does not necessarily carry over to

$$
\begin{gather*}
(T \mathcal{O}) C H \mathcal{A}, \quad(\mathcal{B G}) C H \mathcal{A}, \quad(\mathcal{R B G}) C H \mathcal{A} \\
(T \mathcal{O}) \mathrm{C}, \quad(\mathcal{B G}) \mathcal{C}, \quad(\mathcal{R B G}) \mathrm{C} \tag{3}
\end{gather*}
$$

$(Y \mathcal{O}) H \mathcal{A}, \mathcal{B} \mathcal{A}, \mathcal{R B} \mathcal{A}$
in the intervals $[(T \mathcal{O}) C H \mathcal{A}, K(T \mathcal{O}) C H \mathcal{A}], \ldots,[\mathcal{R B} \mathcal{A}, K \mathcal{R B} \mathcal{A}]$, for (3) are possibly not the greatest varieties in their T-classes. But still some of them may be the least varieties in their K-classes. This remains as an open problem.

All the other intervals in Figure 3 marked with thicker lines in this part are K-classes. The same carries over to all similar intervals in Figure 2 relative to T-classes. They are all marked with thicker lines.

The picture is repeated indefinitely, except possibly for the intervals indicated above; the K- and T-classes alternate going upwards, actually indefinitely, with their join equaling $\mathcal{C R}$. The only problem is which varieties in (3) are minimal in their K-classes, if any? We know that they are not the greatest varieties in their T-classes, and we may not apply the formula $\left(\mathcal{V}^{T}\right)_{K}=\mathcal{V}^{T}$.


To each of the varieties
$H \mathcal{A}, \mathcal{C}, \quad C H \mathcal{A}$
we apply the upper $\mathbf{T}$. To the resulting three varieties

$$
T H \mathcal{A}, \quad T \mathcal{C}, \quad T C H \mathcal{A}
$$

we then apply upper $\mathbf{K}$ obtaining new varieties

$$
K T H \mathcal{A}, \quad K T \mathcal{C}, \quad K T C H \mathcal{A}
$$

and continue this procedure indefinitely.
Note that the resulting sequence is contained in the sequence in Fact 2.2, so that we may conclude, also using properties discussed above, that

$$
[H \mathcal{A}, T H \mathcal{A}], \quad[\mathcal{C}, T \mathcal{C}], \quad[C H \mathcal{A}, T C H \mathcal{A}]
$$

are $\mathbf{T}$-classes, and the next level, that is

$$
[T H \mathcal{A}, K T H \mathcal{A}], \quad[T \mathrm{C}, K T \mathrm{C}], \quad[T C H \mathcal{A}, K T C H \mathcal{A}]
$$

are $\mathbf{K}$-classes, and so on.
That the join of the classes constructed equals $\mathcal{C R}$ is guarantied by Fact 5.3 since the ascending chain constructed above is a subchain of the one in this reference.

We can perform the same kind of analysis starting with the set $\{T \mathcal{O}, \mathcal{B} \mathcal{G}$, $\mathcal{R B} \mathcal{G}\}$, and successively apply the operator $\mathcal{V} \mapsto \mathcal{V}^{K}$, and continue similarly as above.

Any of the procedures can be applied to any of the varieties $[\mathcal{R B} \mathcal{A},(T \mathcal{O}) C H \mathcal{A}]$ $\cap \Gamma$, but we will obtain full $\mathbf{K}$ - or $\mathbf{T}$-classes at most in the second etc. step as above.

The remaining classes would be full. See Figure 3.

## 6. $\mathrm{B}^{\wedge}$-Relation

This relation has a very natural definition as follows.

Definition 6.1. Define the $\mathbf{B}^{\wedge}$-relation on $\mathcal{L}(\mathcal{C R})$ by

$$
\mathcal{U} \mathbf{B}^{\wedge} \mathcal{V} \Leftrightarrow \mathcal{U} \cap \mathcal{B}=\mathcal{V} \cap \mathcal{B}
$$

Now $\mathbf{B}^{\wedge}$ is a complete congruence whose classes can be described in terms of canonical varieties, see Section 2.

