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Supplemented Results of Finite Maximal Codes Related to the Triangle Conjecture Proposed by D. Perrin and M.P. Schützenbeger

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Abstract. In the theory of codes, the well-known triangle conjecture was first proposed by D. Perrin and M.P. Schtzenberger in 1981. Building on the recent work [21], we further study finite maximal codes and present new findings related to triangle conjecture. This paper also provides a more concise set of results from [21] as well as amendments and corrections.

Keywords: Finite maximal codes; Triangle conjecture; Factorization of $\mathbb{Z}_n$; System of factorizations of $\mathbb{Z}_n$.

1. Introduction

Let A^* be a *free monoid* generated by an alphabet A. Then, we call an element $w \in A^*$ a *word* over A and we denote the empty word by 1. Accordingly, a subset of A^* is called a *language* over A. The notation A^+ is $A^* - \{1\}$. Now, we call the language $X \subseteq A^+$ a *code* if it is a basis of a free submonoid of A^*.

Among the codes, the maximal codes have been extensively studied in the literature by many authors because the maximal codes have many interesting properties, for example, see [2], [4]–[6], [9]–[20]. In fact, the codes that are maximal play a crucial role in the theory of codes. Some of the well known conjectures of codes such as the triangle conjecture, the commutatively prefix conjecture and the conjecture of factorizing codes are all false on general codes. Currently, no counter example has been found in finite maximal codes [1, 12]. Thus, the structure of finite maximal codes is an interesting phenomenon. It is well established that the structure of finite maximal codes are closely related to the factorization problem of the additive group $\mathbb{Z}_n$ and the problem of completion of finite codes. If there exists a pair (P, Q) of subsets of $\mathbb{N}$ such that for every $i = 0, 1, \cdots, n-1$, there exists one and only one pair $(p, q) \in P \times Q$ satisfying $p + q = i \bmod n$, then we call the pair (P, Q) a *factorization* of $\mathbb{Z}_n$ (see [7, 8]). By using the factorization of $\mathbb{Z}_n$, we are still able to construct some codes which are finitely incompletable (see [9, 10, 11, 14]).

In [21], we gave a property of the structure of finite maximal codes and by using this property, some partial results can be obtained based on the triangle conjecture for the finitely completable codes. In this paper, we will revise our previous results given in [21] so that the statements of our revised results are more concise than our previous results. Some corrections and supplemented results are also given in this paper.

For those notations and terminology not given in this paper, the reader is referred to the well known monographs of A. Salomaa [16], J. Berstel, D. Perrin and C. Reutenauer [1].

2. On the Structure of Finite Maximal Codes

Throughout this section, we let $\mathbb{P} \subseteq \mathfrak{P}(\mathbb{N})$ and $\mathbb{Q} \subseteq \mathfrak{P}(\mathbb{N})$, where $\mathfrak{P}(\mathbb{N})$ is the power set of $\mathbb{N}$. If for any $P \in \mathbb{P}$ and $Q \in \mathbb{Q}$, (P, Q) are the factorizations of $\mathbb{Z}_n$, then $(\mathbb{P}, \mathbb{Q})$ is said to be a *system of factorizations of* $\mathbb{Z}_n$. If Γ, Λ are the sets of integers, then we write

$$\begin{aligned} a^\Gamma &= \{a^i \mid i \in \Gamma\}, \\ \Gamma + \Lambda &= \{i + j \mid i \in \Gamma, j \in \Lambda\}. \end{aligned}$$

If $\Gamma = \{k\}$, i.e., Γ is a singleton, then for the sake of simplicity, we use $k + \Lambda$ instead of $\{k\} + \Lambda$. For an integer k, we use the notation $\overline{k}$ to denote the smallest nonnegative integer contained in the congruence class of k modulo n, and for a set Γ of integers, we write

$$\overline{\Gamma} = \{\overline{\gamma} | \gamma \in \Gamma\}.$$

In the following, we always denote $T = \{0, \cdots, n-1\}$ and suppose that $P \subseteq T$ and $Q \subseteq T$ for any factorization (P, Q) of $\mathbb{Z}_n$. Clearly, if (P, Q) is a factorization of $\mathbb{Z}_n$, then $(\overline{P - k}, \overline{Q - \ell})$ are also factorizations of $\mathbb{Z}_n$ for all $k, \ell \in \mathbb{N}$. Hence, we call the system $(\mathbb{P}, \mathbb{Q})$ of factorizations of $\mathbb{Z}_n$ perfect if $\{\overline{P-k} | k \in \mathbb{N}, P \in \mathbb{P}\} \subseteq \mathbb{P}$ and $\{\overline{Q-\ell} | \ell \in \mathbb{N}, Q \in \mathbb{Q}\} \subseteq \mathbb{Q}$. If $(\mathbb{P}, \mathbb{Q})$ is a perfect system of factorizations of $\mathbb{Z}_n$, then for any $k \in T$, there exists $P \in \mathbb{P}$ ($Q \in \mathbb{Q}$) such that $k \in P$ ($k \in Q$).

Let $X \subseteq A^*$. Then, we call a word $y \in A^*$ a *right completable word* for X if $yw \in X^*$ for some w in A^*. A word $y \in A^*$ is said to be a *strongly right completable word* for X if $ysA^* \cap X^* \neq \emptyset$ for any $s \in A^*$. One can observe that the set of strongly right completable words is a right ideal of A^* [1]. It is known that every finite maximal code $X \subseteq A^+$ has some strongly right completable words [16], and there exists $n \in \mathbb{N}$ (related to $a \in A$) such that $a^n \in X$. We call this number n the *order* of a (related to X) (see [1, Prop. 2.5.15]).

