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Cubic Closed Ideals of BG-Algebras and Their Products

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Abstract. In this paper, the concept of cubic set to ideals and closed ideals of BG-algebras are introduced and their related properties are studied. Relations among cubic BG-subalgebras, cubic ideals and cubic closed ideals of BG-algebras are investigated. The inverse images of cubic ideals of BG-algebras are defined, and how the inverse images of cubic ideals becomes cubic ideals of BG-algebras are studied. Also, the product of cubic BG-algebras are investigated.

Keywords: BG-algebra; Cubic set; Cubic BG-subalgebra; Cubic ideal; Cubic closed ideal.

1. Introduction

The notion of fuzzy sets introduced by Zadeh [23] in 1965 laid the foundation for the development of fuzzy Mathematics. This theory has a wide range of application in several branches of Mathematics such as logic, set theory, group theory, semigroup theory, real analysis, measure theory, and topology. After a decade, the notion of interval-valued fuzzy sets was introduced by Zadeh [24] in 1975, as an extension of fuzzy sets, that is, fuzzy sets with interval-valued membership

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functions. Cubic set was introduced by Jun et al. [7], as a generalization of fuzzy set and intuitionistic fuzzy set and is characterized by membership degree and non-membership degree. The membership function is an interval fuzzy number, while non-membership function is a fuzzy set. Jun et al. [8] applied the notion of cubic sets to a group, and introduced the notion of cubic subgroups.

The notions of BCK/BCI-algebras [4, 5] were initiated by Imai and Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Senapati et al. [14] introduced intuitionistic fuzzy translations of intuitionistic fuzzy H-ideals in BCK/BCI-algebras. Senapati et al. [22] proposed cubic intuitionistic q-ideals of BCI-algebras. Neggers and Kim [11] introduced a new notion, called a B-algebras which is related to several classes of algebras of interest such as BCI/BCK-algebras. Jun et al. [6] applied the concept of fuzzy sets to B-algebras. Senapati et al. [17, 18, 16] done lot of works on B-algebras. Kim and Kim [9] introduced the notion of BG-algebras, which is a generalization of B-algebras. Ahn and Lee [1] fuzzified BG-algebras. The authors [2, 12, 16, 15, 19, 20, 21] presented the concept and basic properties of bipolar fuzzy subalgebras, intuitionistic fuzzy subalgebras, intuitionistic L-fuzzy ideals, interval-valued intuitionistic fuzzy subalgebras, interval-valued intuitionistic fuzzy closed ideals of BG-algebras.

Senapati et al. [13] introduced cubic structure of BG-subalgebras. In this paper, cubic closed ideals of BG-algebras are defined and lot of properties are investigated. The following section briefly reviewed some background on BG-algebra, BG-subalgebra, refinement of unit interval, cubic BG-subalgebras. In Section 3, the concepts and operations of cubic ideals and cubic closed ideal are introduced. In section 4, properties of cubic ideals under homomorphisms are considered. In section 5, product of cubic BG-subalgebras and some of its properties are studied. Finally, in Section 6, conclusion and scope for future research are given.

2. Preliminaries

In this section, some definitions are recalled which are used in the later sections. A BG-algebra is an important class of logical algebras introduced by Kim and Kim [9] and was extensively investigated by several researchers. This algebra is outlined as follows.

A non-empty set Υ with a constant 0 and a binary operation $*$ is said to be BG-algebra [9] if it fulfills the following axioms:

(BG1) $\zeta * \zeta = 0$,

(BG2) $\zeta * 0 = \zeta$,

(BG3) $(\zeta * \wp) * (0 * \wp) = \zeta$,

for all $\zeta, \wp \in \Upsilon$.

In any BG-algebra Υ, the following hold (see [9]):

(i) the right cancellation law holds in Υ, i.e., $\zeta * \wp = \psi * \wp$ implies $\zeta = \psi$,

(ii) $0 * (0 * \zeta) = \zeta$ for all $\zeta \in \Upsilon$,
(iii) if $\zeta * \wp = 0$, then $\zeta = \wp$ for any $\zeta, \wp \in \Upsilon$,
(iv) if $0 * \zeta = 0 * \wp$, then $\zeta = \wp$ for any $\zeta, \wp \in \Upsilon$,
(v) $(\zeta * (0 * \zeta)) * \zeta = \zeta$ for all $\zeta \in \Upsilon$.

A non-empty subset S of a BG-algebra Υ is said to be a BG-subalgebra [1] of Υ if $\zeta * \wp \in S$, for all $\zeta, \wp \in S$. A function $f : \Upsilon \to \Psi$ of BG-algebras is said to be a homomorphism ([9]) if $f(\zeta * \wp) = f(\zeta) * f(\wp)$ for all $\zeta, \wp \in \Upsilon$. Note that if $f : \Upsilon \to \Psi$ is a BG-homomorphism, then $f(0) = 0$. A non-empty subset K of a BG-algebra Υ is said to be an ideal [10] if it fulfills following conditions:

(i) $0 \in K$,
(ii) $\zeta * \wp \in K$ and $\wp \in K \Rightarrow \zeta \in K$,
(iii) $\zeta \in K$ and $\wp \in \Upsilon \Rightarrow \zeta * \wp \in K$, $K \times \Upsilon \subseteq K$.

An ideal K of a BG-algebra $(\Upsilon, *, 0)$ is said to be closed if $0 * \zeta \in K$, for all $\zeta \in K$.

