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Subdirectly Irreducible Bands Whose Structural Semilattices Have Height 2

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Abstract. We characterize some subdirectly irreducible bands whose structural semilattices have height 2 in terms of fundamental semilattices of semigroups.

Keywords: Subdirectly irreducible bands; Fundmental semilattices of semigroups.

1. Introduction and Preliminaries

We recall that every non-trivial semigroup is a subdirect product of some subdirectly irreducible semigroups [3]. A nontrivial semigroup S is subdirectly irreducible if and only if there exists the least nontrivial congruence on S. In 1973, Gerhard gave a representation of subdirectly irreducible bands in terms of transformations [2]. In 2017, Wang, Leng and Yu characterized subdirectly irreducible regular bands whose structural semilattices are finite chains by using refined semilattices of semigroups [7]. The purpose of this paper is to give a characterization of subdirectly irreducible bands (not necessarily regular) whose structural semilattices have heights 2 in terms of fundamental semilattices of semigroups.

First we introduce some notation and terminology. Let X be a nonempty set, and $\mathscr{T}(X)$ denote the semigroup formed by all transformations on X. The

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symbol $\langle\phi\rangle$ means that ϕ is a constant which maps X onto the element $\langle\phi\rangle \in X$. We write the identity relation on X as ε_X and the universal relation on X as ω_X.

Let ρ be a binary relation on X, and we have $\varepsilon_B \subseteq \rho$. Then we have

$$\rho \subseteq \rho^2 \subseteq \rho^3 \subseteq \cdots .$$

The relation $\rho^\infty = \bigcup\{\rho^n : n \geq 1\}$ is said to be the *transitive closure* of ρ. We denote $\rho^{-1} = \{(x,y) \in X \times X : (y,x) \in \rho\}$. Let ρ be a relation on a semigroup S. We call ρ is *left compatible* if for any $a, b, c \in S$, $(a,b) \in \rho$ implies $(ca, cb) \in \rho$. Right compatibility of a relation is dually defined. An equivalence which is both left and right compatible on a semigroup S is called a *congruence* on S. The set of all congruences on S is denoted by $\mathcal{C}(S)$. By the related discussion in Sections 1.4 and 1.5 of [1], we have

Lemma 1.1. *For every left and right compatible relation ρ on a semigroup S, $(\rho \cup \rho^{-1} \cup \varepsilon_S)^\infty$ is the smallest congruence on S containing ρ.*

Let Y be a semilattice and $\{S_\alpha : \alpha \in Y\}$ be a family of pairwise disjoint semigroups. For any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $\Phi_{\alpha,\beta} : S_\alpha \longrightarrow \mathscr{T}(S_\beta), a \mapsto \phi^a_\beta$ be a mapping. Assume that the following conditions are satisfied:

(1) for any $\alpha \in Y$, $a \in S_\alpha$, $\langle\phi^a_\alpha\rangle = a$;

(2) for any α, $\beta \in Y$, $a \in S_\alpha$, and $b \in S_\beta$, $\phi^a_{\alpha\beta}\phi^b_{\alpha\beta}$ is a constant. Then

$$a * b = \langle\phi^b_{\alpha\beta}\phi^a_{\alpha\beta}\rangle\langle\phi^a_{\alpha\beta}\phi^b_{\alpha\beta}\rangle \qquad (a \in S_\alpha,\, b \in S_\beta);$$

gives a multiplication on $S = \bigcup_{\alpha\in Y} S_\alpha$. Suppose also that

(3) for any α, β, $\gamma \in Y, a \in S_\alpha$, $b \in S_\beta$ and $c \in S_\gamma$,

$$\langle\phi^a_{\alpha\beta\gamma}\phi^{b*c}_{\alpha\beta\gamma}\rangle = \langle\phi^{c*a}_{\alpha\beta\gamma}\phi^b_{\alpha\beta\gamma}\rangle\langle\phi^{a*b}_{\alpha\beta\gamma}\phi^c_{\alpha\beta\gamma}\rangle.$$

Then $(S, *)$ is a semigroup, called a *fundamental semilattice Y of semigroups* $S_\alpha\,(\alpha \in Y)$, denoted by $S = \mathscr{F}(Y; S_\alpha, \Phi_{\alpha,\beta})$.