It is a simple matter to characterize all $\mathbf{B}^{\wedge}$-classes of varieties in $\Gamma$ as follows.

Theorem 6.2. The following intervals form the complete set of $\mathbf{B}^{\wedge}$-classes of varieties in $\Gamma$.

$$
[\mathcal{T}, \mathcal{G}], \quad[\mathcal{R B}, \mathcal{C S}], \quad[\mathcal{S}, \mathcal{S G}], \quad[\mathcal{N B}, \mathcal{N B}], \quad[\mathcal{R} e \mathcal{B}, \mathcal{R B}], \quad[\mathcal{B}, \mathcal{C} \mathcal{R}] .
$$

Proof. By a quick glance at Figure 1 of $\Gamma$, we see that $\Gamma$ contains only the following band varieties

$$
\begin{equation*}
\mathcal{T}, \quad \mathcal{R B}, \quad \mathcal{S}, \quad \mathcal{N B}, \quad \mathcal{R} e \mathcal{B}, \quad \mathcal{B} \tag{4}
\end{equation*}
$$



Figure 5: $\mathbf{B}^{\wedge}$-relation restricted to $\Gamma$ (Theorem 6.2)

Clearly $\mathbf{B}^{\wedge}$-classes are parametrized by band varieties, so that the list (4) consists of the lower ends of the $\mathbf{B}^{\wedge}$-classes of $\Gamma$. From [4, Theorem 5.3], we may now read off what the upper ends of these $\mathbf{B}^{\wedge}$-classes look like.

It is now very easy to delineate in Figure 1 what the $\mathbf{B}^{\wedge}$-classes of the band varieties are. See Figure 5.

## 7. $\mathrm{B}^{\vee}$-relation

We studied the $\mathbf{B}^{\wedge}$-relation in Section 6. The $\mathbf{B}^{\vee}$-relation is only a pale dual to it.

Definition 7.1. Define the $\mathbf{B}^{\vee}$-relation on $\mathcal{L}(\mathcal{C R})$ by

$$
\mathcal{U} \mathbf{B}^{\vee} \mathcal{V} \Leftrightarrow \mathcal{U} \vee \mathcal{B}=\mathcal{V} \vee \mathcal{B} .
$$

Then $\mathbf{B}^{\vee}$ is a complete congruence. We do not have a simple parametrization for its classes. However, all its classes are intervals, and we may use the usual notation for them. Indeed, for any $\mathcal{V} \in \mathcal{L}(\mathcal{C} \mathcal{R})$, we write $\nu \mathbf{B}^{\vee}=\left[\mathcal{V}_{B^{\vee}}, \mathcal{V}^{B^{\vee}}\right]$. Clearly $\mathcal{\nu}^{B^{\vee}}=\mathcal{V} \vee \mathcal{B}$, but about $\mathcal{V}_{B} \vee$ we know very little. For general reference, consult [7]. The first important information is the following.

Fact 7.2. We have $\left.\mathbf{B}^{\vee}\right|_{\mathcal{L}(\mathcal{B G})}=\left.\mathbf{K}\right|_{\mathcal{L}(\mathcal{B} \mathcal{G})}$.
Proof. This was proved in [7, Theorem 6.3(iii)].

Our main result here follows.

## Theorem 7.3.

(i) The following intervals are $\mathbf{B}^{\vee}$-classes of varieties contained in $\Gamma$ :

$$
\begin{gathered}
{[\mathcal{T}, \mathcal{B}],[\mathcal{A}, \mathcal{O}(\mathcal{B} \mathcal{A})],[\mathcal{G}, \mathcal{O}(\mathcal{B G})],[(\mathcal{C S}) H \mathcal{A},(L \mathcal{O}) \mathcal{B} \mathcal{A}]} \\
{[(\mathcal{C S}) \mathcal{C}, L \mathcal{O}(\mathcal{B G}) \mathcal{C}],[(\mathcal{C S}) C H \mathcal{A}, L \mathcal{O}(\mathcal{B G}) C H \mathcal{A}],[\mathcal{C S},(L \mathcal{O}) \mathcal{B G}]}
\end{gathered}
$$