Definition 2.1. [23] (Fuzzy Set) *Let Υ be the collection of objects indicated generally by ζ. Then, a fuzzy set A in Υ is outlined as $A = \{< \zeta, \xi_A(\zeta) >: \zeta \in \Upsilon\}$ where $\xi_A(\zeta)$ is said to be the membership value of ζ in A and $0 \leq \xi_A(\zeta) \leq 1$.*

Combined the definition of BG-subalgebra, ideal over crisp set and the idea of fuzzy set Ahn et al. [1] and Muthuraj et al. [10] outlined fuzzy BG-subalgebra and ideal respectively, which is delineated below.

Definition 2.2. [1, 10] *A fuzzy set $A = \{< \zeta, \xi_A(\zeta) >: \zeta \in \Upsilon\}$ is said to be a fuzzy subalgebra of Υ if $\xi_A(\zeta * \wp) \geq \min\{\xi_A(\zeta), \xi_A(\wp)\}$ for all $\zeta, \wp \in \Upsilon$.*

A fuzzy set $A = \{< \zeta, \xi_A(\zeta) >: \zeta \in \Upsilon\}$ in Υ is said to be a fuzzy ideal of Υ if it fulfills (i) $\xi_A(0) \geq \xi_A(\zeta)$ and (ii) $\xi_A(\zeta) \geq min\{\xi_A(\zeta * \wp), \xi_A(\wp)\}$ for all $\zeta, \wp \in \Upsilon$.

Definition 2.3. [24] (Interval-valued fuzzy set) *An interval-valued fuzzy set A over Υ is an object having the form $A = \{\langle \zeta, \tilde{\xi}_A(\zeta) \rangle : \zeta \in \Upsilon\}$, where $\tilde{\xi}_A(\zeta) : \Upsilon \to D[0, 1]$, where $D[0, 1]$ is the set of all subintervals of $[0, 1]$. The intervals $\tilde{\xi}_A(\zeta)$ indicate the intervals of the degree of membership of the element ζ to the set A, where $\tilde{\xi}_A(\zeta) = [\xi_A^-(\zeta), \xi_A^+(\zeta)]$ for all $\zeta \in \Upsilon$.*

Definition 2.4. [3] (Refinement of two intervals) *Consider two elements $D_1, D_2 \in D[0, 1]$. If $D_1 = [a_1^-, a_1^+]$ and $D_2 = [a_2^-, a_2^+]$, then $rmin(D_1, D_2) = [\min(a_1^-, a_2^-), \min(a_1^+, a_2^+)]$ which is indicated by $D_1 \wedge^r D_2$. Thus, if $D_i = [a_i^-, a_i^+] \in D[0, 1]$ for i=1,2,3,4,..., then we delineate $rsup_i(D_i) = [\sup_i(a_i^-), \sup_i(a_i^+)]$, i.e, $\vee_i^r D_i = [\vee_i a_i^-, \vee_i a_i^+]$. Now we call $D_1 \geq D_2$ iff $a_1^- \geq a_2^-$ and $a_1^+ \geq a_2^+$. Similarly, the relations $D_1 \leq D_2$ and $D_1 = D_2$ are delineated.*

On the basis of the interval-valued fuzzy sets, Jun et al. [7] initiated the idea of (external, internal) cubic sets, and explored few of its properties.

Definition 2.5. [7] *Let Υ be a nonempty set. A cubic set A in Υ is a structure $A = \{\langle \zeta, \tilde{\xi}_A(\zeta), \varphi_A\ (\zeta)\rangle : \zeta \in \Upsilon\}$ which is briefly indicated by $A = (\tilde{\xi}_A, \varphi_A)$ where $\tilde{\xi}_A = [\xi_A^-, \xi_A^+]$ is an interval-valued fuzzy set in Υ and φ_A is a fuzzy set in Υ.*

Extending the idea of fuzzy BG-subalgebra, Senapati et al. [13] outlined cubic BG-subalgebra.

Definition 2.6. [13] *Let $A = (\tilde{\xi}_A, \varphi_A)$ be cubic set in Υ, where Υ is a BG-subalgebra. Then the set A is cubic BG-subalgebra over the binary operator $*$ if it fulfills the following conditions:*

(F1) $\tilde{\xi}_A(\zeta * \wp) \geq rmin\{\tilde{\xi}_A(\zeta), \tilde{\xi}_A(\wp)\}$,
(F2) $\varphi_A(\zeta * \wp) \leq \max\{\varphi_A(\zeta), \varphi_A(\wp)\}$.

for all $\zeta, \wp \in \Upsilon$.

The upper and lower level of a cubic BG-subalgebras is outlined in the earlier paper of Senapati et al [13].

Definition 2.7. [13] *Let $A = (\tilde{\xi}_A, \varphi_A)$ be a cubic BG-subalgebra of Υ. For $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$, the set $U(\tilde{\xi}_A : [s_1, s_2]) = \{\zeta \in \Upsilon | \tilde{\xi}_A(\zeta) \geq [s_1, s_2]\}$ is said to be upper $[s_1, s_2]$-level of A and $L(\varphi_A : t) = \{\zeta \in \Upsilon | \varphi_A(\zeta) \leq t\}$ is said to be lower t-level of A.*

3. Cubic Closed Ideals of BG-Algebras

In this section, cubic ideals and cubic closed ideals of BG-algebras are outlined and proved some related results. In the ensuing paragraphs, let Υ indicate a BG-algebra unless otherwise identified.

Definition 3.1. *A cubic set $A = (\tilde{\xi}_A, \varphi_A)$ in Υ is said to be a cubic ideal of Υ if it fulfills:*

(F3) $\tilde{\xi}_A(0) \geq \tilde{\xi}_A(\zeta)$ *and* $\varphi_A(0) \leq \varphi_A(\zeta)$,
(F4) $\tilde{\xi}_A(\zeta) \geq rmin\{\tilde{\xi}_A(\zeta * \wp), \tilde{\xi}_A(\wp)\}$,
(F5) $\varphi_A(\zeta) \leq \max\{\varphi_A(\zeta * \wp), \varphi_A(\wp)\}$

for all $\zeta, \wp \in \Upsilon$.