Lemma 1.2. [6, Corollary 5.7] *A semigroup S is a band if and only if S is a fundamental semilattice of rectangular bands.*

According to Theorem 4.4, Proposition 4.5 and their proofs in [6], we have

Lemma 1.3. *Let $B = \mathscr{F}(Y; B_\alpha, \Phi_{\alpha,\beta})$ be a band. Suppose that $\alpha, \beta \in Y$ with $\alpha \geq \beta$, $a \in B_\alpha$ and $b \in B_\beta$. Then*

(1) *$b\phi^a_\beta = aba$;*

(2) *$ab = (b\phi^a_\beta)b$ and $ba = b(b\phi^a_\beta)$;*

(3) *$\ker\phi^a_\beta = \{(x,y) \in B_\beta \times B_\beta : axa = aya\}$;*

(4) $\operatorname{im} \phi^a_\beta = \{axa : x \in B_\beta\}$.

Lemma 1.4. [8, Proposition 2.6] *If $B = \mathscr{F}(Y; B_\alpha, \Phi_{\alpha,\beta})$ is a band, then for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, $a, b \in B_\alpha$, $\ker \phi^a_\beta$ is a rectangular band congruence, and $B_\beta / \ker \phi^a_\beta$ is isomorphic to $B_\beta / \ker \phi^b_\beta$.*

Now we prepare to discuss subdirectly irreducible bands. Noticing [7, Remark 1.2 and Lemma 1.3], in order to describe subdirectly irreducible bands, it suffices to consider those having neither identity nor zero.

Lemma 1.5. [5, Theorem 4.7] *If B is a subdirectly irreducible band without zero, then B satisfies one of the following conditions.*

(1) *Let $K = \{k \in B : kb = k \text{ for all } b \in B\}$. Then K is a two-sided ideal of B, and for any $x, y \in B$, $xk = yk$ for all $k \in K$ implies $x = y$.*

(2) *Let $K = \{k \in B : bk = k \text{ for all } b \in B\}$. Then K is a two-sided ideal of B, and for any $x, y \in B$, $kx = ky$ for all $k \in K$ implies $x = y$.*

We conclude from Lemma 1.5 that for a subdirectly irreducible band $B = \mathscr{F}(Y; B_\alpha, \Phi_{\alpha,\beta})$, the above K is either a nontrivial left zero semigroup or a nontrivial right zero semigroup. Moreover, K is a subset of B_θ, where θ is the zero of the structural semilattice Y. If K is a left zero semigroup, then for any $x, y \in B_\theta$ with $x \mathcal{R} y$ and $k \in K$, we have $xk = yxk = yk$ which leads to $x = y$. Therefore, $K = B_\theta$. Similarly, we have $K = B_\theta$ if K is a right zero semigroup.

For notation and terminology not explained in this paper, the reader is referred to [1].

2. Main Results and Proofs

A semilattice Y is said to *have height* 2 if any subchain of Y is isomorphic to Y_2, the 2-element semilattice. In this section, we always suppose that Y is a semilattice of height 2, θ is the zero of Y, $B = \mathscr{F}(Y; B_\alpha, \Phi_{\alpha,\beta})$ is a band (whose structural semilattice has height 2) with neither identity nor zero, B_θ is a right zero semigroup and for any $\alpha \in Y - \{\theta\}$ and $a \in B_\alpha$, $a\Phi_{\alpha,\theta}$ is denote by φ^a_θ.

In the following lemmas and corollaries the band B is *subdirectly irreducible.* Note that B_θ is a right zero semigroup.