In this paper, we always assume that the alphabet A is not a singleton and $a \in A$. Let $B = A - \{a\}$ and $w \in B(a^*B)^*$. Moreover, let $(\mathbb{P}, \mathbb{Q})$ be a system of factorizations of $\mathbb{Z}_n$. If for any assigned $P_0 = \{p_1, \cdots, p_s\} \in \mathbb{P}$, $Q_0 = \{q_1, \cdots, q_t\} \in \mathbb{Q}$, the set of n words $X_w = \{w_1, \cdots, w_n\} \subseteq a^*wa^*$ can be arranged in the following matrix form related to (P_0, Q_0) (Here P_0 (Q_0) can be regarded as a $(1 \times s)$ ($(1 \times t)$) matrix, i.e., every element of P_0 (Q_0) has a fixed location),

$$X_w^{(P_0,Q_0)} = \left\{ \begin{array}{llll} a^{i_{11}}wa^{j_{11}}, & a^{i_{12}}wa^{j_{12}}, & \cdots & a^{i_{1t}}wa^{j_{1t}}, \\ a^{i_{21}}wa^{j_{21}}, & a^{i_{22}}wa^{j_{22}}, & \cdots & a^{i_{2t}}wa^{j_{2t}}, \\ \cdots & \cdots & \cdots & \cdots \\ a^{i_{s1}}wa^{j_{s1}}, & a^{i_{s2}}wa^{j_{s2}}, & \cdots & a^{i_{st}}wa^{j_{st}}, \end{array} \right\}, \tag{1}$$

which satisfies the following conditions:

(i) For $k = 1, \cdots, s$ and $m = 1, \cdots, t$.

$$\overline{q_m - i_{km}} \in P_0, \quad \overline{p_k - j_{km}} \in Q_0. \tag{2}$$

(ii) For $Q_k = \{i_{k1}, \cdots, i_{kt}\}$ and $P_m = \{j_{1m}, \cdots, j_{sm}\}$, where $k = 1, \cdots, s$ and $m = 1, \cdots, t$,

$$\{P_1, \cdots, P_t\} \subseteq \mathbb{P}, \quad \{Q_1, \cdots, Q_s\} \subseteq \mathbb{Q}. \tag{3}$$

Then we call X_w *induced by* $(\mathbb{P}, \mathbb{Q})$ and call *the matrix* $X_w^{(P_0,Q_0)}$ *induced by* $(\mathbb{P}, \mathbb{Q})$ *related to* (P_0, Q_0). In the following, we always regard $P \in \mathbb{P}$ and $Q \in \mathbb{Q}$ as $(1 \times s)$ and $(1 \times t)$ matrices respectively when the matrix induced by $(\mathbb{P}, \mathbb{Q})$ related to (P, Q) is concerned.

Remark 2.1. We remind that the conditions (2) are the equivalent form of conditions (5) and (6) in [21]. Moreover, we claim that the conditions (7) and (8) in [21], i.e.,

$$\begin{aligned} \overline{(\{i_{1m}, \cdots, i_{sm}\} + P_0)} \cap Q_0 &= \{q_m\}, \quad m = 1, \cdots, t; \\ \overline{(\{j_{k1}, \cdots, j_{kt}\} + Q_0)} \cap P_0 &= \{p_k\}, \quad k = 1, \cdots, s. \end{aligned}$$

are redundant. In fact, they are the deductions of conditions (2). Assume that $\overline{q_m - i_{km}} \in P_0$. Then, there is $p \in P_0$ such that $q_m - i_{km} = p + \alpha$, where $\alpha = 0$

mod n. Clearly, we may write

$$i_{km} + p = q_m \bmod n. \tag{4}$$

By (P_0, Q_0) being a factorization of $\mathbb{Z}_n$, we can easily see that $\overline{i_{km} + P_0} \cap Q_0 = \{q_m\}$. The situation of the second equation is similar.

Proposition 2.2. *Let $(\mathbb{P}, \mathbb{Q})$ be a system of factorizations of $\mathbb{Z}_n$, and $P_0 = \{p_1, \cdots, p_s\} \in \mathbb{P}$, $Q_0 = \{q_1, \cdots, q_t\} \in \mathbb{Q}$. If $X_w \subseteq a^*wa^*$ is induced by $(\mathbb{P}, \mathbb{Q})$ and $X_w^{(P_0,Q_0)}$ is the matrix induced by $(\mathbb{P}, \mathbb{Q})$ related to (P_0, Q_0), where $w \in B(a^*B)^*$, $B = A - \{a\}$, then the following statements hold:*

(i) *If a^iwa^j is an element of X_w which satisfies $\overline{q_m - i} \in P_0$, then a^iwa^j must be located in the column m of the matrix $X_w^{(P_0,Q_0)}$.*

(ii) *If a^iwa^j is an element of X_w which satisfies $\overline{p_k - j} \in Q_0$, then a^iwa^j must be located in row k of the matrix $X_w^{(P_0,Q_0)}$.*

(iii) *The matrix $X_w^{(P_0,Q_0)}$ is unique.*

Proof. (i) Let a^iwa^j be in X_w. Then, it suffices to prove the following fact. If $\overline{q_m - i} \in P_0$ and $\overline{q_k - i} \in P_0$, then $q_k = q_m$. We denote $q_m - i = p + \alpha$ and $q_k - i = p' + \beta$ for some $p, p' \in P_0$ and $\alpha, \beta = 0 \bmod n$. Thus, we have $q_m - q_k = p - p' + \alpha - \beta$, i.e., $q_m + p' = q_k + p + (\alpha - \beta)$. Since (P_0, Q_0) is a factorization of $\mathbb{Z}_n$, we see that $q_m = q_k$.

(ii) The proof of this part is similar to (i).

(iii) Let $X_w^{(P_0,Q_0)}$ be a matrix induced by $(\mathbb{P}, \mathbb{Q})$ related to (P_0, Q_0). We denote this matrix as (1). Then, the element (k, m) in the matrix $X_w^{(P_0,Q_0)}$ need satisfy $\overline{q_m - i_{km}} \in P_0$ and $\overline{p_k - j_{km}} \in Q_0$. By (i) and (ii), except $a^{i_{km}}wa^{j_{km}}$ itself, no word in X_w satisfies this condition. Therefore, the construction of the matrix $X_w^{(P_0,Q_0)}$ from X_w is unique. ■

Proposition 2.3. *Let $(\mathbb{P}, \mathbb{Q})$ be a perfect system of factorizations of $\mathbb{Z}_n$ and $X_w \subseteq a^*wa^*$ induced by $(\mathbb{P}, \mathbb{Q})$, where $w \in B(a^*B)^*$, $B = A - \{a\}$. Then the following statements hold:*

