Southeast Asian Bulletin of Mathematics (2021) 45: 873–896

Southeast Asian
Bulletin of
Mathematics

Dyad Fuzzy β-Covering Rough Set Models Based on Fuzzy Information System and Its Generalization over Fuzzy Lattice

Xinli Niu, Tongquan Zhou, Zhenduo Sun and Xiangzhi Kong*
School of Science, Jiangnan University, Wuxi, Jiangsu, 214122, China
Email: xiangzhikong@jiangnan.edu.cn

Received 20 April 2021
Accepted 11 May 2021

Communicated by Wieslaw Dudek

Dedicated to the memory of Professor Yuqi Guo (1940–2019)

AMS Mathematics Subject Classification(2000): 06D72, 03B52, 94D05, 93C42

Abstract. In this paper, we present some aspects in fuzzy decision making theory which is closely related to our daily life applications. In fact, if only one factor is considered when making a decision, it usually leads to one-sided and limited judgment. In order to deal with this problem, we use fuzzy β-co-neighborhood to define a new fuzzy β-covering rough set model based on information system for the first time. At the same time, we extend the concept of twin approximation operators to the information system, and propose a new modeldyad fuzzy β-covering rough set models, which is a combination of covering rough set theory, fuzzy rough set theory and fuzzy information system. This model can analyze and solve practical problems from positive and negative aspects, so as to make more accurate decisions. Therefore, we give an example in medicine to illustrate its practical value. Finally, we generalize the above model to fuzzy lattice and construct L-dyad fuzzy β-covering rough set models.

Keywords: Fuzzy β-covering; Fuzzy information system; Fuzzy rough set; Fuzzy lattice.

1. Introduction

The theory of rough set was initially proposed by Pawlak [31, 32], which is a novel mathematical method to handle with ambiguity and uncertainty in infor-

*Corresponding author.

mation systems. it is well known that Pawlak was the first one to construct the approximate space over rough set theory according to equivalence relations in [33]. Since many authors have generalized the right sets of Pawalk to many other right sets by extending this approximation space to other approximation spaces, such as relation approximation space [6, 66], the neighborhood approximation space [43] and the covering approximation space [67].

By extending Pawlaks rough sets into covering approximation spaces, the study of covering-based rough sets [1, 2, 26] were proposed to manage the kinds of covering data. On the one hand, the rough set based on coverage enriches Pawlak's rough set. For example, several new types of coverage-based rough set models have been established [40, 56], and corresponding axiom systems have been proposed [67, 58]. On the other hand, it is applied to solve some practical problems, such as knowledge reduction [24, 16, 44, 57], decision rule synthesis [12, 17, 37], and other fields [7, 23, 46]. In recent years, many scholars have also combined covering based rough set theory with other theories for research, such as lattice theory [4, 65], matroid theory [21, 47] and fuzzy set theory [8, 22, 64] has become increasing attractive.

In order to better study the relationship between fuzzy rough set and covering based rough set, Li et al. [18] proposed the concept of fuzzy covering approximate space, which was later developed by Deer et al. [9]. In recent years, a variety of fuzzy rough set models are defined on the basis of this concept [13, 38]. Based on the fuzzy covering defined in [13, 19], Ma [29] first proposed the fuzzy β-covering by introducing the parameter β $(0 < \beta \leq 1)$. Moreover, Ma proposed some neighborhood-related covering rough sets in [28]. In addition, Yang et al. [52] proved some properties of fuzzy β-covering on the basis of Mas research, and defined a new rough set model by introducing the concept of complementary β-neighborhood. Accordingly, Yang et al. [49] have given the definition of fuzzy β-minimal description, and discussed some properties of fuzzy covering rough sets. Based on these concepts in the β-fuzzy covering approximation spaces, many scholars have further studied the corresponding applications and generalizations in [15, 45].

The information system proposed in [25, 30] is a platform that represents objects with attributes and values of attributes, which refers to a man-machine interaction computer application system with information processing services as the main activity. Thus, the information systems play an important role in today's research in the field of artificial intelligence. In addition, with the rapid progress and wide application of information technology, information processing, transmission, storage and utilization have become the focus of research. Recently, these problems have attracted the research of many scholars and have been successfully applied in decision making, knowledge discovery and other fields [5, 10, 11, 34, 35, 48, 59].

However, with the advent of the era of big data and the general increase in ambiguity in the information obtained, the drawbacks of traditional tools in processing and analyzing information have gradually emerged. In order to deal with a large amount of inaccurate knowledge at the same time, scholars have successively proposed some methods, including the fuzzy information system which is obtained by replacing the attribute set on the crisp power set in the information system with the attribute set on the fuzzy power set [42], in addition to rough sets and their extensions [31, 32, 33, 55], Dempster-Shafer Theory [39], fuzzy sets [60, 61], granular calculation [62, 53], and S-approximation Spaces [41, 14]. In the covering approximation space, there are several approximation operators similar to [56, 55, 63, 36, 3, 54], which have different structures and properties. L.W. Ma [20] propose the notion of twin approximation operators, which is defined in covering rough set theory. In this paper, we will focus on a special extension of rough sets over fuzzy information system. We first define two fuzzy β-covering rough set models on two universes U and V by using the concept of fuzzy β-neighborhood and fuzzy β-co-neighborhood respectively, then we extend the concept of twin approximation operators and propose a type of novel modeldyad fuzzy β-covering rough set models $[(\underline{APR}, \overline{APR}), (\underline{\underline{APR}}, \overline{\overline{APR}})]$, which is a combination of covering rough set theory, fuzzy rough set theory and fuzzy information system. The above theorems can be used to analyze and solve the practical problems from positive and negative aspects, so as to make more accurate decisions. Finally, we extend the model to dyad L-fuzzy β-covering rough set models which is defined on lattice.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions that will be used in this paper. In Section 3, we propose a new rough set model based on fuzzy information system, properties of the new models are investigated and practical examples are provided. In Section 4, we construct L-dyad fuzzy β-covering rough set models as an extension of dyad fuzzy β-covering rough set models over lattice. We conclude the paper in Section 5.

2. Preliminaries

In this section, we introduce some preliminary definitions in fuzzy set theory and covering rough set theory which will be used in following study.

Definition 2.1. [51] *Let U be a universe of discourse and $\mathcal{C}$ be a family of subsets of U. If no element in $\mathcal{C}$ is empty and $U_{C\in\mathcal{C}} = U$, then $\mathcal{C}$ is called a covering of U, and the ordered pair $(U, \mathcal{C})$ is called a covering approximation space.*

Definition 2.2. [29] *Let (U,C) be a covering approximation space, and $U - X$ be denoted by $-X$ as well as X^c in the covering approximation space (U,C). Then for any $x \in U$, the neighborhood N_x and co-neighborhoods M_x of x are defined as*

$$N_x = \cap \{C \in \mathcal{C} : x \in C\}, M_x = \cap -C : (C \in \mathcal{C}) \wedge (x \notin C)$$

respectively, where $M_X = U$ if $x \in C$ for each $C \in \mathcal{C}$.

Definition 2.3. [51] *Let U be a universe of discourse. Then, a fuzzy set $\widetilde{A}$, or a* *fuzzy subset $\widetilde{A}$ of U, is defined by a function that assigns each element x of U to a value $\widetilde{A}(x) \in [0,1]$. We use $F(U)$ to denote the family of all fuzzy subsets of U, i.e., the set of all functions from U to [0,1], which is called the fuzzy power set of U.*

For any $\widetilde{A}, \widetilde{B} \in F(U)$, we say that $\widetilde{A}$ is contained in $\widetilde{B}$, denoted by $\widetilde{A} \subset \widetilde{B}$, if $\widetilde{A}(x) \leq \widetilde{B}(x)$ for all $x \in U$, and we say that $\widetilde{A} = \widetilde{B}$ if and only if $\widetilde{A} \subset \widetilde{B}$ and $\widetilde{B} \subset \widetilde{A}$.