(ii) The following intervals are $\mathbf{B}^{\vee}$-classes which intersect $\Gamma$ but are not contained in $\Gamma$ :

$$
\begin{gathered}
{\left[\mathcal{R B A}, \mathcal{R}^{K}(\mathcal{B A})\right],\left[(\mathcal{R B G}) \mathcal{C}, \mathcal{R}^{K}(\mathcal{B G}) \mathcal{C}\right],} \\
{\left[(\mathcal{R B G}) C H \mathcal{A}, \mathcal{R}^{K}(\mathcal{B G}) C H \mathcal{A}\right],\left[\mathcal{R B G}, \mathcal{R}^{K}(\mathcal{B G})\right] .}
\end{gathered}
$$

(iii) The $\mathbf{B}^{\vee}$-classes of the varieties containing $\mathcal{B}$ are singletons.

Proof. By simple inspection, we may see that the sets $[\mathcal{T},(L \mathcal{O}) \mathcal{B} \mathcal{G}],[\mathcal{R B} \mathcal{A}, \mathcal{R B G}]$, $[\mathcal{B})$ form a partition of $\Gamma$. We will make good use of Fact 7.2 by profiting from results on the K-relation in Section 4, especially Fact 4.1, with sometimes citing the relevant results explicitly.
(i) The lower ends $\mathcal{V}$ of the classes of the listed intervals satisfy $\mathcal{V}=\mathcal{V}_{K}$ by Fact 4.1 (iii) and we may apply Fact 7.2 to conclude that $\mathcal{V}_{B^{\vee}}=\mathcal{V}$. For the upper ends of these classes, we must calculate $\mathcal{V} \vee \mathcal{B}$.

$$
\mathcal{T} \vee \mathcal{B}=\mathcal{B}, \text { trivially }
$$



Figure 6: $\mathbf{B}^{\vee}$-relation restricted to $\Gamma$ (Theorem 7.3)

$$
\begin{aligned}
\mathcal{G} \vee \mathcal{B} & =\mathcal{O}(\mathcal{B G}) \text { by }[7, \text { Lemma 3.4(i) }], \\
\mathcal{A} \vee \mathcal{B} & =(\mathcal{G} \cap H \mathcal{A}) \vee \mathcal{B}=(G \vee \mathcal{B}) \cap(H \mathcal{A} \vee \mathcal{B})=O(\mathcal{B} \mathcal{A}), \\
\mathcal{C S} \vee \mathcal{B} & =(L \mathcal{O}) \mathcal{B \mathcal { G }} \text { by }[7, \text { Lemma 3.4(ii)], } \\
(\mathcal{S S}) H \mathcal{A} \vee \mathcal{B} & =(\mathcal{C S} \cap H \mathcal{A}) \vee \mathcal{B}=(\mathcal{C S} \vee \mathcal{B}) \cap(H \mathcal{A} \vee \mathcal{B}) \\
& =(L \mathcal{O}) \mathcal{B} \mathcal{G} \cap H \mathcal{A}=(L \mathcal{O}) \mathcal{B} \mathcal{A} .
\end{aligned}
$$

$(\mathcal{C S}) \mathcal{C} \vee \mathcal{B}$ and $(\mathcal{C S}) C H \mathcal{A} \vee \mathcal{B}$ follow the same pattern.
(ii) For the join $\mathcal{R B G} \vee \mathcal{B}$ we must use Polák's theorem. For it we need the ladders of $\mathcal{R B G}$ and $\mathcal{B}$, that is



Using [14, Theorem 8.2 and Proposition 8.4] and [4, Proposition 4.1], we obtain

$$
\mathcal{R B G}_{T_{r}}=\left(\mathcal{J}_{3} \cap \overline{\mathcal{J}}_{3} \cap \mathcal{B G}\right)_{T_{r}}=\left(\mathcal{J}_{3}\right)_{T_{r}} \cap\left(\overline{\mathcal{J}}_{3}\right)_{T_{r}} \cap(\mathcal{B G})_{T_{r}}=\mathcal{J}_{3} \cap \mathcal{J}_{2} \cap \mathcal{B}
$$

so that $\mathcal{R B}_{T_{r} K^{*}}=\mathcal{T}$, which together with its dual yields