(i) *If $a^iwa^j, a^{i'}wa^{j'} \in X_w$ which satisfies $|\overline{i} - \overline{i'}| \in \{|k - \ell| | k, \ell \in P, P \in \mathbb{P}\}$, then $|\overline{j} - \overline{j'}| \in \{|k - \ell| | k, l \in P, P \in \mathbb{P}\}$.*

(ii) *If $a^iwa^j, a^{i'}wa^{j'} \in X_w$ which satisfies $|\overline{j} - \overline{j'}| \in \{|k - \ell| | k, \ell \in Q, Q \in \mathbb{Q}\}$, then we also have $|\overline{i} - \overline{i'}| \in \{|k - \ell| | k, \ell \in Q, Q \in \mathbb{Q}\}$.*

Proof. Let $|\overline{i} - \overline{i'}| = |k - \ell|$, where $k, \ell \in P_0$ with $P_0 \in \mathbb{P}$. Then, we may assume that $\overline{i} \geq \overline{i'}$ and $k \geq \ell$. For the other case, the argument is similar. Thus, we have $\overline{i} - \overline{i'} = k - \ell$. Let us denote $\overline{i} + \ell = \overline{i'} + k = q \bmod n$ (where $q \in T$). Thus, $\overline{q - i_1}, \overline{q - i_2} \in P_0$. Since $(\mathbb{P}, \mathbb{Q})$ is perfect, there exists an element Q_0 of $\mathbb{Q}$ such that $q \in Q_0$. Furthermore, we can arrange X_w as a matrix $X_w^{(P_0,Q_0)}$ which induced by $(\mathbb{P}, \mathbb{Q})$ related to (P_0, Q_0). By using Proposition 2.2 (i), it can be seen that $a^{i_1}wa^{j_1}, a^{i_2}wa^{j_2}$ are located in the same column in the matrix $X_w^{(P_0,Q_0)}$. Thus, the equation (3) of the definition of the matrix $X_w^{(P_0,Q_0)}$ leads to $|\overline{j_1} - \overline{j_2}| \in \{|k-\ell| | k, \ell \in P, P \in \mathbb{P}\}$ and hence (i) is proved. The proof of (ii) is similar and is hence omitted. ∎

Clearly, $\{|k-\ell| | k, \ell \in P, P \in \mathbb{P}\} \cap \{|k-\ell| | k, \ell \in Q, Q \in \mathbb{Q}\} = \{0\}$ because (P,Q) is a factorization of $\mathbb{Z}_n$ for any $P \in \mathbb{P}$ and any $Q \in \mathbb{Q}$. In order to prove the next Theorem, we recall the known concept of *Bernoulli distribution on* A^*. It is a morphism π from A^* into the multiplicative monoid $\mathbb{R}_+$ of some nonnegative real numbers, which satisfies $\Sigma_{a\in A}\pi(a) = 1$ (see [1, pp. 40]). If π is a Bernoulli distribution on A^*, then, it is known that $\pi(X) \leq 1$ for any code $X \subseteq A^+$.

We now prove the following crucial theorem which is closely related to finite maximal codes.

Theorem 2.4. *Let* $(\mathbb{P},\mathbb{Q})$ *be a perfect system of factorizations of* $\mathbb{Z}_n$ *and* $X_w \subseteq a^*wa^*$ *induced by* $(\mathbb{P},\mathbb{Q})$, *where* $w \in B(a^*B)^*$, $B = A - \{a\}$. *Then,* $\{a^n\} \cup X_w$ *is a maximal code in* $a^* \cup a^*wa^*$.

Proof. We first prove that $\{a^n\} \cup X_w$ is a code. If $\{a^n\} \cup X_w$ is not a code, then there exists a word which has different decompositions in $\{a^n\} \cup X_w$, say

$$\begin{aligned}&(a^{\alpha_1})(a^{i_1}wa^{j_1})(a^{\alpha_2})(a^{i_2}wa^{j_2})\cdots(a^{\alpha_k})(a^{i_k}wa^{j_k})(a^{\alpha_{k+1}})\\ =\ &(a^{\alpha'_1})(a^{i'_1}wa^{j'_1})(a^{\alpha'_2})(a^{i'_2}wa^{j'_2})\cdots(a^{\alpha'_k})(a^{i'_k}wa^{j'_k})(a^{\alpha'_{k+1}}),\end{aligned}$$

where $\alpha_m = 0 \bmod n$ and $\alpha'_m = 0 \bmod n$, $m = 1, 2, \cdots, k+1$. We may assume that this word is the shortest which has different decompositions. Thus, we write

$$i_1 = i'_1 \bmod n, \quad j_k = j'_k \bmod n.$$

and

$$\overline{j_1} - \overline{j'_1} = \overline{i'_2} - \overline{i_2} \neq 0, \cdots, \overline{j_{k-1}} - \overline{j'_{k-1}} = \overline{i'_k} - \overline{i_k} \neq 0.$$

Clearly, we have $|\overline{i'_1} - \overline{i_1}| = 0 \in \{|k-\ell| | k, \ell \in P, P \in \mathbb{P}\}$, by applying Proposition 2.3 (i), we know that $|\overline{j_1} - \overline{j'_1}| \in \{|k-\ell| | k, \ell \in P, P \in \mathbb{P}\}$. Since $\overline{i'_2} - \overline{i_2} = \overline{j_1} - \overline{j'_1}$, again by Proposition 2.3 (i), we have $|\overline{j_2} - \overline{j'_2}| \in \{|k-\ell| | k, \ell \in P, P \in \mathbb{P}\}$. Proceeding in this way, we have $= |\overline{i'_k} - \overline{i_k}| = |\overline{j_{k-1}} - \overline{j'_{k-1}}| \in \{|k-\ell| | k, \ell \in P, P \in \mathbb{P}\}$. On the other hand, by using Proposition 2.3 (ii), we see that $|\overline{j_k} - \overline{j'_k}| = 0 \in \{|k-\ell| | k, \ell \in Q, Q \in \mathbb{Q}\}$ leads to $|\overline{i'_k} - \overline{i_k}| \in \{|k-\ell| | k, \ell \in Q, Q \in \mathbb{Q}\}$. Thus, we arrive at a contradiction because (P,Q) is a factorization of $\mathbb{Z}_n$ and $\overline{i'_k} - \overline{i_k} \neq 0$. Therefore, we have proved that $\{a^n\} \cup X_w$ is a code.