For any family $\alpha_i \in [0,1], i \in I, I \subseteq \mathbb{N}^*$ ($\mathbb{N}^*$ is the set of all positive integers), we write $\vee_{i\in I}\alpha_i$ or $\vee\{\alpha_i : i \in I\}$ for the supremum of $\{\alpha_i : i \in I\}$, and $\wedge_{i\in I}\alpha_i$ or $\wedge\{\alpha_i : i \in I\}$ for the infimum of $\{\alpha_i : i \in I\}$. Given $\widetilde{A}, \widetilde{B} \in F(U)$, the union of $\widetilde{A}$ and $\widetilde{B}$, represented as $\widetilde{A} \cup \widetilde{B}$, which is defined as $(\widetilde{A} \cup \widetilde{B})(x) = \widetilde{A}(x) \vee \widetilde{B}(x)$ for all $x \in U$; The intersection of $\widetilde{A}$ and $\widetilde{B}$, denoted as $\widetilde{A} \cap \widetilde{B}$, which is defined as $(\widetilde{A} \cap \widetilde{B})(x) = \widetilde{A}(x) \wedge \widetilde{B}(x)$ for all $x \in U$; The complement of $\widetilde{A}$, denoted as $\widetilde{A}^c$, is defined by $\widetilde{A}^c(x) = 1 - \widetilde{A}(x)$ for all $x \in U$.

Definition 2.4. [51] *Let U be a universe of discourse, and $F(U)$ the fuzzy power set of U. For each $\beta \in (0,1]$, we call $\widetilde{C} = \left\{\widetilde{C}_1, \widetilde{C}_2, \cdots \widetilde{C}_m\right\}$, with $\widetilde{C}_i \in F(U)\,(i = 1, 2, \cdots, m)$ is a fuzzy β-covering of U, if $(\bigcup_{i=1}^{m} \widetilde{C}_i)(x) \geq \beta$ for each $x \in U$. We also call $(U, \widetilde{C})$ a fuzzy β-covering approximation space.*

Definition 2.5. [27] *Let $(U, \widetilde{C})$ a fuzzy β-covering approximation space, where $\widetilde{C} = \left\{\widetilde{C}_1, \widetilde{C}_2, \cdots \widetilde{C}_m\right\}$ is a fuzzy β-covering of U. For each $x \in U$, we define the fuzzy β-neighborhood $\widetilde{N}_x^\beta$ and the fuzzy β-co-neighborhood $\widetilde{M}_x^\beta$ of x as:*

$$\begin{aligned}
\widetilde{N}_x^\beta &= \cap\left\{\widetilde{C}_i \in \widetilde{C} : \widetilde{C}_i(x) \geq \beta\right\}, \\
\widetilde{M}_x^\beta &= - \cup\left\{\widetilde{C}_i : \widetilde{C}_i(x) < \beta, \widetilde{C}_i \in \widetilde{C}\right\} \\
&= \cap\left\{-\widetilde{C}_i : 1 - \widetilde{C}_i(x) > 1 - \beta, \widetilde{C}_i \in \widetilde{C}\right\},
\end{aligned}$$

where $\widetilde{M}_x = \mathbf{1}_U$(i.e., $\widetilde{M}_x^\beta(y) = 1$ for each $y \in U$) if $\widetilde{C}_i(x) \geq \beta$ for each $\widetilde{C}_i \in \widetilde{C}$.

3. Dyad fuzzy β-Covering Rough Set Models Base on Fuzzy Information System

In this section, we use fuzzy β-co-neighborhood $\widetilde{M}_x^\beta$ to define a new fuzzy β-covering rough set model based on information system for the first time. Meanwhile, we also study some properties of this novel type of model. Finally, we use this new model to expand the twin approximation operators, and illustrate its important value with a practical example.

In general, Information system is a quad $IS = (U, AT, V, f)$ that represents objects with attributes and values of attributes, where U is a non-empty finite set of objects; AT is a non-empty finite set of attributes; $V = \bigcup_{x\in U} V_x$, V_x is the domain of the object x; $f : U \longrightarrow V$ called information function, where $f(x) = V_x \in V(x \in U)$.

Definition 3.1. [30] *Let U and V be two finite universes and $f : U \longrightarrow V$ be a mapping from U to V. Then, a fuzzy information system is a quad $FIS = (U, \widetilde{AT}, V, f)$, where $U = \{x_1, x_2, \cdots, x_n\}$ and $V = \{y_1, y_2, \cdots, y_s\}$ are non-empty finite set of objects and $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_m\right\} \in F(U)$ is a set of attributes describing objects.*

For a fuzzy information system $FIS = (U, \widetilde{AT}, V, f)$, if $\wedge_{x\in U}[\vee_{i=1}^{m}\widetilde{A}_i(x)] \neq 0$ for all $x \in U$, then $\widetilde{AT}$ can be seen as a fuzzy β-covering of U, where $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_m\right\}$ and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^{m}\widetilde{A}_i(x)]]$. In the following study, we always suppose that a given fuzzy information system satisfies $\wedge_{x\in U}[\vee_{i=1}^{m}\widetilde{A}_i(x)]$ for all $x \in U$.

Information system plays a significant role of research in the area of artificial intelligence today. In order to study various relations between fuzzy information systems, in [50], Yang and Hu used the concept of fuzzy β-neighborhood to define a fuzzy β-covering rough set model.

Definition 3.2. [50] *Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, where $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^{m}\widetilde{A}_i(x)]]$. Then, for each $\widetilde{X} \in F(U)$, we define the 1^{st} fuzzy covering lower approximation $\underline{APR}(\widetilde{X})$ and 1^{st} fuzzy covering upper approximation $\overline{APR}(\widetilde{X})$ of $\widetilde{X}$ as:*

$$\begin{aligned}\underline{APR}(\widetilde{X})(y) &= \wedge_{x\in f^{-1}(y)}[(1 - \widetilde{N}_x^{\beta}(x)) \vee \widetilde{X}(x)], y \in V,\\ \overline{APR}(\widetilde{X})(y) &= \vee_{x\in f^{-1}(y)}[\widetilde{N}_x^{\beta}(x) \wedge \widetilde{X}(x)], y \in V.\end{aligned}$$

If $\underline{APR}(\widetilde{X}) \neq \overline{APR}(\widetilde{X})$, then $\widetilde{X}$ is called 1^{st} fuzzy β-covering rough set model.

Given $\beta \in (0, 1)$, for every $x \in U$ in fuzzy information system $FIS = (U, \widetilde{AT}, V, f)$, we divide $\widetilde{AT}$ into two parts $\top_x = \left\{\widetilde{A}_i \in \widetilde{AT} : \widetilde{A}_i(x) \geq \beta\right\}$ and $\perp_x = \left\{\widetilde{A}_i \in \widetilde{AT} : \widetilde{A}_i(x) < \beta\right\}$. The 1^{st} fuzzy β-covering rough set model in Definition 3.2 only depend on the first part $\top_x$. In fact, the two parts $\top_x$ and $\perp_x$ both have very important meanings, because they completely describe x from the positive and negative aspects respectively. Therefore, it is necessary for us to use the latter $\perp_x$ to define another rough set model based on fuzzy covering, and then combine it with the model defined in Definition 3.2 to analyze and solve practical problems from both positive and negative aspects so as to make better judgments. This is the main issue to solve in this paper.

Table 1: A fuzzy information system $FIS = (U, \widetilde{AT})$

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
x_1	0.7	0.6	0.5	0.4	0.2
x_2	0.4	0.5	0.2	0.4	0.6
x_3	0.5	0.2	0.4	0.7	0.6
x_4	0.6	0.4	0.2	0.3	0.5
x_5	0.4	0.7	0.5	0.4	0.8
x_6	0.3	0.7	0.2	0.6	0.2

Based on this idea, we defined a fuzzy covering-based rough set model on the basis of fuzzy β-co-neighborhood $\widetilde{M}_x^\beta$ for the first time.