Example 3.2. Think about a BG-algebra Υ=$\{0, v, p, l\}$ with the next Cayley table

$*$	0	v	p	l
0	0	v	p	l
v	v	0	l	p
p	p	l	0	v
l	l	p	v	0

Let $A = (\tilde{\xi}_A, \varphi_A)$ be a cubic set in Υ outlined as $\tilde{\xi}_A(0) = \tilde{\xi}_A(p) = [1,1]$, $\tilde{\xi}_A(v) = \tilde{\xi}_A(l) = [m_1, m_2]$, $\varphi_A(0) = \varphi_A(p) = 0$ and $\varphi_A(v) = \varphi_A(l) = n$, where $[m_1, m_2] \in D[0,1]$ and $n \in [0,1]$. Then $A = (\tilde{\xi}_A, \varphi_A)$ is a cubic ideal of Υ.

A closed ideal of cubic ideal each also be derived from the above definition.

Definition 3.3. *A cubic set $A = (\tilde{\xi}_A, \varphi_A)$ in Υ is said to be a cubic closed ideal of Υ if it fulfills (F4), (F5) and (F6) $\tilde{\xi}_A(0 * \zeta) \geq \tilde{\xi}_A(\zeta)$ and $\varphi_A(0 * \zeta) \leq \varphi_A(\zeta)$, for all $\zeta \in \Upsilon$.*

Example 3.4. Think about a BG-algebra Υ=$\{0, v, p, l, n, r\}$ with the next Cayley table

$*$	0	v	p	l	n	r
0	0	p	v	l	n	r
v	v	0	p	r	l	n
p	p	v	0	n	r	l
l	l	n	r	0	v	p
n	n	r	l	p	0	v
r	r	l	n	v	p	0

We delineate a cubic set $A = (\tilde{\xi}_A, \varphi_A)$ in Υ by, $\tilde{\xi}_A(0) = [0.5, 0.7]$, $\tilde{\xi}_A(v) = \tilde{\xi}_A(p) = [0.4, 0.6]$, $\tilde{\xi}_A(l) = \tilde{\xi}_A(n) = \tilde{\xi}_A(r) = [0.3, 0.4]$, $\varphi_A(0) = 0.2$, $\varphi_A(v) = \varphi_A(p) = 0.3$, and $\varphi_A(l) = \varphi_A(n) = \varphi_A(r) = 0.5$. By routine calculations, one can verify that A is a cubic closed ideal of Υ.

Proposition 3.5. *Every cubic closed ideal is a cubic ideal.*

The converse of Proposition 3.5 is not true in general as noticed in the next example.

Example 3.6. Think about a BG-algebra $\Upsilon = \{0, v, p, l, n, r\}$ with the next table

$*$	0	v	p	l	n	r
0	0	r	n	l	p	v
v	v	0	r	n	l	p
p	p	v	0	r	n	l
l	l	p	v	0	r	n
n	n	l	p	v	0	r
r	r	n	l	p	v	0

We delineate a cubic set $A = (\tilde{\xi}_A, \varphi_A)$ in Υ by $\tilde{\xi}_A(0) = [0.5, 0.7]$, $\tilde{\xi}_A(v) = [0.4, 0.6]$, $\tilde{\xi}_A(p) = \tilde{\xi}_A(l) = \tilde{\xi}_A(n) = \tilde{\xi}_A(r) = [0.3, 0.4]$, $\varphi_A(0) = 0.1$, $\varphi_A(v) = 0.3$, and $\varphi_A(p) = \varphi_A(l) = \varphi_A(n) = \varphi_A(r) = 0.6$. We know that $A = (\tilde{\xi}_A, \varphi_A)$ is a cubic ideal of Υ. But it is not a cubic closed ideal of Υ in view of the fact that $\tilde{\xi}_A(0 * \zeta) \geq \tilde{\xi}_A(\zeta)$ and $\varphi_A(0 * \zeta) \leq \varphi_A(\zeta)$.

Corollary 3.7. *Every cubic BG-subalgebra satisfying (F4) and (F5) is a cubic closed ideal.*

Theorem 3.8. *Every cubic closed ideal of a BG-algebra Υ is a cubic BG-subalgebra of Υ.*

Proof. If $A = (\tilde{\xi}_A, \varphi_A)$ is a cubic closed ideal of Υ, then for any $\zeta \in \Upsilon$ we have $\tilde{\xi}_A(0 * \zeta) \geq \tilde{\xi}_A(\zeta)$ and $\varphi_A(0 * \zeta) \leq \varphi_A(\zeta)$. Now

$$\begin{aligned}
\tilde{\xi}_A(\zeta * \wp) &\geq rmin\{\tilde{\xi}_A((\zeta * \wp) * (0 * \wp)), \tilde{\xi}_A(0 * \wp)\}, \text{ by (F4)} \\
&= rmin\{\tilde{\xi}_A(\zeta), \tilde{\xi}_A(0 * \wp)\} \\
&\geq rmin\{\tilde{\xi}_A(\zeta), \tilde{\xi}_A(\wp)\}, \text{ by (F6)},
\end{aligned}$$

$$\begin{aligned}
\varphi_A(\zeta * \wp) &\leq \max\{\varphi_A((\zeta * \wp) * (0 * \wp)), \varphi_A(0 * \wp)\}, \text{ by (F5)} \\
&= \max\{\varphi_A(\zeta), \varphi_A(0 * \wp)\} \\
&\leq \max\{\varphi_A(\zeta), \varphi_A(\wp)\}, \text{ by (F6)}.
\end{aligned}$$