Lemma 2.1. *For any $\alpha \in Y - \{\theta\}$, $\Phi_{\alpha,\theta}$ is injective.*

Proof. For any $a, b \in B_\alpha$, suppose that $\varphi^a_\theta = \varphi^b_\theta$. Then for any $x \in B_\theta$, we see from Lemma 1.3(1) that $xa = xb$. It follows from Lemma 1.5(2) that $a = b$. That means $\Phi_{\alpha,\theta}$ is injective. ∎

Lemma 2.2. *For any* $\alpha \in Y - \{\theta\}$ *and* $a \in B_\alpha$, $\ker \varphi_\theta^a \neq \omega_{B_\theta}$.

Proof. Suppose that there exist $\beta \in Y - \{\theta\}$ and $b \in B_\beta$ such that $\ker \varphi_\theta^a = \omega_{B_\theta}$. We see from Lemma 1.3(3) that $xa = axa = aya = ya = x(ya)$. It follows from Lemma 1.5(2) that $a = ya$, a contradiction. ∎

Lemma 2.3. *For any* $\alpha \in Y - \{\theta\}$ *and* $a, b \in B_\alpha$, $a \mathcal{R} b$ *implies* $\ker \varphi_\theta^a = \ker \varphi_\theta^b$.

Proof. Arbitrarily take $x, y \in B_\theta$. We observe from Lemma 1.5(3) that $(x, y) \in \ker \varphi_\theta^a$ if and only if $xa = ya$ if and only if $xb = yb$ since $ab = b$ and $ba = a$. It follows that $\ker \varphi_\theta^a = \ker \varphi_\theta^b$. ∎

Lemma 2.4. *For any* $\alpha \in Y - \{\theta\}$ *and* $a, b \in B_\alpha$, $a \mathscr{L} b$ *implies that* $\mathrm{im}\, \varphi_\theta^a = \mathrm{im}\, \varphi_\theta^b$.

Proof. We obtain from Lemma 1.5(4) that $\mathrm{im}\, \varphi_\theta^a = \{xa : x \in B_\theta\} = \{yb : y \in B_\theta\}$ since $xa = xab = (xa)b$ and $xb = xba = (xb)a$ for any $x \in B_\theta$. ∎

For any $\alpha \in Y$, define two relations on B as follows

$$\rho_\alpha = \bigcap_{a \in B_\alpha} \ker \varphi_\theta^a \cup \varepsilon_B,$$

$$\sigma_\alpha = (\bigcup_{a \in B_\alpha} (\mathrm{im}\, \varphi_\theta^a \times \mathrm{im}\, \varphi_\theta^a) \cup \varepsilon_B)^\infty.$$

Lemma 2.5. *For any* $\alpha \in Y$, $\rho_\alpha \in \mathcal{C}(B)$.

Proof. Obviously, ρ_α is an equivalence on B. To show that $\rho \in \mathcal{C}(B)$, it suffice to verify that ρ is right compatible. Arbitrarily take $(u, v) \in \rho_\alpha$ and $b \in B$. If $c \in B_\alpha$, we see from the definition of ρ_α and Lemma 1.3(3) that $uc = vc$. Otherwise, noticing that Y has height 2, we have $uca = ca = vca$ for any $a \in B_\alpha$. It follows from Lemma 1.3(3) that $(uc, vc) \in \bigcap_{a \in B_\alpha} \ker \varphi_\theta^a$. Hence, $\rho_\alpha \in \mathcal{C}(B)$. ∎

Lemma 2.6. *For any* $\alpha \in Y$, $\sigma_\alpha \in \mathcal{C}(B)$.