Definition 3.3. *Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, where $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^m \widetilde{A}_i(x)]]$. For each $\widetilde{X} \in F(U)$, we define the 2^{nd} fuzzy covering lower approximation $\underline{\underline{APR}}(\widetilde{X})$ and 2^{nd} fuzzy covering upper approximation $\overline{\overline{APR}}(\widetilde{X})$ of $\widetilde{X}$ as:*

$$\begin{aligned}\underline{\underline{APR}}(\widetilde{X})(y) &= \wedge_{x\in f^{-1}(y)}[(1-\widetilde{M}_x^\beta(x)) \vee \widetilde{X}(x)], y \in V,\\ \overline{\overline{APR}}(\widetilde{X})(y) &= \vee_{x\in f^{-1}(y)}[\widetilde{M}_x^\beta(x) \wedge \widetilde{X}(x)], y \in V.\end{aligned}$$

If $\underline{\underline{APR}}(\widetilde{X}) \neq \overline{\overline{APR}}(\widetilde{X})$, then $\widetilde{X}$ is called 2^{nd} fuzzy β-covering rough set model.

Example 3.4. Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5\right\}$ is listed as follows (Table 1).

Then $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5\right\}$ is a fuzzy β-covering of U and $\beta \in (0, 0.6]$. Let $\beta = 0.5$.

Let $V = \{y_1, y_2, y_3, y_4\}$, $f : U \longrightarrow V$ be a mapping from U to V, and

$$f(x) = \begin{cases} y_1 & \text{if } x \in \{x_1, x_4\},\\ y_2 & \text{if } x = x_2,\\ y_3 & \text{if } x \in \{x_3, x_6\},\\ y_4 & \text{if } x = x_5.\end{cases}$$

For

$$\widetilde{X} = \frac{0.4}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.5}{x_4} + \frac{0.8}{x_5} + \frac{0.6}{x_6}$$

Table 2: fuzzy 0.5-neighborhood $\widetilde{N}_{x_1}^{0.5}$

	x_1	x_2	x_3	x_4	x_5	x_6
$\widetilde{N}_{x_1}^{0.5}$	0.5	0.2	0.2	0.2	0.4	0.2
$\widetilde{N}_{x_2}^{0.5}$	0.2	0.5	0.2	0.4	0.7	0.2
$\widetilde{N}_{x_3}^{0.5}$	0.2	0.4	0.5	0.3	0.4	0.2
$\widetilde{N}_{x_4}^{0.5}$	0.2	0.4	0.5	0.5	0.4	0.2
$\widetilde{N}_{x_5}^{0.5}$	0.2	0.2	0.2	0.2	0.5	0.2
$\widetilde{N}_{x_6}^{0.5}$	0.4	0.4	0.2	0.3	0.4	0.6

First, we compute the 1^{st} fuzzy covering lower and upper approximation of $\widetilde{X}$. It is easy to find that

$$\begin{aligned}\widetilde{N}_{x_1}^{0.5} &= \widetilde{A}_1 \cap \widetilde{A}_2 \cap \widetilde{A}_3, \widetilde{N}_{x_2}^{0.5} = \widetilde{A}_2 \cap \widetilde{A}_5, \widetilde{N}_{x_3}^{0.5} = \widetilde{A}_1 \cap \widetilde{A}_4 \cap \widetilde{A}_5,\\ \widetilde{N}_{x_4}^{0.5} &= \widetilde{A}_1 \cap \widetilde{A}_5, \widetilde{N}_{x_5}^{0.5} = \widetilde{A}_2 \cap \widetilde{A}_3 \cap \widetilde{A}_5, \widetilde{N}_{x_6}^{0.5} = \widetilde{A}_2 \cap \widetilde{A}_4.\end{aligned}$$

So, we can obtain the fuzzy β-neighborhood $\widetilde{N}_{x_1}^{0.5}$ of $x_i(i = 1, 2, 3, 4, 5, 6)$ as follows (Table 2).

Then we have

$$\begin{aligned}\underline{APR}(\widetilde{X}) &= \frac{0.5}{y_1} + \frac{0.7}{y_2} + \frac{0.6}{y_3} + \frac{0.8}{y_4},\\ \overline{APR}(\widetilde{X}) &= \frac{0.5}{y_1} + \frac{0.5}{y_2} + \frac{0.6}{y_3} + \frac{0.5}{y_4}.\end{aligned}$$

Next, we compute the 2^{nd} fuzzy covering lower and upper approximation of $\widetilde{X}$. It is easy to find that

$$\begin{aligned}\widetilde{M}_{x_1}^{0.5} &= -\widetilde{A}_4 \cap -\widetilde{A}_5, \widetilde{M}_{x_2}^{0.5} = -\widetilde{A}_1 \cap -\widetilde{A}_3 - \cap\widetilde{A}_4, \widetilde{M}_{x_3}^{0.5} = -\widetilde{A}_2 \cap -\widetilde{A}_3,\\ \widetilde{M}_{x_4}^{0.5} &= -\widetilde{A}_2 \cap -\widetilde{A}_3 - \cap\widetilde{A}_4, \widetilde{M}_{x_5}^{0.5} = -\widetilde{A}_1 \cap -\widetilde{A}_4, \widetilde{M}_{x_6}^{0.5} = -\widetilde{A}_1 \cap -\widetilde{A}_3 - \cap\widetilde{A}_5.\end{aligned}$$

Thus, we can calculate the fuzzy β-neighborhood $\widetilde{M}_{x_1}^{0.5}$ of $x_i(i = 1, 2, 3, 4, 5, 6)$ as follows (Table 3).

So, we have

$$\begin{aligned}\underline{\underline{APR}}(\widetilde{X}) &= \frac{0.4}{y_1} + \frac{0.7}{y_2} + \frac{0.6}{y_3} + \frac{0.8}{y_4},\\ \overline{\overline{APR}}(\widetilde{X}) &= \frac{0.5}{y_1} + \frac{0.6}{y_2} + \frac{0.6}{y_3} + \frac{0.6}{y_4}.\end{aligned}$$

Furthermore, the properties of the models in the above two definitions can be proposed through the following propositions.

Table 3: fuzzy 0.5-co-neighborhood $\widetilde{M}_{x_1}^{0.5}$

	x_1	x_2	x_3	x_4	x_5	x_6
$\widetilde{M}_{x_1}^{0.5}$	0.6	0.4	0.3	0.5	0.2	0.4
$\widetilde{M}_{x_2}^{0.5}$	0.3	0.6	0.3	0.4	0.5	0.4
$\widetilde{M}_{x_3}^{0.5}$	0.4	0.5	0.6	0.6	0.3	0.3
$\widetilde{M}_{x_4}^{0.5}$	0.4	0.5	0.3	0.6	0.3	0.3
$\widetilde{M}_{x_5}^{0.5}$	0.3	0.6	0.3	0.4	0.6	0.4
$\widetilde{M}_{x_6}^{0.5}$	0.3	0.4	0.4	0.4	0.2	0.7

Proposition 3.5. [50] *Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^{m}\widetilde{A}_i(x)]]$. For each $\widetilde{X}, \widetilde{Y} \in F(U)$, we have the following relationships.*

(1) $\underline{APR}(\widetilde{X}^c) = (\overline{APR}(\widetilde{X}))^c, \overline{APR}(\widetilde{X}^c) = (\underline{APR}(\widetilde{X}))^c$.
(2) $\overline{APR}(\emptyset) = \emptyset, \underline{APR}(U) = V$.
(3) $\underline{APR}(\widetilde{X}\cap\widetilde{Y}) = \underline{APR}(\widetilde{X})\cap\underline{APR}(\widetilde{Y}), \overline{APR}(\widetilde{X}\cup\widetilde{Y}) = \overline{APR}(\widetilde{X})\cup\overline{APR}(\widetilde{Y})$.
(4) *If* $\widetilde{X} \subseteq \widetilde{Y}$, *then* $\underline{APR}(\widetilde{X}) \subseteq \underline{APR}(\widetilde{Y})$ *and* $\overline{APR}(\widetilde{X}) \subseteq \overline{APR}(\widetilde{Y})$.
(5) $\underline{APR}(\widetilde{X}\cup\widetilde{Y}) \supseteq \underline{APR}(\widetilde{X})\cup\underline{APR}(\widetilde{Y}), \overline{APR}(\widetilde{X}\cap\widetilde{Y}) \subseteq \overline{APR}(\widetilde{X})\cap\overline{APR}(\widetilde{Y})$.
(6) *If* $1 - \widetilde{N}_x^{\beta}(x) \leq \widetilde{X}(x) \leq \widetilde{N}_x^{\beta}(x)$ *for any* $x \in U$, *then* $\underline{APR}(\widetilde{X}) \subseteq \overline{APR}(\widetilde{X})$.