Hence the theorem. ■

Proposition 3.9. *If a cubic set $A = (\tilde{\xi}_A, \varphi_A)$ in Υ is a cubic closed ideal, then for all $\zeta \in \Upsilon$, $\tilde{\xi}_A(0) \geq \tilde{\xi}_A(\zeta)$ and $\varphi_A(0) \leq \varphi_A(\zeta)$.*

Theorem 3.10. *A cubic set $A = (\tilde{\xi}_A, \varphi_A)$ in Υ is a cubic ideal of Υ iff ξ_A^-, ξ_A^+ and φ_A are fuzzy ideals of Υ.*

Proof. Let $\zeta, \wp \in \Upsilon$. Considering $\xi_A^-(0) \geq \xi_A^-(\zeta)$, $\xi_A^+(0) \geq \xi_A^+(\zeta)$, we get, $\tilde{\xi}_A(0) \geq \tilde{\xi}_A(\zeta)$. Also $\varphi_A(0) \leq \varphi_A(\zeta)$. It is assumed that ξ_A^-, ξ_A^+ and φ_A are fuzzy ideals of Υ. Then $\tilde{\xi}_A(\zeta) = [\xi_A^-(\zeta), \xi_A^+(\zeta)] \geq [min\{\xi_A^-(\zeta * \wp), \xi_A^-(\wp)\}, min\{\xi_A^+(\zeta * \wp), \xi_A^+(\wp)\}] = rmin\Big\{[\xi_A^-(\zeta * \wp), \xi_A^+(\zeta * \wp)], [\xi_A^-(\wp), \xi_A^+(\wp)]\Big\} = rmin\{\tilde{\xi}_A(\zeta * \wp), \tilde{\xi}_A(\wp)\}$ and $\varphi_A(\zeta) \leq \max\{\varphi_A(\zeta * \wp), \varphi_A(\wp)\}$. Therefore, A is a cubic ideal of Υ.

On the contrary, it is assumed that, A is a cubic ideal of Υ. For any $\zeta, \wp \in \Upsilon$, we have $[\xi_A^-(\zeta), \xi_A^+(\zeta)] = \tilde{\xi}_A(\zeta) \geq rmin\{\tilde{\xi}_A(\zeta * \wp), \tilde{\xi}_A(\wp)\} = rmin\{[\xi_A^-(\zeta * \wp), \xi_A^+(\zeta * \wp)], [\xi_A^-(\wp), \xi_A^+(\wp)] = [\min\{\xi_A^-(\zeta * \wp), \xi_A^-(\wp)\}, \min\{\xi_A^+(\zeta * \wp), \xi_A^+(\wp)\}]$. Thus, $\xi_A^-(\zeta) \geq \min\{\xi_A^-(\zeta * \wp), \xi_A^-(\wp)\}$, $\xi_A^+(\zeta) \geq \min\{\xi_A^+(\zeta * \wp), \xi_A^+(\wp)\}$ and $\varphi_A(\zeta) \leq \max\{\varphi_A(\zeta * \wp), \varphi_A(\wp)\}$. Consequently, ξ_A^-, ξ_A^+ and φ_A are fuzzy ideals of Υ. ■

Lemma 3.11. *Let $A = (\tilde{\xi}_A, \varphi_A)$ be a cubic ideal of Υ. The following statements hold:*

(i) *If $\zeta * \wp \leq \psi$ then $\tilde{\xi}_A(\zeta) \geq rmin\{\tilde{\xi}_A(\wp), \tilde{\xi}_A(\psi)\}$ and $\varphi_A(\zeta) \leq \max\{\varphi_A(\wp), \varphi_A(\psi)\}$.*

(ii) *If $\zeta \leq \wp$ then $\tilde{\xi}_A(\zeta) \geq \tilde{\xi}_A(\wp)$ and $\varphi_A(\zeta) \leq \varphi_A(\wp)$.*

Proof. (i) Let $\zeta, \wp, \psi \in \Upsilon$ be such that $\zeta * \wp \leq \psi$. Then $(\zeta * \wp) * \psi = 0$ and thus $\tilde{\xi}_A(\zeta) \geq rmin\{\tilde{\xi}_A(\zeta * \wp), \tilde{\xi}_A(\wp)\} \geq rmin\{rmin\{\tilde{\xi}_A\{((\zeta * \wp) * \psi), \tilde{\xi}_A(\psi)\}, \tilde{\xi}_A(\wp)\} = rmin\{rmin\{\tilde{\xi}_A(0), \tilde{\xi}_A(\psi)\}, \tilde{\xi}_A(\wp)\} = rmin\{\tilde{\xi}_A(\wp), \tilde{\xi}_A(\psi)\}$ and $\varphi_A(\zeta) \leq \max\{\varphi_A(\zeta * \wp), \varphi_A(\wp)\} \leq \max\{\max\{\varphi_A\{((\zeta * \wp) * \psi), \varphi_A(\psi)\}, \varphi_A(\wp)\} = \max\{\max\{\varphi_A(0), \varphi_A(\psi)\}, \varphi_A(\wp)\} = \max\{\varphi_A(\wp), \varphi_A(\psi)\}$.