Now, we proceed to prove that $Y = \{a^n\} \cup X_w$ is a maximal code in $a^* \cup a^*wa^*$. Assume that there is $x \notin Y$ such that $\widetilde{Y} = \{x\} \cup Y$ is again a code. Then, it is clear that $x \notin a^*$ and hence $x \in a^*wa^*$. In fact, we may regard $\widetilde{Y}$ as a code over the alphabet $\widetilde{A} = \{a, w\}$. Let π be a Bernoulli distribution on $(\widetilde{A})^*$ and $\pi(a) = \theta$. Then, we have the following situation:

$$\begin{aligned}\pi(\widetilde{Y}) - 1 &= \pi(w)[(\pi(a))^{\alpha_1} + \cdots + (\pi(a))^{\alpha_n} + (\pi(a))^{\alpha_{n+1}}] - [1 - (\pi(a)^n]\\ &= (1-\theta)[(\theta^{\alpha_1} + \cdots + \theta^{\alpha_n} + \theta^{\alpha_{n+1}}) - (1 + \theta + \theta^2 + \cdots + \theta^{n-1})],\end{aligned}$$

where $\alpha_1, \cdots, \alpha_n$ is the number of appearance of a in the word belonging to X_w, respectively (Notice that X_w contains n words), and α_{n+1} is the number of appearance of a in the word x. By applying the following result

$$\lim_{\theta \to 1} [(\theta^{\alpha_1} + \cdots + \theta^{\alpha_n} + \theta^{\alpha_{n+1}}) - (1 + \theta + \theta^2 + \cdots + \theta^{n-1})] = 1,$$

there exists π' such that $\pi'(\widetilde{Y}) > 1$. Clearly, this is a contradiction since $\pi'(X) \leq 1$ for every code $X \subseteq (\widetilde{A})^+$. Therefore, we now know that $\widetilde{Y}$ is not a code and hence we have shown that $Y = \{a^n\} \cup X_w$ is a maximal code in $a^* \cup a^* w a^*$. ■

Let $X \subseteq A^+$ be a finite maximal code, and n the order of $a \in A$. If $x \in A^*$ be such that the set

$$Q = \{k \in T | a^{k+2n|X|} x A^* \cap X^* \neq \emptyset\} \tag{5}$$

satisfies the following condition

$$a^{j-i} \notin (X^*)^{-1} X^*, \text{ for } i, j \in Q \text{ with } i < j. \tag{6}$$

Then we say that a^Q is a *right set* of X (related to $a \in A$) and x is its generator. Clearly, if $i \in \mathbb{N}$ and $Q' = \overline{Q - i}$, then $a^{Q'}$ is also a right set. The word $a^i x$ is the generator of $a^{Q'}$. Let $y \in A^*$ be a strongly right completable word of X and let $P = \{k \in T | y a^{2n|X|+k} \in X^*\}$.

Then, we say that a^P is a *left set* of X (related to $a \in A$) and y is its generator. Similarly, if $j \in \mathbb{N}$ and $P' = \overline{P - j}$ then $a^{P'}$ is also a left set. The word $y a^j$ is clearly the generator of $a^{P'}$.

Remark 2.5. The following is a statement given in [21]:

If $i \in Q$ and $Q' = \overline{Q - i}$, then $a^{Q'}$ is also a right set, and if $j \in P$ and $P' = \overline{P - j}$, then $a^{P'}$ is also a left set. In fact, the restrictions $i \in Q$ and $j \in P$ are unnecessary.

Proposition 2.6. [21, Props. 2.1 and 2.2] *Let $X \subseteq A^+$ be a finite maximal code and n the order of $a \in A$. Then there exist some right sets of X related to a.*

For $w \in A^*$, we denote the number of the letter a appearing in the word w by $|w|_a$, and in general, for $B \subseteq A$, we write $|w|_B = \sum_{b \in B} |w|_b$, specially, we write $|w|_A = |w|$, and it is called the *length* of w. For $X \subseteq A^*$, we denote the largest length of the words in X by $|X|$.

Proposition 2.7. [21, Prop. 2.6] *Let $X \subseteq A^+$ be a finite maximal code and n the order of $a \in A$. If y is a strongly right completable word of X and a^Q is a right set of X (related to a), then for any $u \in A^*$, there exists a unique $j \in Q$ such that $yua^{2n|X|-j} \in X^*$.*

Proposition 2.8. *Let $X \subseteq A^+$ be a finite maximal code and n the order of $a \in A$, a^P (a^Q) a left (right) set of X related to a. Then for any $u \in A^*$, there exists an unique pair $(p, q) \in P \times Q$ such that $a^{2n|X|-p}ua^{2n|X|-q} \in X^*$.*

Proof. See Proposition 2.8 of [21]. ■

Theorem 2.9. *Let $X \subseteq A^+$ be a finite maximal code and n the order of $a \in A$, a^P (a^Q) a left (right) set of X related to a. Then, (P, Q) is a factorization of $\mathbb{Z}_n$.*

Proof. See Theorem 2.7 of [21]. ■

Corollary 2.10. *Let $X \subseteq A^+$ be a finite maximal code and n the order of $a \in A$. Let*

$$\begin{aligned}\mathbb{P}_X &= \{P \subseteq \mathbb{N} | a^P \textit{ is a left set of } X\},\\ \mathbb{Q}_X &= \{Q \subseteq \mathbb{N} | a^Q \textit{ is a right set of } X\}.\end{aligned}$$

Then $(\mathbb{P}_X, \mathbb{Q}_X)$ is a perfect system of factorizations of $\mathbb{Z}_n$.

Proof. By Theorem 2.9, we know that $(\mathbb{P}_X, \mathbb{Q}_X)$ is a system of factorizations of $\mathbb{Z}_n$. Also, by the definitions of the left set and the right set, we easily see that the system $(\mathbb{P}_X, \mathbb{Q}_X)$ is perfect. ■

Now, we call the $(\mathbb{P}_X, \mathbb{Q}_X)$ in Corollary 2.10 *the system of factorizations of $\mathbb{Z}_n$ induced by X related to a*. In the following, we always suppose that $X \subseteq A^+$ with $a^n \in X$, and for $w \in B(a^*B)^*$, denote

$$X_w = X'_w - (a^n X'_w \cup X'_w a^n), \tag{7}$$

where

$$X'_w = (a^*wa^* \cap X^*)\,.$$

Clearly, if $a^iwa^j \in X_w$, then for $i' = i \bmod n, j' = j \bmod n$ with $(i, j) \neq (i', j')$, $a^{i'}wa^{j'} \notin X_w$.