Now taking into account that $\mathcal{V}^{T}=\mathcal{V}^{T_{l}} \cap \mathcal{V}^{T_{r}}$, and Fact 4.1(ii), the evaluation becomes

$$
\begin{aligned}
\mathcal{R B G} \vee \mathcal{B} & =\left(\mathcal{R B} \mathcal{G}_{K}\right)^{K} \cap \mathcal{T}^{K T_{r}} \cap \mathcal{T}^{K T_{l}} \\
& =(\mathcal{R B G})^{K} \cap \mathcal{T}^{K T}=\mathcal{R}^{K} \cap \mathcal{B G}^{K} \cap \mathcal{B G}=\mathcal{R}^{K}(\mathcal{B G}) .
\end{aligned}
$$

This takes care of the last interval in part (ii). For the first interval, we obtain

$$
\mathcal{R B G} \cap H \mathcal{A}=\mathcal{R B} \mathcal{A}, \quad \mathcal{R}^{K}(\mathcal{B G}) \cap H \mathcal{A}=\mathcal{R}^{K}(\mathcal{B} \mathcal{A})
$$

and thus the interval $\left[\mathcal{R} \mathcal{B} \mathcal{A}, \mathcal{R}^{K}(\mathcal{B} \mathcal{A})\right]$. The argument for the remaining two intervals is essentially the same.
(iii) If $\mathcal{V} \supseteq \mathcal{B}$, then $\mathcal{V}^{B^{\vee}}=\mathcal{V} \cap \mathcal{B}=\mathcal{V}$. Hence all of $\mathcal{V} \supseteq \mathcal{B}$ have the property $\mathcal{V}=\mathcal{V}^{B^{\vee}}$, and thus no two of them are $\mathbf{B}^{\vee}$-related, and the assertion follows. Simple inspection will show that all $\mathbf{B}^{\vee}$-classes in part (i) are contained in $\Gamma$.

## 8. Subdirect Decompositions of $\mathcal{L}(\mathcal{C R})$

From the varietal version of [15, Proposition VII.2.10(i)], we know that $\mathbf{K} \cap$ $\mathbf{T}=\varepsilon$. This induces a subdirect decomposition of $\mathcal{L}(\mathcal{C})$ along $\mathcal{L}(\mathcal{C}) / \mathbf{K}$ and $\mathcal{L}(\mathcal{C} \mathcal{R}) / \mathbf{T}$. What does this decomposition look like? The second part of this reference, in the varietal garb, has the form

$$
\begin{equation*}
\mathcal{V}=\mathcal{V}_{K} \vee \mathcal{V}_{T}=\mathcal{V}^{K} \cap \mathcal{V}^{T} \quad(\mathcal{V} \in \mathcal{L}(\mathcal{C R})) \tag{5}
\end{equation*}
$$

Theorem 8.1. Set

$$
\Phi=\left\{(\mathcal{U} \mathbf{K}, \mathcal{W} \mathbf{T}) \in \mathcal{L}(\mathcal{C R}) / \mathbf{K} \times \mathcal{L}(\mathcal{C R}) / \mathbf{T} \mid \mathcal{U}_{K} \vee \mathcal{W}_{T}=\mathcal{U}^{K} \cap \mathcal{W}^{T}\right\}
$$