(ii) Again, let $\zeta, \wp \in \Upsilon$ be such that $\zeta \leq \wp$. Then $\zeta * \wp = 0$ and thus $\tilde{\xi}_A(\zeta) \geq rmin\{\tilde{\xi}_A(\zeta * \wp), \tilde{\xi}_A(\wp)\} = rmin\{\tilde{\xi}_A(0), \tilde{\xi}_A(\wp)\} = \tilde{\xi}_A(\zeta)$ and $\varphi_A(\zeta) \leq \max\{\varphi_A(\zeta * \wp), \varphi_A(\wp)\} = \max\{\varphi_A(0), \varphi_A(\wp)\} = \varphi_A(\zeta)$. ∎

Using induction on n and by Lemma 3.11 we can easily prove the following theorem.

Theorem 3.12. *If $A = (\tilde{\xi}_A, \varphi_A)$ is a cubic ideal of Υ, then $(\cdots((\zeta * a_1) * a_2) * \cdots) * a_n = 0$ for any $\zeta, a_1, a_2, \ldots, a_n \in \Upsilon$, implies $\tilde{\xi}_A(\zeta) \geq rmin\{\tilde{\xi}_A(a_1), \tilde{\xi}_A(a_2), \ldots, \tilde{\xi}_A(a_n)\}$ and $\varphi_A(\zeta) \leq \max\{\varphi_A(a_1), \varphi_A(a_2), \ldots, \varphi_A(a_n)\}$.*

Theorem 3.13. *A cubic set $A = (\tilde{\xi}_A, \varphi_A)$ is a cubic closed ideal of Υ iff the sets $U(\tilde{\xi}_A : [s_1, s_2])$ and $L(\varphi_A : t)$ are closed ideals of Υ for every $[s_1, s_2] \in D[0, 1]$ and $t \in [0, 1]$.*

Proof. Suppose that $A = (\tilde{\xi}_A, \varphi_A)$ is a cubic closed ideal of Υ. For $[s_1, s_2] \in D[0, 1]$, obviously, $0 * \zeta \in U(\tilde{\xi}_A : [s_1, s_2])$, where $\zeta \in \Upsilon$. Let $\zeta, \wp \in \Upsilon$ be organized in such a way $\zeta * \wp \in U(\tilde{\xi}_A : [s_1, s_2])$ and $\wp \in U(\tilde{\xi}_A : [s_1, s_2])$. Then $\tilde{\xi}_A(\zeta) \geq rmin\{\tilde{\xi}_A(\zeta * \wp), \tilde{\xi}_A(\wp)\} \geq [s_1, s_2]$, which implies $\zeta \in U(\tilde{\xi}_A : [s_1, s_2])$. Consequently, $U(\tilde{\xi}_A : [s_1, s_2])$ is a closed ideal of Υ.

For $t \in [0, 1]$, obviously, $0 * \zeta \in L(\varphi_A : t)$. Let $\zeta, \wp \in \Upsilon$ be such that $\zeta * \wp \in L(\varphi_A : t)$ and $\wp \in L(\varphi_A : t)$. Then $\varphi_A(\zeta) \leq \max\{\varphi_A(\zeta * \wp), \varphi_A(\wp)\} \leq t$, which implies $\zeta \in L(\varphi_A : t)$. Consequently, $L(\varphi_A : t)$ is a closed ideal of Υ.

On the contrary, it is assumed that each non-empty level subset $U(\tilde{\xi}_A : [s_1, s_2])$ and $L(\varphi_A : t)$ are closed ideals of Υ. For any $\zeta \in \Upsilon$, let $\tilde{\xi}_A(\zeta) = [s_1, s_2]$ and $\varphi_A(\zeta) = t$. Then $\zeta \in U(\tilde{\xi}_A : [s_1, s_2])$ and $\zeta \in L(\varphi_A : t)$. Bearing in mind that $0 * \zeta \in U(\tilde{\xi}_A : [s_1, s_2]) \cap L(\varphi_A : t)$, we obtain $\tilde{\xi}_A(0 * \zeta) \geq [s_1, s_2] = \tilde{\xi}_A(\zeta)$ and $\varphi_A(\zeta) \leq t = \varphi_A(\zeta)$, for all $\zeta \in \Upsilon$.

If there exist $\alpha, \beta \in \Upsilon$ ensure that $\tilde{\xi}_A(\alpha) < rmin\{\tilde{\xi}_A(\alpha * \beta), \tilde{\xi}_A(\beta)\}$, then by taking $[s_1', s_2'] = \frac{1}{2}\Big[\tilde{\xi}_A(\alpha * \beta) + rmin\{\tilde{\xi}_A(\alpha), \tilde{\xi}_A(\beta)\}\Big]$, as a consequence $\alpha * \beta \in U(\tilde{\xi}_A : [s_1', s_2'])$ and $\beta \in U(\tilde{\xi}_A : [s_1', s_2'])$, but $\alpha \notin U(\tilde{\xi}_A : [s_1', s_2'])$, which is a contradiction. Consequently, $U(\tilde{\xi}_A : [s_1', s_2'])$ is not closed ideal of Υ.

Again, if there exist $\gamma, \delta \in \Upsilon$ ensure that $\varphi_A(\gamma) > \max\{\varphi_A(\gamma * \delta), \varphi_A(\delta)\}$, then by taking $t' = \frac{1}{2}\Big[\varphi_A(\gamma * \delta) + \max\{\varphi_A(\gamma), \varphi_A(\delta)\}\Big]$, as a consequence $\gamma * \delta \in U(\varphi_A : t')$ and $\delta \in L(\varphi_A : t')$, but $\gamma \notin L(\varphi_A : t')$, which is a contradiction. Consequently, $L(\varphi_A : t')$ is not closed ideal of Υ.

Consequently, $A = (\tilde{\xi}_A, \varphi_A)$ is a cubic closed ideal of Υ because it fulfills (F3) and (F4). ■

4. Investigation of Cubic Ideals under Homomorphisms

In this section, homomorphism of cubic BG-subalgebras are outlined and some results are studied.