For finite maximal codes, we have the following theorem.

Theorem 2.11. *Let $X \subseteq A^+$ be a finite maximal code, n the order of $a \in A$, and $(\mathbb{P}_X, \mathbb{Q}_X)$ the system of factorizations of $\mathbb{Z}_n$ induced by X related to a. Then, for $B = A - \{a\}$ and $w \in B(a^*B)^*$, X_w is induced by $(\mathbb{P}_X, \mathbb{Q}_X)$.*

Proof. (i) Let $w \in B(a^*B)^*$ and $P_0 = \{p_1, \cdots, p_s\} \in \mathbb{P}$, $Q_0 = \{q_1, \cdots, q_t\} \in \mathbb{Q}$. Then, we fix $(p_k, q_m) \in P_0 \times Q_0$ and observe that $u = a^{q_m} w a^{p_k}$. By applying Proposition 2.8, there exists an unique pair $(p, q) \in P_0 \times Q_0$ such that

$$a^{2n|X|-p}(a^{q_m} w a^{p_k})a^{2n|X|-q} \in X^*. \tag{8}$$

Hence, there is a function $\psi : P_0 \times Q_0 \to P_0 \times Q_0$, $(p_k, q_m) \mapsto (p, q)$. It is convenient to denote this (p, q) as (p_{km}, q_{km}). The word in (8) can only be factorized in X as follows:

$$(a^{\alpha_1})(a^{\alpha_2+q_m-p_{km}} w a^{\beta_2+p_k-q_{km}})(a^{\beta_1}),$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 = 0 \bmod n$ and $a^{\alpha_2+q_m-p_{km}} w a^{\beta_2+p_k-q_{km}} \in X_w$. We write

$$\alpha_2 + q_m - p_{km} = i_{km}, \quad \beta_2 + p_k - q_{km} = j_{km}, \tag{9}$$

and define a function $\phi : P_0 \times Q_0 \to X_w$, $(p_k, q_m) \mapsto a^{i_{km}} w a^{j_{km}}$. We now show that ϕ is an injection. If there are $(p_k, q_m), (p_{k'}, q_{m'}) \in P_0 \times Q_0$, with $(p_k, q_m) \neq (p_{k'}, q_{m'})$, such that $\phi((p_k, q_m)) = \phi((p_{k'}, q_{m'}))$, then

$$\begin{aligned}\phi((p_k, q_m)) &= a^{\alpha_2+q_m-p_{km}} w a^{\beta_2+p_k-q_{km}} \\ &= a^{\alpha_2'+q_{m'}-p_{k'm'}} w a^{\beta_2'+p_{k'}-q_{k'm'}} = \phi((p_{k'}, q_{m'})).\end{aligned}$$

This result is clearly impossible since (P_0, Q_0) is a factorization of $\mathbb{Z}_n$. Hence, we have shown that ϕ is an injection. Clearly, $(n - P_0, Q_0)$ and $(P_0, n - Q_0)$ are also the factorizations of $\mathbb{Z}_n$. Thus, for any $a^i w a^j \in X_w$, there exist some $p, p' \in P_0$, $q, q' \in Q_0$ such that $(n - p') + q = i \bmod n$ and $p + (n - q') = j \bmod n$. Hence, we obtain the following equality.

$$a^{2n|X|-p'}(a^q w a^p)a^{2n|X|-q'} = a^\alpha(a^i w a^j)a^\beta,$$

where $\alpha, \beta = 0 \bmod n$. Now, by the definition of ϕ, we have $\phi(p, q) = a^i w a^j$. This shows that ϕ is a surjection and so that ϕ is a bijection. Let $k = 1, \cdots, s$ and $m = 1, \cdots, t$. Then all words $a^{i_{km}} w a^{j_{km}}$ constitute the matrix (1) and the equation (2) follow from (8).

(ii) Let $Q_k = \{i_{k1}, \cdots, i_{kt}\}$ and $P_m = \{j_{1m}, \cdots, j_{sm}\}$, where $k = 1, \cdots, s$ and $m = 1, \cdots, t$. Then, we prove that $a^{\overline{P_1}}$ is a left set and $a^{\overline{Q_1}}$ is a right set. For other $P \in \{P_1, \cdots, P_t\}$ and $Q \in \{Q_1, \cdots, Q_s\}$, the proofs are similar. Now, we let y be a strongly right completable word generating a^{P_0} and x the generator of a^{Q_0}. Then, we see immediately that $y_1 = y a^{q_1+2n|X|} w$ is also a strongly right completable word. Observing (2), there exists $p_{k1} \in P_0$, $k = 1, \cdots, s$, such that

$$\begin{pmatrix} i_{11} \\ i_{21} \\ \vdots \\ i_{s1} \end{pmatrix} + \begin{pmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{s1} \end{pmatrix} = \begin{pmatrix} q_1 \\ q_1 \\ \vdots \\ q_1 \end{pmatrix} \bmod\ n.$$

We have

$$y_1 a^{j_{k1}} = (ya^{q_1+2n|X|}w)a^{j_{k1}} = (ya^{p_{k1}})(a^{\alpha})(a^{i_{k1}}wa^{j_{k1}}) \in X^*, \quad k = 1, \cdots, s,$$

where $\alpha = 0 \bmod n$, $ya^{p_{k1}} \in X^*$, $a^{i_{k1}}wa^{j_{k1}} \in X_w$. Clearly, $y_1 a^{2n|X|+\overline{j_{k1}}} \in X^*$ and hence, we have

$$\overline{P_1} = \overline{\{j_{11}, \cdots, j_{s1}\}} \subseteq \{i \in T | y_1 a^{2n|X|+i} \in X^*\}.$$