Then the mappings

$$
\begin{aligned}
\varphi & : \mathcal{V} \mapsto(\mathcal{V} K, \mathcal{V} T) \quad(\mathcal{V} \in \mathcal{L}(\mathcal{C} \mathcal{R})), \\
\psi:(\mathcal{U K}, \mathcal{W} \mathbf{T}) & \mapsto \mathcal{U}_{K} \vee \mathcal{W}_{T}=\mathcal{U}^{K} \cap \mathcal{W}^{T} \quad((\mathcal{U K}, \mathcal{W} \mathbf{T}) \in \Phi)
\end{aligned}
$$

are mutually inverse isomorphisms between $\mathcal{L}(\mathcal{C R})$ and $\Phi$.
Proof. By (5), $\varphi$ maps $\mathcal{L}(\mathcal{C})$ into $\Phi$. Trivially $\psi$ maps $\Phi$ into $\mathcal{L}(\mathcal{C})$. For $\mathcal{V} \in \mathcal{L}(\mathcal{C R})$, we have

$$
\mathcal{V} \varphi \psi=(\mathcal{V}, \mathcal{V} \mathbf{T}) \psi=\mathcal{V} \quad \text { by }(5)
$$

For $(\mathcal{U K}, \mathcal{W} \mathbf{T}) \in \Phi$, let $\mathcal{V}=\mathcal{U}_{K} \vee \mathcal{W}_{T}=\mathcal{U}^{K} \cap \mathcal{W}^{T}$. Then

$$
\begin{gathered}
\mathcal{V}_{K}=\left(\mathcal{U}_{K} \vee \mathcal{W}_{T}\right)_{K}=\mathcal{U}_{K} \vee \mathcal{W}_{T K} \supseteq \mathcal{U}_{K} \\
\mathcal{V}_{K}=\left(\mathcal{U}^{K} \cap \mathcal{W}^{T}\right)_{K} \subseteq\left(\mathcal{U}^{K}\right)_{K}=\mathcal{U}_{K}
\end{gathered}
$$

and thus $\mathcal{V}_{K}=\mathcal{U}_{K}$. Similarly, we get $\mathcal{V}_{T}=\mathcal{W}_{T}$. Therefore $\mathcal{U K} \mathcal{V}$ and $\mathcal{W} \mathbf{T} \mathcal{V}$, that is $(\mathcal{U K}, \mathcal{W} \mathbf{T})=(\mathcal{V} \mathbf{K}, \mathcal{V} \mathbf{T})$. Thus

$$
(\mathcal{U K}, \mathcal{W} \mathbf{T}) \psi \varphi=\mathcal{V} \varphi=(\mathcal{V} \mathbf{K}, \mathcal{V} \mathbf{T})=(\mathcal{U} \mathbf{K}, \mathcal{W} \mathbf{T})
$$

It follows that $\varphi$ and $\psi$ are mutually inverse bijections between $\mathcal{L}(\mathcal{C} \mathcal{R})$ and $\Phi$.
Trivially $\varphi$ is inclusion preserving. In order to show that $\psi$ is inclusion preserving, it suffices to assume that $(\mathcal{V} \mathbf{K}, \mathcal{V} \mathbf{T}) \subseteq\left(\mathcal{V}^{\prime} \mathbf{K}, \mathcal{V}^{\prime} \mathbf{T}\right)$ for $\mathcal{V}, \mathcal{V}^{\prime} \in \mathcal{L}(\mathcal{C R})$. Then $\mathcal{V} \mathbf{K} \subseteq \mathcal{V}^{\prime} \mathbf{K}$ and $\mathcal{V} \mathbf{T} \subseteq \mathcal{V}^{\prime} \mathbf{T}$. But then $\mathcal{V} \subseteq \mathcal{V}^{\prime}$ and therefore $\psi$ is order preserving as well. The assertion of the theorem follows.

From [7, Lemma 4.2], we have

$$
\mathcal{V}=\mathcal{V}_{B^{\wedge}} \vee \mathcal{V}_{B^{\vee}}=\mathcal{V}^{B^{\wedge}} \cap \mathcal{V}^{B^{\vee}} \quad(\mathcal{V} \in \mathcal{L}(\mathcal{R}))
$$

and in [8] we saw that $\mathbf{B}^{\wedge} \cap \mathbf{B}^{\vee}=\varepsilon$. This induces another subdirect decomposition of $\mathcal{L}(\mathcal{C} \mathcal{R})$.