Proof. By Lemma 1.1, it suffice to verify that $\bigcup_{a \in B_\alpha} (\mathrm{im}\, \varphi_\theta^a \times \mathrm{im}\, \varphi_\theta^a)$ is right compatible. For any $a \in B_\alpha$ and $b \in B$, suppose that $(u, v) \in \mathrm{im}\, \varphi_\theta^a \times \mathrm{im}\, \varphi_\theta^a$. We see from Lemma 1.3(4) that $u = sa, v = ta$ for some $s, t \in B_\theta$ and $a \in B_\alpha$. If $b \in B_\alpha$, then we obtain again from Lemma 1.3(4) that $(ub, vb) \in \bigcup_{a \in B_\alpha} (\mathrm{im}\, \varphi_\theta^a \times \mathrm{im}\, \varphi_\theta^a)$. Otherwise, if $b \notin B_\alpha$, then we have $ub = xab = ab$ and $vb = yab = ab$ and hence $(ub, vb) \in \bigcup_{a \in B_\alpha} (\mathrm{im}\, \varphi_\theta^a \times \mathrm{im}\, \varphi_\theta^a)$. ∎

Lemma 2.7. *For any* $\alpha, \beta \in Y - \{\theta\}$ *with* $\alpha \neq \beta$, $\sigma_\alpha \subseteq \rho_\beta$.

Proof. It suffices to prove that $\bigcup_{a\in B_\alpha}(\operatorname{im}\varphi^a_\theta \times \operatorname{im}\varphi^a_\theta) \subseteq \rho_\alpha$ since σ_α is the least congruence containing $\bigcup_{a\in B_\alpha}(\operatorname{im}\varphi^a_\theta \times \operatorname{im}\varphi^a_\theta)$. Arbitrarily taking $(u,v) \in \operatorname{im}\varphi^a_\theta \times \operatorname{im}\varphi^a_\theta$ for some $a \in B_\alpha$, it follows from condition (2) of the definition of a fundamental semilattice of semigroups, $\operatorname{im}\varphi^a_\theta \subseteq \ker\varphi^b_\theta$ for any $b \in B_\beta$ which leads to $(u,v) \in \rho_\beta$. ■

Lemma 2.8. *There exist distinct $x, y \in B_\theta$ such that $\bigcap_{\alpha\in Y}\rho_\alpha \subseteq \omega_{\{x,y\}} \cup \varepsilon_B$.*

Proof. First we see from Lemma 1.3(3) that for any $u, v \in B_\theta$, if $(u,v) \in \bigcap_{\alpha\in Y}\rho_\alpha$, then $\omega_{\{u,v\}} \cup \varepsilon_B \in \mathcal{C}(B)$. Therefore if there exist $x_1, y_1, x_2, y_2 \in B_\theta$ such that $\{x_1, y_1\} \neq \{x_2, y_2\}$ and $(x_1, y_1), (x_2, y_2) \in \bigcap_{\alpha\in Y}\rho_\alpha$, then both $\rho_1 = \omega_{\{x_1,y_1\}} \cup \varepsilon_B$ and $\rho_2 = \omega_{\{x_2,y_2\}} \cup \varepsilon_B$ are nontrivial congruences on B. However, $\rho_1 \cap \rho_2 = \varepsilon_B$, contradicting the fact that B is subdirectly irreducible. Thus, there must exist distinct $x, y \in B_\theta$ such that $\bigcap_{\alpha\in Y}\rho_\alpha \subseteq \omega_{\{x,y\}} \cup \varepsilon_B$. ■

Lemma 2.9. *If there exists $\alpha \in Y - \{\theta\}$ such that B_α is a left zero semigroup, then $Y = Y_2$, $\rho_\alpha = \varepsilon_B$ and there exist distinct $x, y \in B_\theta$ such that $\operatorname{im}\varphi^a_\theta = \{x, y\}$ for any $a \in B_\alpha$.*

Proof. It follows from Lemma 2.4 and the definition of σ_α that $\sigma_\alpha = \omega_{\{\operatorname{im}\varphi^a_\theta\}} \cup \varepsilon_B$ for some $a \in B_\alpha$. If there exists $\beta \in Y - \{\theta\}$ such that $\beta \neq \alpha$, then we obtain from condition (2) of the definition of a fundamental semilattice of semigroups that $\operatorname{im}\varphi^b_\theta \subseteq u \ker\varphi^a_\theta$ for any $u \in B_\beta$. This implies that $\sigma_\alpha \cap \sigma_\beta = \varepsilon_B$. However, we know from Lemma 2.2 that σ_α and σ_β are nontrivial, a contradiction. So we get $Y = Y_2$.