In [50], Yang and Hu have discussed the properties of the model defined in Definition 3.2 in detail and have given the proofs of Proposition 3.5. Therefore, we omit these results in this paper and only study the properties of the new model defined in Definition 3.3 and give some proofs.

Proposition 3.6. *Let $FIS = (U, \widetilde{AT}, V, f)$ be a fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^{m}\widetilde{A}_i(x)]]$. Then, for each $\widetilde{X}, \widetilde{Y} \in F(U)$, we have the following relationships.*

(1) $\underline{\underline{APR}}(\widetilde{X}^c) = (\overline{\overline{APR}}(\widetilde{X}))^c, \overline{\overline{APR}}(\widetilde{X}^c) = (\underline{\underline{APR}}(\widetilde{X}))^c$.
(2) $\overline{\overline{APR}}(\emptyset) = \emptyset, \underline{\underline{APR}}(U) = V$.
(3) $\underline{\underline{APR}}(\widetilde{X}\cap\widetilde{Y}) = \underline{\underline{APR}}(\widetilde{X})\cap\underline{\underline{APR}}(\widetilde{Y}), \overline{\overline{APR}}(\widetilde{X}\cup\widetilde{Y}) = \overline{\overline{APR}}(\widetilde{X})\cup\overline{\overline{APR}}(\widetilde{Y})$.
(4) *If* $\widetilde{X} \subseteq \widetilde{Y}$, *then* $\underline{\underline{APR}}(\widetilde{X}) \subseteq \underline{\underline{APR}}(\widetilde{Y})$ *and* $\overline{\overline{APR}}(\widetilde{X}) \subseteq \overline{\overline{APR}}(\widetilde{Y})$.
(5) $\underline{\underline{APR}}(\widetilde{X}\cup\widetilde{Y}) \supseteq \underline{\underline{APR}}(\widetilde{X})\cup\underline{\underline{APR}}(\widetilde{Y}), \overline{\overline{APR}}(\widetilde{X}\cap\widetilde{Y}) \subseteq \overline{\overline{APR}}(\widetilde{X})\cap\overline{\overline{APR}}(\widetilde{Y})$.
(6) *If* $1 - \widetilde{M}_x^{\beta}(x) \leq \widetilde{X}(x) \leq \widetilde{M}_x^{\beta}(x)$ *for any* $x \in U$, *then* $\underline{\underline{APR}}(\widetilde{X}) \subseteq \overline{\overline{APR}}(\widetilde{X})$.

Proof. (1) For any $y \in V$, we have

$$\begin{aligned}
&\underline{\underline{APR}}(\widetilde{X}^c)(y)\\
&= \wedge_{x\in f^{-1}(y)}\,[(1-\widetilde{M}_x^{\beta}(x)) \vee \widetilde{X}^c(x)] = \wedge_{x\in f^{-1}(y)}[1-(\widetilde{M}_x^{\beta}(x)\wedge \widetilde{X}(x))]\\
&=1-\vee_{x\in f^{-1}(y)}[\widetilde{M}_x^{\beta}(x)\wedge \widetilde{X}(x)] = 1-\overline{\overline{APR}}(\widetilde{X})(y)\\
&=(\overline{\overline{APR}}(\widetilde{X}))^c(y)
\end{aligned}$$

and

$$\overline{\overline{APR}}(\widetilde{X}^c)(y) = 1-\underline{\underline{APR}}((\widetilde{X}^c)^c(y) = 1-\underline{\underline{APR}}(\widetilde{X})(y) = (\underline{\underline{APR}}(\widetilde{X}))^c(y).$$

Then

$$\underline{\underline{APR}}(\widetilde{X}^c) = (\overline{\overline{APR}}(\widetilde{X}))^c, \overline{\overline{APR}}(\widetilde{X}^c) = (\underline{\underline{APR}}(\widetilde{X}))^c.$$

(2) Since $U(x)=1$ and $\emptyset(x)=0$ for every $x\in U$, we have

$$\overline{\overline{APR}}(\emptyset)(y) = \vee_{x\in f^{-1}(y)}[\widetilde{M}_x^{\beta}(x)\wedge\emptyset(x)] = 0$$

and

$$\underline{\underline{APR}}(U)(y) = \wedge_{x\in f^{-1}(y)}[(1-\widetilde{M}_x^{\beta}(x))\vee U(x)] = 1.$$

Therefore,

$$\overline{\overline{APR}}(\emptyset)=\emptyset, \underline{\underline{APR}}(U)=V.$$

(3) For any $y\in V$, we have

$$\begin{aligned}
&\underline{\underline{APR}}(\widetilde{X}\cap\widetilde{Y})(y)\\
&= \wedge_{x\in f^{-1}(y)}\,[(1-\widetilde{M}_x^{\beta}(x))\vee(\widetilde{X}\cap\widetilde{Y})(x)]\\
&= \wedge_{x\in f^{-1}(y)}\,[((1-\widetilde{M}_x^{\beta}(x))\vee\widetilde{X}(x))\wedge((1-\widetilde{M}_x^{\beta}(x))\vee\widetilde{Y}(x))]\\
&=(\wedge_{x\in f^{-1}(y)}[(1-\widetilde{M}_x^{\beta}(x))\vee\widetilde{X}(x)])\wedge(\wedge_{x\in f^{-1}(y)}[(1-\widetilde{M}_x^{\beta}(x))\vee\widetilde{Y}(x)])\\
&=\underline{\underline{APR}}(\widetilde{X})(y)\wedge\underline{\underline{APR}}(\widetilde{Y})(y) = (\underline{\underline{APR}}(\widetilde{X})\cap\underline{\underline{APR}}(\widetilde{Y}))(y)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\overline{APR}}(\widetilde{X}\cup\widetilde{Y})(y) &= \vee_{x\in f^{-1}(y)}\,[\widetilde{M}_x^{\beta}(x)\wedge(\widetilde{X}\cup\widetilde{Y})(x)]\\
&= \vee_{x\in f^{-1}(y)}\,[(\widetilde{M}_x^{\beta}(x)\wedge\widetilde{X}(x))\vee(\widetilde{M}_x^{\beta}(x)\wedge\widetilde{Y}(x))]\\
&=(\vee_{x\in f^{-1}(y)}[\widetilde{M}_x^{\beta}(x)\wedge\widetilde{X}(x)])\vee(\vee_{x\in f^{-1}(y)}[\widetilde{M}_x^{\beta}(x)\wedge\widetilde{Y}(x)])\\
&=\overline{\overline{APR}}(\widetilde{X})(y)\vee\overline{\overline{APR}}(\widetilde{Y})(y) = (\overline{\overline{APR}}(\widetilde{X})\cup\overline{\overline{APR}}(\widetilde{Y}))(y).
\end{aligned}$$

Then we have $\underline{\underline{APR}}(\widetilde{X}\cap\widetilde{Y}) = \underline{\underline{APR}}(\widetilde{X})\cap\underline{\underline{APR}}(\widetilde{Y}), \overline{\overline{APR}}(\widetilde{X}\cup\widetilde{Y}) = \overline{\overline{APR}}(\widetilde{X})\cup \overline{\overline{APR}}(\widetilde{Y})$.