Let f be a function from a set Υ into a set Ψ. Let B be a cubic set in Ψ. Then the inverse image of B, is outlined as $f^{-1}(B) = (f^{-1}(\tilde{\xi}_B), f^{-1}(\varphi_B))$ with the membership function and non-membership function respectively are given by $f^{-1}(\tilde{\xi}_B)(\zeta) = \tilde{\xi}_B(f(\zeta))$ and $f^{-1}(\varphi_B)(\zeta) = \varphi_B(f(\zeta))$. It can be shown that $f^{-1}(B)$ is a cubic set.

Theorem 4.1. *Let $f : \Upsilon \to \Psi$ be a homomorphism of BG-algebras. If $B = (\tilde{\xi}_B, \varphi_B)$ is a cubic ideal of Ψ, then the pre-image $f^{-1}(B) = (f^{-1}(\tilde{\xi}_B), f^{-1}(\varphi_B))$ of B under f in Υ is a cubic ideal of Υ.*

Proof. For all $\zeta \in \Upsilon$, $f^{-1}(\tilde{\xi}_B)(\zeta) = \tilde{\xi}_B(f(\zeta)) \leq \tilde{\xi}_B(0) = \tilde{\xi}_B(f(0)) = f^{-1}(\tilde{\xi}_B)(0)$ and $f^{-1}(\varphi_B)(\zeta) = \varphi_B(f(\zeta)) \geq \varphi_B(0) = \varphi_B(f(0)) = f^{-1}(\varphi_B)(0)$.

Let $\zeta, \wp \in \Upsilon$. Then $f^{-1}(\tilde{\xi}_B)(\zeta) = \tilde{\xi}_B(f(\zeta)) \geq rmin\{\tilde{\xi}_B((f(\zeta) * f(\wp)), \tilde{\xi}_B(f(\wp))\} \geq rmin\{\tilde{\xi}_B\ (f(\zeta * \wp), \tilde{\xi}_B(f(\wp))\} = rmin\{f^{-1}(\tilde{\xi}_B)(\zeta * \wp), f^{-1}(\tilde{\xi}_B)(\wp)\}$ and $f^{-1}(\varphi_B)(\zeta) = \varphi_B(f(\zeta)) \leq \max\{\varphi_B\ ((f(\zeta) * f(\wp)), \varphi_B(f(\wp))\} \leq \max\{\varphi_B(f(\zeta * \wp), \varphi_B(f(\wp))\} = \max\{f^{-1}(\varphi_B)(\zeta * \wp), f^{-1}(\varphi_B)(\wp)\}$.

Consequently, $f^{-1}(B) = (f^{-1}(\tilde{\xi}_B), f^{-1}(\varphi_B))$ is a cubic ideal of Υ. ■

Theorem 4.2. *Let $f : \Upsilon \to \Psi$ be an epimorphism of BG-algebras. Then $B = (\tilde{\xi}_B, \varphi_B)$ is a cubic ideal of Ψ, if $f^{-1}(B) = (f^{-1}(\tilde{\xi}_B), f^{-1}(\varphi_B))$ of B under f in Υ is a cubic ideal of Υ.*

Proof. For any $\zeta \in \Psi$, there exists $a \in \Upsilon$ ensure that $f(a) = \zeta$. Then $\tilde{\xi}_B(\zeta) = \tilde{\xi}_B(f(a)) = f^{-1}(\tilde{\xi}_B)(a) \leq f^{-1}(\tilde{\xi}_B)(0) = \tilde{\xi}_B(f(0)) = \tilde{\xi}_B(0)$ and $\varphi_B(\zeta) = \varphi_B(f(a)) = f^{-1}(\varphi_B)(a) \geq f^{-1}(\varphi_B)(0) = \varphi_B(f(0)) = \varphi_B(0)$.

Let $\zeta, \wp \in \Psi$. Then $f(a) = \zeta$ and $f(b) = \wp$ for some $a, b \in \Upsilon$. Thus $\tilde{\xi}_B(\zeta) = \tilde{\xi}_B(f(a)) = f^{-1}(\tilde{\xi}_B)(a) \geq rmin\{f^{-1}(\tilde{\xi}_B)(a * b), f^{-1}(\tilde{\xi}_B)(b)\} = rmin\{\tilde{\xi}_B(f(a * b)), \tilde{\xi}_B(f(b))\} = rmin\{\tilde{\xi}_B(f(a) * f(b)), \tilde{\xi}_B(f(b))\} = rmin\{\tilde{\xi}_B(\zeta * \wp), \tilde{\xi}_B(\wp)\}$ and $\varphi_B(\zeta) = \varphi_B(f(a)) = f^{-1}(\varphi_B)(a) \leq \max\{f^{-1}(\varphi_B)(a * b), f^{-1}(\varphi_B)(b)\} = \max\{\varphi_B(f(a * b)), \varphi_B(f(b))\} = \max\{\varphi_B(f(a) * f(b)), \varphi_B(f(b))\} = \max\{\varphi_B(\zeta * \wp), \varphi_B(\wp)\}$. Consequently, $B = (\tilde{\xi}_B, \varphi_B)$ is a cubic ideal of Ψ. ■

5. Product of Cubic BG-algebras

In this section, product of cubic BG-algebras are outlined and some results are studied.