We now claim that $\overline{j_{k1}} \neq \overline{j_{m1}}$ for $k \neq m$. If $\overline{j_{k1}} = \overline{j_{m1}}$, then $y_1 a^{\overline{j_{k1}}} = y_1 a^{\overline{j_{m1}}}$, and

$$(ya^{p_{k1}})(a^{\alpha})(a^{i_{k1}}wa^{j_{k1}})(a^{\beta}) = (ya^{p_{m1}})(a^{\alpha'})(a^{i_{m1}}wa^{j_{m1}})(a^{\beta'}),$$

where $\alpha, \alpha', \beta, \beta' = 0 \bmod n$. Since X is a code, we see that $p_{k1} = p_{m1}$ and $\overline{i_{k1}} = \overline{i_{m1}}$. Furthermore, we have $a^{\overline{i_{k1}}}wa^{\overline{j_{k1}}} = a^{\overline{i_{m1}}}wa^{\overline{j_{m1}}}$. This is a contradiction because $\phi : P_0 \times Q_0 \to X_w$ is a bijection and hence the words in X_w are different to each other (Noting that the definition of X_w, there are no different words $a^i wa^j, a^{i'}wa^{j'} \in X_w$ such that $i = i' \bmod n$ and $j = j' \bmod n$). Since $P_{y_1} = \{i \in T | y_1 a^{2n|X|+i} \in X^*\}$ contains s elements, we see that $P_1 = P_{y_1}$ and $a^{\overline{P_1}}$ is a left set.

Now, we consider the word $x_1 = wa^{p_1+4n|X|}x$. By using (2), there exists $q_{1m} \in Q_0$, $m = 1, \cdots, t$ such that

$$\begin{pmatrix} j_{11} \\ j_{12} \\ \vdots \\ j_{1t} \end{pmatrix} + \begin{pmatrix} q_{11} \\ q_{12} \\ \vdots \\ q_{1t} \end{pmatrix} = \begin{pmatrix} p_1 \\ p_1 \\ \vdots \\ p_1 \end{pmatrix} \bmod \ n.$$

We have

$$a^{2n|X|+\overline{i_{1k}}} x_1 = a^{2n|X|+\overline{i_{1k}}}(wa^{p_1+4n|X|}x) = (a^{\alpha})(a^{i_{1k}}wa^{j_{1k}})(a^{\beta})(a^{2n|X|+q_{1k}}x),$$

where $\alpha, \beta = 0 \bmod n$, $a^{i_{1k}}wa^{j_{1k}} \in X_w$ and $a^{2n|X|+q_{1k}}xA^* \cap X^* \neq \emptyset$ (since $q_{ik} \in Q_0$). Let

$$Q_{x_1} = \{k \in T | a^{2n|X|+k} x_1 A^* \cap X^* \neq \emptyset\}.$$

Then, we deduce that $\overline{Q_1} \subseteq Q_{x_1}$. If $k \in Q_{x_1}$, then there exists $z \in A^*$ such that $a^{2n|X|+k}x_1 z \in X^*$. This word has a factorization in X as follows:

$$a^{2n|X|+k}x_1 z = a^{2n|X|+k}wa^{4n|X|+p_1}xz = (a^{2n|X|+k-i_{1\ell}})(a^{i_{1\ell}}wa^{j_{1\ell}})(a^{\alpha})(a^{q_{1\ell}}xz),$$

where all words in the parentheses belong to X^*. Clearly, $k = \overline{i_{1\ell}} \in \overline{Q_1}$. Therefore, we have $\overline{Q_1} = Q_{x_1}$. In order to show that $a^{\overline{Q_1}}$ is a right set, we still need verify that $\overline{Q_1}$ satisfies condition (6). If there exist $\overline{i_{1k}}, \overline{i_{1\ell}} \in \overline{Q_1}$ with $\overline{i_{1k}} \neq \overline{i_{1\ell}}$ such that $a^{|\overline{i_{1k}}-\overline{i_{1\ell}}|} \in (X^*)^{-1}X^*$, then $a^{|i_{1k}-i_{1\ell}|} \in (X^*)^{-1}X^*$ too and there are $u, v \in X^*$ such that $u = va^{i_{1k}-i_{1\ell}}$ (Assume that $i_{1k} > i_{1\ell}$). Furthermore, we have $ua^{i_{1\ell}}x_1 = va^{i_{1k}}x_1$. This word has the following factorizations:

$$ua^{i_{1\ell}}x_1 = ua^{i_{i\ell}}wa^{p_1+4n|X|}x = (u)(a^{i_{1\ell}}wa^{j_{1\ell}})(a^{\beta})(a^{q_{1\ell}}x),$$

$$va^{i_{1k}}x_1 = va^{i_{ik}}wa^{p_1+4n|X|}x = (v)(a^{i_{1k}}wa^{j_{1k}})(a^{\beta'})(a^{q_{1k}}x),$$

where $\beta, \beta' = 0 \bmod n$, $a^{i_{1\ell}}wa^{j_{1\ell}}, a^{i_{1k}}wa^{j_{1k}} \in X_w$. For the sake of convenience, we assume that $q_{1\ell} < q_{1k}$. Thus, we have

$$((v)(a^{i_{1k}}wa^{j_{1k}})(a^{\beta'}))^{-1}((u)(a^{i_{1\ell}}wa^{j_{1\ell}})(a^{\beta})) = a^{q_{1k}-q_{1\ell}}$$

and hence, we have $a^{q_{1k}-q_{1\ell}} \in (X^*)^{-1}X^*$. This result contradicts to a^{Q_0} being a right set. Therefore, we see that $\overline{Q_1}$ satisfies condition (6) and $a^{\overline{Q_1}}$ is also a right set.

Since (P_0, Q_0) is arbitrarily taken, we have proved that X_w is induced by $(\mathbb{P}_X, \mathbb{Q}_X)$. ■

Remark 2.12. Let $(\mathbb{P}, \mathbb{Q})$ be a perfect system of factorizations of $\mathbb{Z}_n$ and $X \subseteq \cup_{k=1}^{m} a^*B(a^*B)^{k-1}a^*$ with $a^n \in X$, where $B = A - \{a\}$. If $X_w \subseteq a^*wa^*$ is induced by $(\mathbb{P}, \mathbb{Q})$ for each $w \in \cup_{k=1}^{m} B(a^*B)^{k-1}$, where X_w is defined as (7). By Theorem 2.4, each X_w is a maximal code in $a^* \cup a^*wa^*$. We may ask the following question: Is X a maximal code in $\cup_{k=1}^{m} a^*B(a^*B)^{k-1}a^*$? In fact, when X is a code, the answer to this question is positive. However, X may not be a code, see the following example.