(4) It follows from $\widetilde{X} \subseteq \widetilde{Y}$ that $\widetilde{X}(x) \leq \widetilde{Y}(x)$ for any $x \in U$. For each $y \in V$, so we have

$$\begin{aligned}\underline{\underline{APR}}(\widetilde{X})(y) &= \wedge_{x\in f^{-1}(y)}[(1-\widetilde{M}_x^\beta(x)) \vee \widetilde{X}(x)] \\ &\leq \wedge_{x\in f^{-1}(y)}[(1-\widetilde{M}_x^\beta(x)) \vee \widetilde{Y}(x)] = \underline{\underline{APR}}(\widetilde{Y})(y),\end{aligned}$$

$$\begin{aligned}\overline{\overline{APR}}(\widetilde{X})(y) &= \vee_{x\in f^{-1}(y)}[\widetilde{M}_x^\beta(x) \wedge \widetilde{X}(x)] \\ &\leq \vee_{x\in f^{-1}(y)}[\widetilde{M}_x^\beta(x) \wedge \widetilde{Y}(x)] = \overline{\overline{APR}}(\widetilde{Y})(y).\end{aligned}$$

Hence

$$\underline{\underline{APR}}(\widetilde{X}) \subseteq \underline{\underline{APR}}(\widetilde{Y}),$$

$$\overline{\overline{APR}}(\widetilde{X}) \subseteq \overline{\overline{APR}}(\widetilde{Y}).$$

(5) Because of $\underline{\underline{APR}}(\widetilde{X}) \subseteq \underline{\underline{APR}}(\widetilde{X} \cup \widetilde{Y}), \underline{\underline{APR}}(\widetilde{Y}) \subseteq \underline{\underline{APR}}(\widetilde{X} \cup \widetilde{Y})$, follows from (4), so we have

$$\underline{\underline{APR}}(\widetilde{X}) \cup \underline{\underline{APR}}(\widetilde{Y}) \subseteq \underline{\underline{APR}}(\widetilde{X} \cup \widetilde{Y}),$$

Similarly, since $\overline{\overline{APR}}(\widetilde{X} \cap \widetilde{Y}) \subseteq \underline{\underline{APR}}(\widetilde{X})$ and $\overline{\overline{APR}}(\widetilde{X} \cap \widetilde{Y}) \subseteq \underline{\underline{APR}}(\widetilde{Y})$, we have

$$\overline{\overline{APR}}(\widetilde{X} \cap \widetilde{Y}) \subseteq \overline{\overline{APR}}(\widetilde{X}) \cap \overline{\overline{APR}}(\widetilde{Y}).$$

(6) For any $y \in V$, if there is $1 - \widetilde{M}_x^\beta(x) \leq \widetilde{X}(x) \leq \widetilde{M}_x^\beta(x)$ for any $x \in U$, then

$$\begin{aligned}\underline{\underline{APR}}(\widetilde{X})(y) &= \wedge_{x\in f^{-1}(y)}[(1-\widetilde{M}_x^\beta(x)) \vee \widetilde{X}(x)] \\ &= \wedge_{x\in f^{-1}(y)}\widetilde{X}(x) \leq \vee_{x\in f^{-1}(y)}\widetilde{X}(x),\end{aligned}$$

$$\overline{\overline{APR}}(\widetilde{X})(y) = \vee_{x\in f^{-1}(y)}[\widetilde{M}_x^\beta(x) \wedge \widetilde{X}(x)] = \vee_{x\in f^{-1}(y)}\widetilde{X}(x).$$

Thus

$$\underline{\underline{APR}}(\widetilde{X}) \subseteq \overline{\overline{APR}}(\widetilde{X}).$$ ■

In order to further study the properties of $\underline{\underline{APR}}(\widetilde{X})$ and $\overline{\overline{APR}}(\widetilde{X})$, lets see an example as follows.

Example 3.7. For the fuzzy information system $FIS = (U, \widetilde{AT}, V, f)$ in Example 3.4, let

$$\begin{aligned}\widetilde{A} &= \frac{0.8}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.7}{x_4} + \frac{0.3}{x_5} + \frac{0.5}{x_6}, \\ \widetilde{B} &= \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.6}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.5}{x_6}.\end{aligned}$$

Thus

$$\widetilde{A}\cap\widetilde{B} = \frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{0.5}{x_3}+\frac{0.5}{x_4}+\frac{0.3}{x_5}+\frac{0.5}{x_6},$$
$$\widetilde{A}\cup\widetilde{B} = \frac{0.8}{x_1}+\frac{0.6}{x_2}+\frac{0.6}{x_3}+\frac{0.7}{x_4}+\frac{0.6}{x_5}+\frac{0.5}{x_6}.$$

Then we have

$$\overline{\overline{APR}}(\widetilde{A}) = \frac{0.6}{y_1}+\frac{0.6}{y_2}+\frac{0.5}{y_3}+\frac{0.3}{y_4},$$
$$\overline{\overline{APR}}(\widetilde{B}) = \frac{0.6}{y_1}+\frac{0.3}{y_2}+\frac{0.6}{y_3}+\frac{0.63}{y_4},$$
$$\overline{\overline{APR}}(\widetilde{A}\cap\widetilde{B}) = \frac{0.6}{y_1}+\frac{0.3}{y_2}+\frac{0.5}{y_3}+\frac{0.3}{y_4},$$
$$\underline{\underline{APR}}(\widetilde{A}) = \frac{0.7}{y_1}+\frac{0.6}{y_2}+\frac{0.5}{y_3}+\frac{0.4}{y_4},$$
$$\underline{\underline{APR}}(\widetilde{B}) = \frac{0.5}{y_1}+\frac{0.4}{y_2}+\frac{0.5}{y_3}+\frac{0.6}{y_4},$$
$$\underline{\underline{APR}}(\widetilde{A}\cup\widetilde{B}) = \frac{0.7}{y_1}+\frac{0.6}{y_2}+\frac{0.5}{y_3}+\frac{0.6}{y_4}.$$

We can see that $\overline{\overline{APR}}(\widetilde{A}\cap\widetilde{B}) = \overline{\overline{APR}}(\widetilde{A})\cap\overline{\overline{APR}}(\widetilde{B})$ and $\underline{\underline{APR}}(\widetilde{A}\cup\widetilde{B}) = \underline{\underline{APR}}(\widetilde{A})\cup\underline{\underline{APR}}(\widetilde{B})$ clearly. That means the equalities of item (5) in Proposition 3.6 can be hold under some conditions.

Now we give a sufficient condition under which the equalities of item (5) hold. First we introduce the consistency of mapping.

Definition 3.8. [5] *Let U and V be two finite universes, $f: U \longrightarrow V$ be a mapping from U to V, and $\widetilde{A},\widetilde{B}\in F(U)$. If $[x]_f = \{y\in U : f(y)=f(x)\}$, then $\{[x]_f : x\in U\}$ is a partition of U. For any $x\in U$, if one of the following statements holds:*

(1) *$\widetilde{A}(u)\leq\widetilde{B}(u)$ for any $u\in[x]_f$,*

(2) *$\widetilde{A}(u)\geq\widetilde{B}(u)$ for any $u\in[x]_f$,*

then f is called consistent with respect to $\widetilde{A}$ and $\widetilde{B}$.