Definition 5.1. *Let $A = (\tilde{\xi}_A, \varphi_A)$ and $B = (\tilde{\xi}_B, \varphi_B)$ be two cubic sets of Υ. The cartesian product $A \times B = (\Upsilon \times \Upsilon, \tilde{\xi}_A \times \tilde{\xi}_B, \varphi_A \times \varphi_B)$ is delineated by $(\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta, \wp) = rmin\{\tilde{\xi}_A(\zeta), \tilde{\xi}_B(\wp)\}$ and $(\varphi_A \times \varphi_B)(\zeta, \wp) = \max\{\varphi_A(\zeta), \varphi_B(\wp)\}$, where $\tilde{\xi}_A \times \tilde{\xi}_B : \Upsilon \times \Upsilon \to D[0, 1]$ and $\varphi_A \times \varphi_B : \Upsilon \times \Upsilon \to [0, 1]$ for all $\zeta, \wp \in \Upsilon$.*

Proposition 5.2. *Let $A = (\tilde{\xi}_A, \varphi_A)$ and $B = (\tilde{\xi}_B, \varphi_B)$ be cubic ideals of Υ. Then $A \times B$ is a cubic ideal of $\Upsilon \times \Upsilon$.*

Proof. For any $(\zeta, \wp) \in \Upsilon \times \Upsilon$, we have $(\tilde{\xi}_A \times \tilde{\xi}_B)(0, 0) = rmin\{\tilde{\xi}_A(0), \tilde{\xi}_B(0)\} \geq rmin\{\tilde{\xi}_A(\zeta),\ \tilde{\xi}_B(\wp)\} = (\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta, \wp)$ and $(\varphi_A \times \varphi_B)(0, 0) = \max\{\varphi_A(0), \varphi_B(0)\} \leq \max\{\varphi_A(\zeta), \varphi_B(\wp)\} = (\varphi_A \times \varphi_B)(\zeta, \wp)$.

Let $(\zeta_1, \wp_1)$ and $(\zeta_2, \wp_2) \in \Upsilon \times \Upsilon$. Then

$$\begin{aligned}
&(\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta_1, \wp_1) \\
&= rmin\{\tilde{\xi}_A(\zeta_1), \tilde{\xi}_B(\wp_1)\} \\
&\geq rmin\{rmin\{\tilde{\xi}_A(\zeta_1 * \zeta_2), \tilde{\xi}_A(\zeta_2)\}, rmin\{\tilde{\xi}_B(\wp_1 * \wp_2), \tilde{\xi}_B(\wp_2)\}\} \\
&= rmin\{rmin\{\tilde{\xi}_A(\zeta_1 * \zeta_2), \tilde{\xi}_B(\wp_1 * \wp_2)\}, rmin\{\tilde{\xi}_A(\zeta_2), \tilde{\xi}_B(\wp_2)\}\} \\
&= rmin\{(\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta_1 * \zeta_2, \wp_1 * \wp_2), (\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta_2, \wp_2)\} \\
&= rmin\{(\tilde{\xi}_A \times \tilde{\xi}_B)((\zeta_1, \wp_1) * (\zeta_2, \wp_2)), (\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta_2, \wp_2)\},
\end{aligned}$$

$$\begin{aligned}
&(\varphi_A \times \varphi_B)(\zeta_1, \wp_1) \\
&= \max\{\varphi_A(\zeta_1), \varphi_B(\wp_1)\} \\
&\leq \max\{\max\{\varphi_A(\zeta_1 * \zeta_2), \varphi_A(\zeta_2)\}, \max\{\varphi_B(\wp_1 * \wp_2), \varphi_B(\wp_2)\}\} \\
&= \max\{\max\{\varphi_A(\zeta_1 * \zeta_2), \varphi_B(\wp_1 * \wp_2)\}, \max\{\varphi_A(\zeta_2), \varphi_B(\wp_2)\}\} \\
&= \max\{(\varphi_A \times \varphi_B)(\zeta_1 * \zeta_2, \wp_1 * \wp_2), (\varphi_A \times \varphi_B)(\zeta_2, \wp_2)\} \\
&= \max\{(\varphi_A \times \varphi_B)((\zeta_1, \wp_1) * (\zeta_2, \wp_2)), (\varphi_A \times \varphi_B)(\zeta_2, \wp_2)\}.
\end{aligned}$$

Consequently, $A \times B$ is a cubic ideal of $\Upsilon \times \Upsilon$. ■

Proposition 5.3. *Let $A = (\tilde{\xi}_A, \varphi_A)$ and $B = (\tilde{\xi}_B, \varphi_B)$ be cubic closed ideals of Υ. Then $A \times B$ is a cubic closed ideal of $\Upsilon \times \Upsilon$.*

Proof. Now, $(\tilde{\xi}_A \times \tilde{\xi}_B)((0, 0) * (\zeta, \wp)) = (\tilde{\xi}_A \times \tilde{\xi}_B)(0 * \zeta, 0 * \wp) = rmin\{\tilde{\xi}_A(0 * \zeta), \tilde{\xi}_B(0 * \wp)\} \geq rmin\{\tilde{\xi}_A(\zeta), \tilde{\xi}_B(\wp)\} = (\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta, \wp)$ and $(\varphi_A \times \varphi_B)((0, 0) * (\zeta, \wp)) = (\varphi_A \times \varphi_B)(0 * \zeta, 0 * \wp) = \max\{\varphi_A(0 * \zeta), \varphi_B(0 * \wp)\} \leq \max\{\varphi_A(\zeta), \varphi_B(\wp)\} = (\varphi_A \times \varphi_B)(\zeta, \wp)$. Consequently, $A \times B$ is a cubic closed ideal of $\Upsilon \times \Upsilon$. ■