Noticing that $\sigma_\alpha = \omega_{\{\operatorname{im}\varphi^a_\theta\}} \cup \varepsilon_B$, we observe that $\sigma_\alpha \cap \rho_\alpha = \varepsilon_B$. Then we must have $\rho_\alpha = \varepsilon_B$ since B is subdirectly irreducible and σ_α is nontrivial.

According to Lemma 1.3(4), we can take ua, va from $\operatorname{im}\varphi^a_\theta$. For any $b \in B$, if $b \in B_\alpha$, then we have $uab = ua$ and $vab = vb$; if $b \in B_\theta$, then we have $uab = vab = b$. That is, $\omega_{\{ua,va\}} \cup \varepsilon_B \in \mathcal{C}(B)$. Noticing that B is subdirectly irreducible, there must exist distinct $x, y \in B_\theta$ such that $\operatorname{im}\varphi^a_\theta = \{x, y\}$. ■

Lemma 2.10. *If for any $\alpha \in Y - \{\theta\}$, B_α is not a left zero semigroup, then there exist distinct $x, y \in B_\theta$ such that $\bigcap_{\alpha\in Y}\rho_\alpha = \omega_{\{x,y\}} \cup \varepsilon_B$.*

Proof. If $Y \neq Y_2$, then we see from Lemma 2.7 that for any $\alpha \in Y - \{\theta\}$, there exists $\beta \in Y - \{\theta\}$ such that $\sigma_\beta \subseteq \rho_\alpha$. Note from Lemma 2.2 and the definition of σ_β that σ_β is nontrivial. We obtain from condition (1) of the definition of a fundamental semilattice of semigroups and Lemma 2.8 that there exist distinct $x, y \in B_\theta$ such that $\bigcap_{\alpha\in Y}\rho_\alpha = \omega_{\{x,y\}} \cup \varepsilon_B$.

Now suppose that $Y = Y_2$. Note that for any $\alpha \in Y - \{\theta\}$, there exist distinct $a, b \in B_\alpha$ such that $a \mathcal{R} b$. It follows from Lemma 2.3 that $\ker\varphi^a_\theta = \ker\varphi^b_\theta$. We easily see from Lemma 1.3(4) that for every $u \in B_\theta$, $(u, u\varphi^a_\theta) \in \ker\varphi^a_\theta$. So we obtain from Lemma 2.1 that $\operatorname{im}\varphi^a_\theta \neq \operatorname{im}\varphi^b_\theta$ and hence there exists $z \in B_\theta$ such that $za = z\varphi^a_\theta \neq z\varphi^b_\theta = zb$. Let $\rho_1 = \omega_{\{za,zb\}} \cup \varepsilon_B$. For any $c \in B$, if $c \in B_\alpha$, then we have $zac = zbac = zbc$ since B_α is a rectangular band; if $c \in B_\theta$, then we have $zac = c = zbc$ since B_θ is a right zero semigroup. Therefore, $\rho_1 \in \mathcal{C}(B)$. Note that B is a subdirectly irreducible band. For any $v \in B_\theta$ such that $(v, z) \notin \ker \varphi_\theta^a$, we must have $va = vb$. Now for any $a', b' \in B_\alpha$ with $a \,\mathcal{L}\, a'$, $b \,\mathcal{L}\, b'$ and $a' \,\mathcal{R}\, b'$, we have $\operatorname{im} \varphi_\theta^a = \operatorname{im} \varphi_\theta^{a'}$, $\operatorname{im} \varphi_\theta^b = \operatorname{im} \varphi_\theta^{b'}$ and $\ker \varphi_\theta^{a'} = \ker \varphi_\theta^{b'}$. Hence we obtain that $(za, zb) \in \ker \varphi_\theta^{a'}$ which means that ρ_α is nontrivial. Again it follows from condition (1) of the definition of a fundamental semilattice of semigroups and Lemma 2.8 that there exist distinct $x, y \in B_\theta$ such that $\bigcap_{\alpha \in Y} \rho_\alpha = \omega_{\{x,y\}} \cup \varepsilon_B$. ■