Proposition 3.9. *Let $FIS = (U,\widetilde{AT},V,f)$ be a fuzzy information system and $f: U\longrightarrow V$ be a surjective mapping from U to V, $\widetilde{AT} = \left\{\widetilde{A}_1,\widetilde{A}_2,\cdots,\widetilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta\in(0,\wedge_{x\in U}[\vee_{i=1}^{m}\widetilde{A}_i(x)]]$. For $\widetilde{A},\widetilde{B}\in F(U)$, if f is consistent with respect to $\widetilde{A}$ and $\widetilde{B}$, then $\overline{\overline{APR}}(\widetilde{A}\cap\widetilde{B}) = \overline{\overline{APR}}(\widetilde{A})\cap\overline{\overline{APR}}(\widetilde{B})$ and $\underline{\underline{APR}}(\widetilde{A}\cup\widetilde{B}) = \underline{\underline{APR}}(\widetilde{A})\cup\underline{\underline{APR}}(\widetilde{B})$.*

Proof. It follows from Proposition 3.6 that $\underline{\underline{APR}}(\widetilde{A}\cup\widetilde{B})\supseteq\underline{\underline{APR}}(\widetilde{A})\cup\underline{\underline{APR}}(\widetilde{B})$ and $\overline{\overline{APR}}(\widetilde{A}\cap\widetilde{B})\subseteq\overline{\overline{APR}}(\widetilde{A})\cap\overline{\overline{APR}}(\widetilde{B})$. Then we only need to prove that $\underline{\underline{APR}}(\widetilde{A}\cup\widetilde{B})\subseteq\underline{\underline{APR}}(\widetilde{A})\cup\underline{\underline{APR}}(\widetilde{B})$ and $\overline{\overline{APR}}(\widetilde{A}\cap\widetilde{B})\supseteq\overline{\overline{APR}}(\widetilde{A})\cap\overline{\overline{APR}}(\widetilde{B})$ hold. For any $y\in V$, if f is consistent with respect to $\widetilde{A}$ and $\widetilde{B}$, it follows from Definition 3.8 that one of the following conditions holds:

(1) $\widetilde{A}(x)\le\widetilde{B}(x)$ for any $x\in f^{-1}(y)$,

(2) $\widetilde{A}(x)\ge\widetilde{B}(x)$ for any $x\in f^{-1}(y)$.

Then we have

$$\begin{aligned}\underline{\underline{APR}}(\widetilde{A}\cup\widetilde{B})(y) &= \wedge_{x\in f^{-1}(y)}[(1-\widetilde{M}^{\beta}_{x}(x))\vee(\widetilde{A}\cup\widetilde{B})(x)]\\ &= \wedge_{x\in f^{-1}(y)}[(1-\widetilde{M}^{\beta}_{x}(x))\vee\widetilde{A}(x)]=\underline{\underline{APR}}(\widetilde{A})(y)\end{aligned}$$

or

$$\begin{aligned}\underline{\underline{APR}}(\widetilde{A}\cup\widetilde{B})(y) &= \wedge_{x\in f^{-1}(y)}[(1-\widetilde{M}^{\beta}_{x}(x))\vee(\widetilde{A}\cup\widetilde{B})(x)]\\ &= \wedge_{x\in f^{-1}(y)}[(1-\widetilde{M}^{\beta}_{x}(x))\vee\widetilde{B}(x)]=\underline{\underline{APR}}(\widetilde{B})(y).\end{aligned}$$

Therefore $\underline{\underline{APR}}(\widetilde{A}\cup\widetilde{B})\subseteq\underline{\underline{APR}}(\widetilde{A})\cup\underline{\underline{APR}}(\widetilde{B})$.

Similarly, we have

$$\begin{aligned}\overline{\overline{APR}}(\widetilde{A}\cap\widetilde{B})(y) &= \vee_{x\in f^{-1}(y)}[\widetilde{M}^{\beta}_{x}(x)\wedge(\widetilde{A}\cap\widetilde{B})(x)]\\ &= \vee_{x\in f^{-1}(y)}[\widetilde{M}^{\beta}_{x}(x)\wedge\widetilde{A}(x)]=\overline{\overline{APR}}(\widetilde{A})(y)\end{aligned}$$

or

$$\begin{aligned}\overline{\overline{APR}}(\widetilde{A}\cap\widetilde{B})(y) &= \vee_{x\in f^{-1}(y)}[\widetilde{M}^{\beta}_{x}(x)\wedge(\widetilde{A}\cap\widetilde{B})(x)]\\ &= \vee_{x\in f^{-1}(y)}[\widetilde{M}^{\beta}_{x}(x)\wedge\widetilde{B}(x)]=\overline{\overline{APR}}(\widetilde{B})(y).\end{aligned}$$

Therefore $\overline{\overline{APR}}(\widetilde{A}\cap\widetilde{B})\supseteq\overline{\overline{APR}}(\widetilde{A})\cap\overline{\overline{APR}}(\widetilde{B})$.

This concludes the proof of this proposition. ■

Through Proposition 3.9, we can get the following corollary.

Corollary 3.10. *Let $FIS=(U,\widetilde{AT},V,f)$ be a fuzzy information system and $f:U\longrightarrow V$ be a surjective mapping from U to V, $\widetilde{AT}=\left\{\widetilde{A}_1,\widetilde{A}_2,\cdots,\widetilde{A}_m\right\}$ is a fuzzy β-covering of U and $\beta\in(0,\wedge_{x\in U}[\vee_{i=1}^{m}\widetilde{A}_i(x)]]$. For $\widetilde{X}_1,\widetilde{X}_2,\cdots,\widetilde{X}_n\in F(U)$, if f is consistent with any two of fuzzy sets $\widetilde{X}_1,\widetilde{X}_2,\cdots,\widetilde{X}_n$, then $\underline{\underline{APR}}(\bigcup_{i=1}^{n}\widetilde{X}_i)=\bigcup_{i=1}^{n}\underline{\underline{APR}}(\widetilde{X}_i)$ and $\overline{\overline{APR}}(\bigcap_{i=1}^{n}\widetilde{X}_i)=\bigcap_{i=1}^{n}\overline{\overline{APR}}(\widetilde{X}_i)$.*

In [28], L.W. Ma propose the notion of twin approximation operators, which is defined in the covering rough set theory. In this paper, we generalize this concept and propose a new modeldyad fuzzy β-covering rough set models, which is a combination of covering rough set theory, fuzzy rough set theory and fuzzy information system.

Definition 3.11. *Let* $FIS = (U, \widetilde{AT}, V, f)$ *be a fuzzy information system and* $f : U \longrightarrow V$ *be a surjective mapping from* U *to* V*,* $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_m\right\}$ *is a fuzzy* β*-covering of* U *and* $\beta \in (0, \wedge_{x\in U}[\vee_{i=1}^{m} \widetilde{A}_i(x)]]$*. Then, for any* $\widetilde{X} \in F(U)$*,* $\underline{APR}(\widetilde{X})$ *and* $\overline{APR}(\widetilde{X})$ *are* 1^{st} *fuzzy covering lower and upper approximation of* $\widetilde{X}$ *respectively,* $\underline{\underline{APR}}(\widetilde{X})$ *and* $\overline{\overline{APR}}(\widetilde{X})$ *are* 2^{nd} *fuzzy covering lower and upper approximation of* $\widetilde{X}$ *respectively, we call the two-pair of models* $[(\underline{APR}, \overline{APR}), (\underline{\underline{APR}}, \overline{\overline{APR}})]$ *a dyad fuzzy* β*-covering rough set models.*

The dyad fuzzy β-covering rough set models is on the basis of fuzzy β-neighborhood $\widetilde{N}_x^{\beta}$ and fuzzy β-co-neighborhood $\widetilde{M}_x^{\beta}$ at the same time, which are not only closely related but also complementary to each other. Thereby, the positive and negative aspects can be integrated to analyze and solve practical problems so as to make more accurate decisions.

Here giving a practical example to show their important value.