Definition 5.4. *Let $A = (\tilde{\xi}_A, \varphi_A)$ and $B = (\tilde{\xi}_B, \varphi_B)$ be cubic BG-subalgebras of Υ. For $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$, the set $U(\tilde{\xi}_A \times \tilde{\xi}_B : [s_1, s_2]) = \{(\zeta, \wp) \in \Upsilon \times \Upsilon | (\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta, \wp) \geq [s_1, s_2]\}$ is said to be a upper $[s_1, s_2]$-level of $A \times B$ and $L(\varphi_A \times \varphi_B : t) = \{(\zeta, \wp) \in \Upsilon \times \Upsilon | (\varphi_A \times \varphi_B)(\zeta, \wp) \leq t\}$ is said to be lower t-level of $A \times B$.*

Theorem 5.5. *For any two cubic sets $A = (\tilde{\xi}_A, \varphi_A)$ and $B = (\tilde{\xi}_B, \varphi_B)$, $A \times B$ is a cubic closed ideals of $\Upsilon \times \Upsilon$ iff the non-empty upper $[s_1, s_2]$-level cut $U(\tilde{\xi}_A \times \tilde{\xi}_B : [s_1, s_2])$ and the non-empty lower t-level cut $L(\varphi_A \times \varphi_B : t)$ are closed ideals of $\Upsilon \times \Upsilon$ for any $[s_1, s_2] \in D[0,1]$ and $t \in [0,1]$.*

Proof. Let $A = (\tilde{\xi}_A, \varphi_A)$ and $B = (\tilde{\xi}_B, \varphi_B)$ be cubic closed ideals of Υ, therefore, for any $(\zeta, \wp) \in \Upsilon \times \Upsilon$, $(\tilde{\xi}_A \times \tilde{\xi}_B)((0,0) * (\zeta, \wp)) \geq (\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta, \wp)$ and $(\varphi_A \times \varphi_B)((0,0) * (\zeta, \wp)) \leq (\varphi_A \times \varphi_B)(\zeta, \wp)$. For $[s_1, s_2] \in D[0,1]$, if $(\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta, \wp) \geq [s_1, s_2]$, then $(\tilde{\xi}_A \times \tilde{\xi}_B)((0,0) * (\zeta, \wp)) \geq [s_1, s_2]$. This implies, $(0,0) * (\zeta, \wp) \in U(\tilde{\xi}_A \times \tilde{\xi}_B : [s_1, s_2])$. Let $(\zeta, \wp), (\zeta', \wp') \in \Upsilon \times \Upsilon$ be designed in such a way that $(\zeta, \wp) * (\zeta', \wp') \in U(\tilde{\xi}_A \times \tilde{\xi}_B : [s_1, s_2])$ and $(\zeta', \wp') \in U(\tilde{\xi}_A \times \tilde{\xi}_B : [s_1, s_2])$. Now, $(\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta, \wp) \geq rmin\{(\tilde{\xi}_A \times \tilde{\xi}_B)((\zeta, \wp) * (\zeta', \wp')), (\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta', \wp')\} \geq rmin([s_1, s_2], [s_1, s_2]) = [s_1, s_2]$. This implies, $(\zeta, \wp) \in U(\tilde{\xi}_A \times \tilde{\xi}_B : [s_1, s_2])$. Thus $U(\tilde{\xi}_A \times \tilde{\xi}_B : [s_1, s_2])$ is closed ideal of $\Upsilon \times \Upsilon$. Similarly, $L(\varphi_A \times \varphi_B : t)$ is closed ideal of $\Upsilon \times \Upsilon$.

On the contrary, let $(\zeta, \wp) \in \Upsilon \times \Upsilon$ be designed in such a way that $(\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta, \wp) = [s_1, s_2]$ and $(\varphi_A \times \varphi_B)(\zeta, \wp) = t$. This implies, $(\zeta, \wp) \in U(\tilde{\xi}_A \times \tilde{\xi}_B : [s_1, s_2])$ and $(\zeta, \wp) \in L(\varphi_A \times \varphi_B : t)$. In view of the fact that $(0,0) * (\zeta, \wp) \in U(\tilde{\xi}_A \times \tilde{\xi}_B : [s_1, s_2])$ and $(0,0) * (\zeta, \wp) \in L(\varphi_A \times \varphi_B : t)$ (by definition of closed ideal), thus, $(\tilde{\xi}_A \times \tilde{\xi}_B)((0,0) * (\zeta, \wp)) \geq [s_1, s_2]$ and $(\varphi_A \times \varphi_B)((0,0) * (\zeta, \wp)) \leq t$. This gives, $(\tilde{\xi}_A \times \tilde{\xi}_B)((0,0) * (\zeta, \wp)) \geq (\tilde{\xi}_A \times \tilde{\xi}_B)(\zeta, \wp)$ and $(\varphi_A \times \varphi_B)((0,0) * (\zeta, \wp)) \leq (\varphi_A \times \varphi_B)(\zeta, \wp)$. Consequently, $A \times B$ is a cubic closed ideal of $\Upsilon \times \Upsilon$. ∎

6. Conclusions and Future Work

Recently, in [13], the authors have studied cubic BG-subalgebras of BG-algebras. In this paper, we have introduced the concept of cubic ideal and cubic closed ideal of BG-algebras and investigated some of their useful properties. The product of cubic BG-subalgebras has been introduced and some important properties are of it are also studied. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BF-algebras, lattices and Lie algebras. In our future study of fuzzy structure of BG-algebra, maybe the following topics should be considered:

(i) To find cubic (T, S)-fuzzy BG-subalgebras and ideals, where T and S are given imaginable triangular norms and conorms;

(ii) To get more results in cubic closed ideals of BG-algebra and application;

(iii) To find cubic intuitionistic BG-subalgebras and ideals, and their properties.

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