Theorem 8.2. If in Theorem 8.1, we replace $(\mathbf{K}, \mathbf{T})$ by $\left(\mathbf{B}^{\wedge}, \mathbf{B}^{\vee}\right)$, or alternatively by $\left(\mathbf{B}^{\vee}, \mathbf{B}^{\wedge}\right)$, the statement and the proof of Theorem 8.1 remain valid giving the decomposition of $\mathcal{L}(\mathcal{C})$ into a subdirect product of $(\mathcal{B}]$ and $[\mathcal{B})$.

Proof. Straightforward translation.

The third subdirect decomposition also concerns $(\mathcal{B}]$ and $[\mathcal{B})$, but with different functions. It is a specialization of [8, Theorem 3.1] which concerns an
arbitrary lattice $L$ and its neutral element $a$. Letting $L=\mathcal{L}(\mathcal{C R})$ and $a=\mathcal{B}$, we arrive at

Theorem 8.3. The mappings

$$
\varphi: \mathcal{V} \mapsto\left(\mathcal{V}_{B^{\wedge}}, \mathcal{V}_{B^{\vee}}\right), \quad \psi:(\mathcal{U}, \mathcal{W}) \mapsto \mathcal{U} \vee \mathcal{W}_{B^{\vee}}=\mathcal{U}^{B^{\wedge}} \cap \mathcal{W}
$$

are mutually inverse isomorphisms between $\mathcal{L}(\mathcal{C})$ and the lattice

$$
\left\{(\mathcal{U}, \mathcal{W}) \in(\mathcal{B}] \times[\mathcal{B}) \mid \mathcal{U} \vee \mathcal{W}_{B^{\vee}}=\mathcal{U}^{\left.B^{\wedge} \cap \mathcal{W}\right\}}\right.
$$

which is a subdirect product of $(\mathcal{B}]$ and $[\mathcal{B})$.

## References

[1] P.R. Jones, Mal'cev product of varieties of completely regular semigroups, J. Aust. Math. Soc. 42 (1987) 227-246.
[2] J. Kadourek, On the word problem for bands of groups and for free objects in some other varieties of completely regular semigroups, Semigroup Forum 38 (1989) 1-55.
[3] F. Pastijn, The lattice of completely regular semigroup varieties, J. Aust. Math. Soc. A 49 (1990) 24-42.
[4] M. Petrich, Canonical varieties of completely regular semigroups, J. Aust. Math. Soc. 83 (2007) 87-104.
[5] M. Petrich, A lattice of varieties of completely regular semigroups, Comm. Algebra 42 (2014) 1397-1413.
[6] M. Petrich, Varieties of completely regular semigroups related to canonical varieties, Semigroup Forum 90 (2015) 53-99.
[7] M. Petrich, New operators for varieties of completely regular semigroups, Semigroup Forum 91 (2015) 415-449.
[8] M. Petrich, On the varieties of bands in completely regular semigroups, Publ. Math. Debrecen 89 (2016) 43-61.
[9] M. Petrich, Some relations on a semilattice of varieties of completely regular semigroups, Semigroup Forum 93 (2016) 607-628.
[10] M. Petrich, A semilattice of varieties of completely regular semigroups, Math. Bohem. 145 (1) (2020) 1-14.
[11] M. Petrich, Bases of certain varieties of completely regular semigroups, Comment. Math. Univ. Carolin. 62 (1) (2021) 41-65.
[12] M. Petrich and N.R. Reilly, Semigroups generated by certain operators on varieties of completely regular semigroups, Pacific. J. Math. 132 (1988) 151-175.
[13] M. Petrich and N.R. Reilly, Operators related to idempotent generated and monoid completely regular semigroups, J. Aust. Math. Soc. A 49 (1990) 1-23.
[14] M. Petrich and N.R. Reilly, Operators related to $E$-disjunctive and fundamental completely regular semigroups, J. Algebra 134 (1990) 1-27.
[15] M. Petrich and N.R. Reilly, Completely Regular Semigroups, Wiley, New York, 1999.
[16] L. Polák, Varieties of completely regular semigroups I, Semigroup Forum 32 (1985) 97-123.
[17] N.R. Reilly, Varieties of completely regular semigroups, J. Aust. Math. Soc. A 38 (1985) 372-393.