Example 2.13. Let $\mathbb{P} = \{P_0\}$, where $P_0 = (0, 2, 4)$ and $\mathbb{Q} = \{Q_0, Q_1\}$, where $Q_0 = (0.1), Q_1 = (1, 2)$. Clearly, $(\mathbb{P}, \mathbb{Q})$ is a system of factorizations of $\mathbb{Z}_6$. Moreover, let $A = \{a, b\}$ and

$$X = \{a^2ba^6, ab, a^2ba^2, aba^2, a^2ba^4, aba^4, ba^2b, aba^2b, ba^2ba^2, ba^2ba^4\}.$$

Then X_b and X_{ba^2b} are induced by $(\mathbb{P}, \mathbb{Q})$. The following are the matrices $X_b^{(P_0,Q_0)}$ and $X_{ba^2b}^{(P_0,Q_0)}$ induced by $(\mathbb{P}, \mathbb{Q})$ related to (P_0, Q_0).

$$X_b^{(P_0,Q_0)} = \left\{ \begin{matrix} a^2ba^6, & ab, \\ a^2ba^2, & aba^2, \\ a^2ba^4, & aba^4, \end{matrix} \right\},$$

$$X_{ba^2b}^{(P_0,Q_0)} = \left\{ \begin{matrix} ba^2b, & aba^2b, \\ ba^2ba^2, & aba^2ba^2, \\ ba^2ba^4, & aba^2ba^4, \end{matrix} \right\}.$$

(Notice that $aba^2ba^2, aba^2ba^4 \in X_b^2$.) However, we have the following equality:

$$(aba^2)(a^2ba^2)(ba^2ba^2) = (aba^4)(ba^2b)(a^2ba^2),$$

where all the words in the parentheses belong to X. This equality shows that X is not a code.

3. Triangle Conjecture

In this section, we return to consider the well known triangle conjecture. Let $A = \{a, b\}$ and $X \subseteq a^*ba^*$ be a code. Then the triangle conjecture asserts that

$$\mathrm{Card}(\{a^iba^j \in X | i + j \le K\}) \le K + 1,$$

for any $K \in \mathbb{N}$.

In the literature, the triangle conjecture was first proposed by D. Perrin and M.P. Schützenberger in 1981 [12]. It has already been proved by P. Shor in 1983 that this conjecture is in general not true [17]. In this section, our work is mainly to prove that the triangle conjecture holds for finitely completable codes under an additional condition. Thus, our result in this section can be regarded as a supplemented result of the well known triangle conjecture.

We first recall a kind of factorizations of $\mathbb{Z}_n$ introduced in [8]. Consider a factorization (P, Q) of $\mathbb{Z}_n$ satisfying the following condition: For every $i = 0, 1, \cdots, n-1$, there exists one and only one pair $(p, q) \in P \times Q$ such that $p + q = i$ not modulo n. Then we call such factorization a *strong factorization* or call it *Krasner factorization*.

For strong factorizations, we state the following theorem.

Theorem 3.1. *Let $X \subseteq A^+$ be a finite maximal code and n the order of $a \in A$. Moreover, let $(\mathbb{P}, \mathbb{Q})$ be the system of factorizations of $\mathbb{Z}_n$ induced by X. If there exists $P \subseteq \mathbb{N}$ ($Q \subseteq \mathbb{N}$) such that $((\{P\} \cup \mathbb{P}, \mathbb{Q}))$ ($(\mathbb{P}, \{Q\} \cup \mathbb{Q}))$) is again a system factorizations of $\mathbb{Z}_n$, and it contains a strong factorization, then for any $w \in B(a^*B)^*$, where $B = A - \{a\}$, the following inequalities hold:*

$$\mathrm{Card}(\{a^i w a^j \in X_w \mid i + j \leq K\}) \leq K + 1, \quad K = 0, 1, 2, \cdots. \tag{10}$$

Proof. We only consider the condition that $(\{P\} \cup \mathbb{P}, \mathbb{Q})$ contains a strong factorization of $\mathbb{Z}_n$. For the condition that $(\mathbb{P}, \{Q\} \cup \mathbb{Q})$ contains a strong factorization of $\mathbb{Z}_n$, the proof is similar. Let (P, Q) be the strong factorization in $(\{P\} \cup \mathbb{P}, \mathbb{Q})$. Clearly, any $k \in \mathbb{N}$ may be factorized uniquely as $k = p + q + \alpha$ such that $(p, q) \in P \times Q$ and $\alpha = 0 \bmod n$ with $\alpha > 0$. Therefore, we have two functions: $\lambda : \mathbb{N} \to P, k \mapsto p$ and $\sigma : \mathbb{N} \to Q, k \mapsto q$. Furthermore, we define a function $\Phi : X_w \to a^Q w a^P, a^i w a^j \mapsto a^{\sigma(i)} w a^{\lambda(j)}$.

In fact, Φ is an injection. If this assertion is not true, then there exist $a^{i_1} w a^{j_1}, a^{i_2} w a^{j_2} \in X_w$ with $a^{i_1} w a^{j_1} \neq a^{i_2} w a^{j_2}$ such that $\Phi(a^{i_1} w a^{j_1}) = \Phi(a^{i_2} w a^{j_2}) = a^q w a^p$, where $(p, q) \in P \times Q$. We now factorize i_1, j_1, i_2, j_2 as follows:

$$i_1 = p_1 + q + \alpha_1, \quad i_2 = p_2 + q + \alpha_2, \quad j_1 = p + q_1 + \beta_1, \quad j_2 = p + q_2 + \beta_2,$$