Corollary 2.11. *There exist distinct $x, y \in B_\theta$ such that $\omega_{\{x,y\}} \cup \varepsilon_B$ is the least nontrivial congruence on B.*

Proof. This directly follows Lemmas 2.6, 2.8, 2.9 and 2.10. ■

In the following lemmas, x and y will be used to represent the least nontrivial congruence on B. Noticing Lemma 2.2 and the definition of σ_α for $\alpha \in Y$, we have

Corollary 2.12. *For any $\alpha \in Y - \{\theta\}$, $(x, y) \in \bigcap_{\alpha \in Y - \{\theta\}} \sigma_\alpha$.*

Lemma 2.13. *For any $\alpha \in Y - \{\theta\}$ and $a \in B_\alpha$,*

$$\xi = \Big(\bigcup_{c \in R_a} ((\operatorname{im} \varphi_\theta^c \setminus x \ker \varphi_\theta^c) \times (\operatorname{im} \varphi_\theta^c \setminus x \ker \varphi_\theta^c)) \cup \varepsilon_B\Big)^\infty \in \mathcal{C}(B).$$

Proof. It follows from Lemma 1.1 that we only need to show that the symmetric relation $\bigcup_{c \in R_a} ((\operatorname{im} \varphi_\theta^c \setminus x \ker \varphi_\theta^c) \times (\operatorname{im} \varphi_\theta^c \setminus x \ker \varphi_\theta^c))$ is right compatible. Arbitrarily take $c \in R_a$ and $u, v \in \operatorname{im} \varphi_\theta^c \setminus x \ker \varphi_\theta^c$. Then we see from Lemma 1.3(4) that $uc = u$ and $vc = v$. For any $b \in B$, if $b \in B_\alpha$, then we have $ub = ucb$ and $vb = vcb$. Note that $cb \,\mathcal{R}\, c$. Then we obtain that $ub, vb \in \operatorname{im} \varphi_\theta^{cb} \setminus x \ker \varphi_\theta^{cb}$ with $cb \,\mathcal{R}\, a$. If $b \notin B_\alpha$, then we have $ub = ucb = cb = vcb = vb$. Hence we obtain that $\bigcup_{c \in R_a} ((\operatorname{im} \varphi_\theta^c \setminus x \ker \varphi_\theta^c) \times (\operatorname{im} \varphi_\theta^c \setminus x \ker \varphi_\theta^c))$ is right compatible. ■

Lemma 2.14. *For any $\alpha \in Y - \{\theta\}$ and $a \in B_\alpha$, $|\operatorname{im} \varphi_\theta^a| = 2$.*

Proof. If there exists $\alpha \in Y - \{\theta\}$ such that B_α is a left zero semigroup, then we see from Lemma 2.9 that $Y = \{\alpha, \theta\}$ and for any $a \in B_\alpha$, $|\operatorname{im} \varphi_\theta^a| = 2$. If for any $\alpha \in Y - \{\theta\}$ such that B_α is not a left zero semigroup, then we see from Lemma 2.10 that $\bigcap_{b \in B} \ker \varphi_\theta^a \cup \varepsilon_B = \omega_{\{x,y\}} \cup \varepsilon_B$, the least nontrivial congruence on B. Note from Lemma 1.4 that for any $s, t \in B$ with $s \,\mathcal{D}\, t$, we always have $|\operatorname{im} \varphi_\theta^s| = |\operatorname{im} \varphi_\theta^t|$. Suppose that there exists some $\beta \in Y - \{\theta\}$ and $d \in B_\beta$ such that $|\operatorname{im}\varphi_\theta^d| \geq 3$. Then the following congruence