Example 3.12. In the application of medicine, we may assume that one medicine is usually useful for many symptoms. Let $U = \{x_i : i = 1, 2, \cdots, n\}$ be the universe of n kinds of symptoms, $\widetilde{AT} = \left\{\widetilde{A}_j : j = 1, 2, \cdots, m\right\}$ be the universe of m kinds of medicines, and $\widetilde{A}_j(x_i)$ represents the effect of the medicine $\widetilde{A}_j$ on the symptom $x_i (i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$. Let β be the critical value. If we assume that for each symptom $x_i \in U$, there is at least one medicine $\widetilde{A}_j \in \widetilde{AT}$ such that the effect of the medicine $\widetilde{A}_j$ on the symptom x_i is not less than β, then $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_m\right\}$ is a fuzzy β-covering of U.

We suppose $FIS = (U, \widetilde{AT})$ be a fuzzy information system, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5\right\}$ is listed as follows (Table 4).

Then $\widetilde{AT} = \left\{\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5\right\}$ is a fuzzy β-covering of U and $\beta \in (0, 0.6]$. Let $\beta = 0.5$.

Let $V = \{y_1, y_2, y_3, y_4\}$ be the universe of four patients, $f : U \longrightarrow V$ be a mapping from U to V, and

$$f(x) = \begin{cases} y_1 & \text{if } \ x \in \{x_1, x_2, x_6\}, \\ y_2 & \text{if } \ x \in \{x_3, x_5\}, \\ y_3 & \text{if } \ x \in \{x_4, x_8\}, \\ y_4 & \text{if } \ x = x_7 \end{cases}$$

Table 4: A fuzzy information system $FIS = (U, \widetilde{AT})$

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
x_1	0.7	0.4	0.1	0.3	0.6
x_2	0.6	0.2	0.3	0.4	0.5
x_3	0.1	0.7	0.6	0.4	0.2
x_4	0.3	0.6	0.7	0.3	0.1
x_5	0.4	0.8	0.6	0.6	0.2
x_6	0.8	0.3	0.2	0.5	0.3
x_7	0.2	0.1	0.3	0.8	0.4
x_8	0.4	0.2	0.8	0.3	0.7

and $f(x_i) = y_s$ denote the patient y_s is suffering from the symptom x_i.

If a fuzzy set $\widetilde{X}$ denotes a kind of new medicines ability to cure the symptoms $x_i(i = 1, 2, \cdots, 8)$ according to many experiments, then we can consider which patient can use this kind of new medicine.

For

$$\widetilde{X} = \frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.2}{x_3} + \frac{0.4}{x_4} + \frac{0.5}{x_5} + \frac{0.4}{x_6} + \frac{0.7}{x_7} + \frac{0.3}{x_8}$$

we can compute that

$$\underline{APR}(\widetilde{X}) = \frac{0.5}{y_1} + \frac{0.4}{y_2} + \frac{0.6}{y_3} + \frac{0.7}{y_4}, \overline{APR}(\widetilde{X}) = \frac{0.6}{y_1} + \frac{0.5}{y_2} + \frac{0.4}{y_3} + \frac{0.7}{y_4}$$

and we can also obtain

$$\underline{\underline{APR}}(\widetilde{X}) = \frac{0.4}{y_1} + \frac{0.4}{y_2} + \frac{0.4}{y_3} + \frac{0.7}{y_4}, \overline{\overline{APR}}(\widetilde{X}) = \frac{0.6}{y_1} + \frac{0.5}{y_2} + \frac{0.4}{y_3} + \frac{0.6}{y_4}$$

First, we can make a positive decision by analyzing $\underline{APR}(\widetilde{X})$ and $\overline{APR}(\widetilde{X})$ under the critical value β=0.5:

(1). Since $\underline{APR}(\widetilde{X})(y_i) \geq 0.5$ and $\overline{APR}(\widetilde{X})(y_i) \geq 0.5(i = 1, 4)$, we conclude that this kind of new medicine has great benefits for the treatment of the patient y_1 and y_4.

(2). As $\underline{APR}(\widetilde{X})(y_2) \geq 0.5$ and $\overline{APR}(\widetilde{X})(y_2) < 0.5$, we conclude that this kind of new medicine has some help in the treatment of the patient y_2.

(3). According to $\underline{APR}(\widetilde{X})(y_3) < 0.5$ and $\overline{APR}(\widetilde{X})(y_3) < 0.5$, we conclude that this kind of new medicine has a little help in the treatment of the patient y_3.

Then, we can make a negative decision by analyzing $\underline{\underline{APR}}(\widetilde{X})$ and $\overline{\overline{APR}}(\widetilde{X})$under the critical value β=0.5:

(1). Since $\underline{\underline{APR}}(\widetilde{X})(y_4) \geq 0.5$ and $\overline{\overline{APR}}(\widetilde{X})(y_4) \geq 0.5$, we conclude that this kind of new medicine has great benefits for the treatment of the patient y_4.

(2). As $\underline{\underline{APR}}(\widetilde{X})(y_i) \geq 0.5$ and $\overline{\overline{APR}}(\widetilde{X})(y_i) < 0.5(i = 1, 2)$, we conclude that this kind of new medicine has some help in the treatment of the patient y_1 and y_2.

(3). According to $\underline{\underline{APR}}(\widetilde{X})(y_3) < 0.5$ and $\overline{\overline{APR}}(\widetilde{X})(y_3) < 0.5$, we conclude that this kind of new medicine has a little help in the treatment of the patient y_3.

We can see that there have some differences between the positive and negative decisions. In order to make a more accurate decision. Now we use dyad fuzzy β-covering rough set models $[(\underline{APR}, \overline{APR}), (\underline{\underline{APR}}, \overline{\overline{APR}})]$ to analyze this problem. By computing the$\underline{APR}(\widetilde{X})$(we say lower positive decision), $\overline{APR}(\widetilde{X})$(we say upper positive decision), $\underline{\underline{APR}}(\widetilde{X})$(we say lower negative decision), $\overline{\overline{APR}}(\widetilde{X})$(we say upper negative decision), under the critical value β=0.5, relationship among these decisions can be describe in the following Fig. 1.

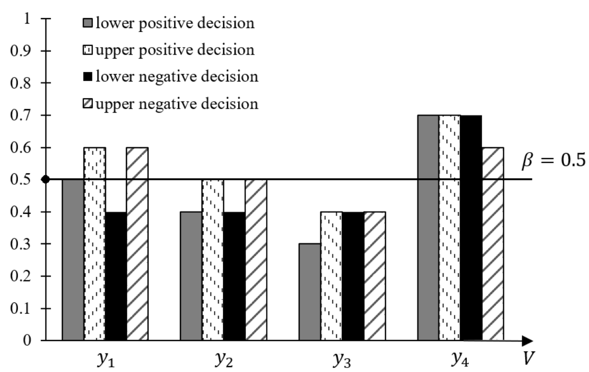

Figure 1: Four types of decision in Example 3.12 under the critical value β=0.5.

(1). Since all the decisions of patient y_4 are higher than the critical value, this kind of new medicine must be useful for patient y_4, so it should be given to him immediately.

(2). Because the three decisions of patient y_1 are not less than the critical value, this kind of new medicine has a great possibility to be useful to patient y_1, so suggest to take this new medicine and observe the effect.

(3). According to all the decisions of patient y_2 are no more than the critical value, and with two decisions is equal to the critical value, then this kind of new medicine may not be of great help to the treatment of patient y_2, its recommended to follow the doctor's advice and not blindly try this new medicine.

(4). Since all the decisions of patient y_3 are less than the critical value, this kind of new medicine has little help to patient y_3, so it is not recommended to try this new medicine.

Now, we can rank the efficacy of this new medicine on patients under the dyad fuzzy β-covering rough set models:

$$y_4 > y_1 > y_2 > y_3.$$

This result is more efficient for us to use new medicine to treatment patients.

In fact, in order to solve practical problems and make more accurate decisions, if we can analyze the issues using different methods from more aspects, then we can obtain a better conclusion. The Example 3.12 shows that analyzing the same problem from positive and negative aspects respectively usually get different decisions, and a more comprehensive decision can be obtained by considering two aspects at the same time, which fully reflects the advantages of dyad fuzzy β-covering rough set models to deal with practical problems.