where $p_1, p_2 \in P$ and $q_1, q_2 \in Q$, $\alpha_1, \alpha_2, \beta_1, \beta_2 = 0 \bmod n$ with $\alpha_1, \alpha_2, \beta_1, \beta_2 < 2n|X|$. Since $a^{i_1} w a^{j_1} \neq a^{i_2} w a^{j_2}$, we see easily that $\overline{i_1} - \overline{i_2} \neq 0$ or $\overline{j_1} - \overline{j_2} \neq 0$. Assume that $\overline{j_1} - \overline{j_2} \neq 0$. By Proposition 2.3, $|\overline{i_1} - \overline{i_2}| = |p_1 - p_2| \in \{|k - \ell||k, \ell \in P, P \in \mathbb{P}\}$ leads to $|\overline{j_1} - \overline{j_2}| \in \{|k - \ell||, k, \ell \in P, P \in \mathbb{P}\}$. This contradicts to $|\overline{j_1} - \overline{j_2}| = |q_1 - q_2| \in \{|k - \ell||, k, \ell \in Q, Q \in \mathbb{Q}\}$ and $\overline{j_1} - \overline{j_2} \neq 0$. For $\overline{i_1} - \overline{i_2} \neq 0$, again by Proposition 2.3, $|\overline{j_1} - \overline{j_2}| = |q_1 - q_2| \in \{|k - \ell||k, \ell \in Q, Q \in \mathbb{Q}\}$ leads to $|\overline{i_1} - \overline{i_2}| \in \{|k - \ell||, k, \ell \in Q, Q \in \mathbb{Q}\}$. Similarly, we see that this result contradicts to $|\overline{i_1} - \overline{i_2}| = |p_1 - p_2| \in \{|k - \ell||, k, \ell \in P, P \in \mathbb{P}\}$ and $\overline{i_1} - \overline{i_2} \neq 0$. Therefore, we have proved that Φ is an injection. Since (P, Q) is a strong factorization, we obtain immediately that

$$\mathrm{Card}(\{a^q wa^p \in a^Q wa^P \mid p+q \le K\}) \le K+1, \quad K = 0,1,2,\cdots.$$

Clearly, $\sigma(i) \le i$ and $\lambda(j) \le j$. Hence, we deduce that

$$\begin{aligned}\{a^{\sigma(i)} wa^{\lambda(j)} \in \Phi(X_w) | i+j \le K\} &\subseteq \{a^{\sigma(i)} wa^{\lambda(j)} \in \Phi(X_w) | \sigma(i)+\lambda(j) \le K\} \\ &\subseteq \{a^q wa^p \in a^Q wa^P | q+p \le K|\}.\end{aligned}$$

The above result implies that

$$\begin{aligned}\mathrm{Card}(\{a^i ba^j \in X_w \mid i+j \le K\}) &= \mathrm{Card}(\{a^{\sigma(i)} ba^{\lambda(j)} \in \Phi(X_w) \mid i+j \le K\}) \\ &\le \mathrm{Card}(\{a^q ba^p \in a^Q wa^P \mid q+p \le K\}) \\ &\le K+1,\end{aligned}$$

where $K = 0,1,2,\cdots$. Therefore, we have proved that the inequality (9) holds. ■

In closing this paper, we give the following Remark.

Remark 3.2. For the system of factorizations $(\mathbb{P},\mathbb{Q})$ of $\mathbb{Z}_n$, we list three types as following:

(i) There exists a strong factorization (P,Q) of $\mathbb{Z}_n$ which is contained in $(\mathbb{P},\mathbb{Q})$.

(ii) There exists a strong factorization (P,Q) of $\mathbb{Z}_n$ such that $(\{P\}\cup\mathbb{P},\{Q\}\cup\mathbb{Q})$ is again a system factorizations of $\mathbb{Z}_n$. Moreover, at least one of P and Q belong to $\mathbb{P}\cup\mathbb{Q}$ (Clearly, if $P \in \mathbb{P}\cup\mathbb{Q}$, then $P \in \mathbb{P}$ and $\{P\}\cup\mathbb{P} = \mathbb{P}$. For $Q \in \mathbb{P}\cup\mathbb{Q}$, the situation is analogous.)

(iii) There exists a strong factorization (P,Q) of $\mathbb{Z}_n$ such that $(\{P\}\cup\mathbb{P},\{Q\}\cup\mathbb{Q})$ is again a system factorizations of $\mathbb{Z}_n$.

Clearly,

$$Type(\mathrm{i}) \subseteq Type(\mathrm{ii}) \subseteq Type(\mathrm{iii}).$$

In [21], we prove that the conclusion of Theorem 3.1 holds if the system of factorizations $(\mathbb{P},\mathbb{Q})$ of $\mathbb{Z}_n$ induced by X belong to type (i), and in this paper, we prove that the conclusion holds if $(\mathbb{P},\mathbb{Q})$ belong to type (ii). We can not prove that the conclusion holds for $(\mathbb{P},\mathbb{Q})$ belonging to type (iii).

Remark 3.3. One naturally call two codes X and Y *commutative equivalent* if there exists a bijection between these codes such that a word and its image only differs by the ordering of their letters. Hence, a code X is said to be a *commutative prefix code* if X is commutative equivalent to a prefix code.

Now, we let $(\mathbb{P},\mathbb{Q})$ be a perfect system of factorizations of $\mathbb{Z}_n$ and the code $X \subseteq \cup_{k=1}^m a^*B(a^*B)^{k-1}a^*$ with $a^n \in X$, where $B = A - \{a\}$. If for each $w \in B(a^*B)^{k-1}$, $k = 1,\cdots,m$, X_w satisfies the inequalities (10), then we know that all X_w are commutative prefix codes (see [1, Prop. 14.6.3]). Accordingly, we ask the following natural question.

Is X a commutative prefix code?

The answer to this question is however negative. We now provide the following example.

Example 3.4. Consider the following Shor's code in [17]

$$X_S = b\{1, a, a^7, a^{13}, a^{14}\} \cup \{a^3, a^8\}b\{1, a^2, a^4, a^6\} \cup a^{11}b\{1, a, a^2\}.$$

In keeping the notation given in this paper, we exchange the position of a and b, i.e, let

$$X = a\{1, b, b^7, b^{13}, b^{14}\} \cup \{b^3, b^8\}a\{1, b^2, b^4, b^6\} \cup b^{11}a\{1, b, b^2\}.$$

Since $a \in X$, we see that for each $w \in b(a^*b)^*$, X_w has at most one element and hence all X_w (if they are not empty) are prefix codes. However, it is clear that X is not a commutative prefix code. This illustrates the situation.

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