$$\xi = (\bigcup_{e\in R_d}((\operatorname{im}\varphi_\theta^d \setminus x\ker\varphi_\theta^d)\times(\operatorname{im}\varphi_\theta^d \setminus x\ker\varphi_\theta^d))\cup\varepsilon_B)^\infty$$

as constructed in Lemma 2.13 is nontrivial. However, $(\omega_{\{x,y\}}\cup\varepsilon_B)\cap\xi = \varepsilon_B$, contradicting the fact that B is subdirectly irreducible. ∎

Lemma 2.15. *For $\alpha \in Y - \{\theta\}$, if B_α is not a left zero semigroup, then there exist $a, b \in B_\alpha$ and $z \in B_\theta$ such that $\operatorname{im}\varphi_\theta^a = \{x, z\}$ and $\operatorname{im}\varphi_\theta^b = \{y, z\}$.*

Proof. It follows from Lemmas 2.9, 2.10 and Corollary 2.12 that $(x, y) \in \ker\varphi_\theta^c$ for any $c \in B_\alpha$ and $(x, y) \in \sigma_\alpha$. Noticing Lemmas 2.3 and 2.4, we see that there exist $a, b \in B_\alpha$ with $a\,\mathcal{R}\,b$ such that $x = x\varphi_\theta^a$ and $y = y\varphi_\theta^b$. Suppose $u \notin x\ker\varphi_\theta^a$. We claim that $ua = u\varphi_\theta^a = u\varphi_\theta^b = ub$. Otherwise, $\omega_{\{ua,ub\}}\cup\varepsilon_B \in \mathcal{C}(B)$. In fact, for any $e \in B$, if $c \in B_\alpha$, then we have $uac = ubac = ubc$ since B_α is a rectangular band. If $c \notin B_\alpha$, then we see from the multiplication in a fundamental semilattice of semigroups that $uac = ac = \langle\varphi_\theta^a\varphi_\theta^c\rangle = x\varphi_\theta^c$ since $\operatorname{im}\varphi_\theta^a \subseteq x\ker\varphi_\theta^c$; similarly, $ubc = bc = \langle\varphi_\theta^b\varphi_\theta^c\rangle = y\varphi_\theta^c = x\varphi_\theta^c$. However, $(\omega_{\{ua,ub\}}\cup\varepsilon_B)\cap(\omega_{\{x,y\}}\cup\varepsilon_B) = \varepsilon_B$, contradicting the fact that B is subdirectly irreducible band. Hence, we obtain from Lemma 2.14 that there exists $z \in B_\theta$ such that $\operatorname{im}\varphi_\theta^a = \{x, z\}$ and $\operatorname{im}\varphi_\theta^b = \{y, z\}$. ∎

To end the paper, we give our main results. In the next two proofs, for any $s, t \in B$, $\Theta(s, t)$ represents the congruence generated by $\{(s, t)\}$.

Theorem 2.16. *If there exists $\alpha \in Y - \{\theta\}$ such that B_α is a left zero semigroup, then B is subdirectly irreducible if and only if the following conditions are satisfied:*

(1) *$Y = Y_2$ and $\Phi_{\alpha,\theta}$ is injective;*

(2) *$\bigcap_{a\in B_\alpha}\ker\varphi_\theta^a = \varepsilon_{B_\theta}$;*

(3) *there exist distinct $x, y \in B_\theta$ such that $\operatorname{im}\varphi_\theta^a = \{x, y\}$ for all $a \in B_\alpha$.*

Proof. The necessity part follows from Lemmas 2.1, 2.9 and the definition of ρ_α.

To prove sufficiency part, arbitrarily take distinct $u, v \in B$ with $\{u, v\} \neq \{x, y\}$. It suffices to prove $(x, y) \in \Theta(u, v)$. We complete the proof via discussing the following cases:

Case 1. $u, v \in B_\theta$. According to condition (2), there exists $a \in B_\alpha$ such that $(u, v) \notin \ker\varphi_\theta^a$. Note condition (3) and suppose $u \in x\ker\varphi_\theta^a, v \in y\ker\varphi_\theta^a$. We obtain from Lemma 1.3(4) that $ua = x$ and $va = y$ which means that $(x, y) \in \Theta(u, v)$.

Case 2. $u \in B_\theta, v \in B_\alpha$. Note from condition (3) that $\ker\varphi_\theta^v \neq \omega_{B_\theta}$ and take $z \notin u\ker\varphi_\theta^v$, we have $zu = u$ and $zv \notin u\ker\varphi_\theta^v$ so that $zv \neq u$. This case is reduced to Case 1.

Case 3. $u, v \in B_\alpha$. We see from condition (1) and Lemma 1.3(1) that there exists $z \in B_\theta$ such that $zu \neq zv$. This case is also reduced to Case 1. ∎

Theorem 2.17. *If for any $\alpha \in Y - \{\theta\}$, B_α is not a left zero semigroup, then B is subdirectly irreducible if and only if the following conditions are satisfied:*

(1) *for all $\alpha \in Y - \{\theta\}$, $\Phi_{\alpha,\theta}$ is injective;*

(2) *there exist distinct $x, y \in B_\theta$ such that $\bigcap_{a \in B} \ker \varphi_\theta^a = \omega_{\{x,y\}} \cup \varepsilon_{B_\theta}$;*

(3) *for all $\alpha \in Y - \{\theta\}$, there exist $a, b \in B_\alpha$ and $z \in B_\theta$ such that $\operatorname{im} \varphi_\theta^a = \{x, z\}$ and $\operatorname{im} \varphi_\theta^b = \{y, z\}$.*

Proof. The necessity part follows from Lemmas 2.1, 2.10, 2.15 and the definition of ρ_α for $\alpha \in Y$.

We now prove the sufficiency part. Arbitrarily take distinct $u, v \in B_\theta$ with $\{u, v\} \neq \{x, y\}$. We see from condition (2) that it suffices to prove $(x, y) \in \Theta(u, v)$. We complete the proof via discussing the following cases:

Case 1. $u, v \in B_\theta$. We see from condition (2) that there exist $\beta \in Y - \{\theta\}$ and $c \in B_\beta$ such that $u \in x \ker \varphi_\theta^c$ and $v \notin x \ker \varphi_\theta^c$. According to condition (3), Lemmas 2.3 and 2.4, we see that there exist $d, e \in B_\beta$ with $c \,\mathcal{R}\, d \,\mathscr{R}\, e$ such that $ud = x, ue = y, vd = ve = z$. It follows that $(x, y) \in \Theta(u, v)$.

Case 2. $u \in B_\theta, v \in B_\beta$ for some $\beta \in Y - \{\theta\}$. Note from condition (3) that $\ker \varphi_\theta^v \neq \omega_{B_\theta}$ and take $z \notin u \ker \varphi_\theta^v$, we have $zu = u$ and $zv \notin u \ker \varphi_\theta^v$ so that $zv \neq u$. This case is reduced to Case 1.

Case 3. $u, v \notin B_\theta$. If $(u, v) \notin \mathcal{D}$, then we see from condition (2) of the definition of a fundamental semilattice of semigroups, $\operatorname{im} \varphi_\theta^u \subseteq \ker \varphi_\theta^v$ which means that $\varphi_\theta^u \neq \varphi_\theta^v$; if $(u, v) \in \mathcal{D}$, then we see from condition (1) that $\varphi_\theta^u \neq \varphi_\theta^v$ either. So there exists $w \in B_\theta$ such that $w\varphi_\theta^u \neq w\varphi_\theta^v$ and hence $wu \neq wv$. This case is also reduced to Case 1. ∎

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