4. Dyad L-Fuzzy β-Covering Rough Set Models

In this section, we generalize the model in Section 3 to fuzzy lattice and construct dyad L-fuzzy β-covering rough set models.

4.1. L-Fuzzy β-Covering Approximation Space

In this subsection, we introduce some preliminary definitions about lattice and define the concepts of L-fuzzy β-neighborhood and L-fuzzy β-co-neighborhood on this basis.

Definition 4.1. [29] *A lattice is a partially ordered set in which any two elements a and b have a minimum upper bound $a \vee b$ and a maximum lower bound $a \wedge b$.*

A lattice L is complete if any subset $A \subset L$ has a minimum upper bound $\vee A$ and a maximum lower bound $\wedge A$ in L.

A lattice L is completely distributive if the following conditions are met:

$$\wedge_{i\in I}(\vee_{j\in J_i} a_{ij}) = \vee_{f\in\Pi_{i\in I}J_i}(\wedge_{i\in I} a_{if(i)}),$$
$$\vee_{i\in I}(\wedge_{j\in J_i} a_{ij}) = \wedge_{f\in\Pi_{i\in I}J_i}(\vee_{i\in I} a_{if(i)}),$$

where $a_{ij} \in L$, I and J_i are nonempty index sets, and $f \in \Pi_{i\in I}J_i$ means that f is a mapping $f : I \longrightarrow \cup_{i\in I}J_i$ such that $f(i) \in J_i$ for each $i \in I$.

We simply call the complete completely distributive lattice CCD lattice.

Definition 4.2. [27] *Let $(L, \wedge, \vee, 0, 1)$ be a CCD lattice, where 0 and 1 represent the smallest element and largest element of L respectively. If there is an order-reversing involution $'\,: L \longrightarrow L$ on it, then the CCD lattice L is called a quasi-complemented lattice, and represented as $(L, \wedge, \vee, ', 0, 1)$. Through the order-reversing involution of a lattice L we refer to a mapping $'\,: L \longrightarrow L$ satisfying the following properties:*

(1).$'$ is an involution correspondence, i.e., $\forall a \in L.(a')' = a$;

(2).$'$ is an order-reversing map, i.e., $\forall a, b \in L, a \le b \Longleftrightarrow b' \le a'$;

Obviously, if $(L, \wedge, \vee, ', 0, 1)$ is a quasi-complemented lattice, the a' is unique for each $a \in L$.

Definition 4.3. [27] *Let U be the domain of discourse, and L be the quasi-complemented lattice. An L-fuzzy set $\widehat{X}$ of U is a mapping $\widehat{X} : U \longrightarrow L$.*

We use $\widehat{F}(U)$ to represent the set of all L-fuzzy sets in U. For $\widehat{X}, \widehat{Y} \in \widehat{F}(U)$, we define the relation $\widehat{X} \le \widehat{Y}$ (or $\widehat{X} \ge \widehat{Y}$), $\widehat{X} < \widehat{Y}$ (or $\widehat{X} > \widehat{Y}$) as

$$
\begin{aligned}
\widehat{X}(x) &\le \widehat{Y}(x) \quad (or \;\; \widehat{X}(x) \ge \widehat{Y}(x)), x \in U, \\
\widehat{X}(x) &< \widehat{Y}(x) \quad (or \;\; \widehat{X}(x) > \widehat{Y}(x)), x \in U,
\end{aligned}
$$

and operations $\sqcup$,$\sqcap$ and $'$ on $\widehat{F}(U)$ as

$$
\begin{aligned}
(\widehat{X} \sqcup \widehat{Y})(x) &= \widehat{X}(x) \vee \widehat{Y}(x), (\widehat{X} \sqcap \widehat{Y})(x) \\
&= \widehat{X}(x) \wedge \widehat{Y}(x), (\widehat{X})'(x) = (\widehat{X}(x))', x \in U.
\end{aligned}
$$

Obviously, L-fuzzy sets of U is an extension of the fuzzy set of U, which extends the range from interval [0,1] to a quasi-complement lattice L.

Definition 4.4. [27] *Let U be an arbitrary universal set and $L = (L, \wedge, \vee, ', 0, 1)$ be a quasi-complemented lattice. For a family of L-fuzzy sets $\widehat{C} = \left\{\widehat{C}_1, \widehat{C}_2, \cdots, \widehat{C}_m\right\}$ and some $\beta > 0 (\beta \in L)$, if $(\sqcup_{i=1}^{m} \widehat{C}_i)(x) \ge \beta$ for each $x \in U$. we call $\widehat{C}$ is the L-fuzzy β-covering of U, and $(U, \widehat{C})$ is called the L-fuzzy β-covering approximation space.*

Definition 4.5. *Let $(U, \widehat{C})$ a L-fuzzy β-covering approximation space, where $\widehat{C} = \left\{\widehat{C}_1, \widehat{C}_2, \cdots \widehat{C}_m\right\}$ is a L-fuzzy β-covering of U. For each $x \in U$, we define the L-fuzzy β-neighborhood $\widehat{N}_x^\beta$ and the L-fuzzy β-co-neighborhood $\widehat{M}_x^\beta$ of x as:*

$$
\begin{aligned}
\widehat{N}_x^\beta &= \sqcap \left\{\widehat{C}_i \in \widehat{C} : \widehat{C}_i(x) \ge \beta\right\}, \\
\widehat{M}_x^\beta &= - \sqcup \left\{\widehat{C}_i : \widehat{C}_i(x) < \beta, \widehat{C}_i \in \widehat{C}\right\} \\
&= \sqcap \left\{-\widehat{C}_i : 1 - \widehat{C}_i(x) > 1 - \beta, \widehat{C}_i \in \widehat{C}\right\}.
\end{aligned}
$$

4.2. Dyad L-Fuzzy β-Covering Rough Set Models

In this subsection, we first define the L-fuzzy information system. In addition, based on L-fuzzy β-neighborhood $\widehat{N}_x^\beta$ and L-fuzzy β-co-neighborhood $\widehat{M}_x^\beta$, we construct two novel L- rough set models respectively. Finally, we integrated these two models to define the dyad L-fuzzy β-covering rough set models and gives an example to illustrate.

Definition 4.6. *Let U and V be two arbitrary universal sets, and $f : U \longrightarrow V$ be a mapping from U to V. Then, a L-fuzzy information system is a quad $FIS_L = (U, \widehat{AT}, V, f)$, where $U = \{x_1, x_2, \cdots, x_n\}$ is a not empty limited set of objects, $\widehat{AT} = \left\{\widehat{A}_1, \widehat{A}_2, \cdots, \widehat{A}_m\right\} \in \widehat{F}(U)$ is a set of attributes describing the objects.*

Definition 4.7. *Let $FIS_L = (U, \widehat{AT}, V, f)$ be a L-fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, where $\widehat{AT} = \left\{\widehat{A}_1, \widehat{A}_2, \cdots, \widehat{A}_m\right\}$ is a L-fuzzy β-covering of U. Then, for each $\widehat{X} \in \widehat{F}(U)$, we define the 1^{st} L-fuzzy covering lower approximation $\underline{apr}(\widehat{X})$ and 1^{st} L-fuzzy covering upper approximation $\overline{apr}(\widehat{X})$ of $\widehat{X}$ as:*

$$\underline{apr}(\widehat{X})(y) = \wedge_{x \in f^{-1}(y)}[(\widehat{N}_x^{\beta}(x))' \vee \widehat{X}(x)], y \in V,$$
$$\overline{apr}(\widehat{X})(y) = \vee_{x \in f^{-1}(y)}[\widehat{N}_x^{\beta}(x) \wedge \widehat{X}(x)], y \in V.$$

If $\underline{apr}(\widehat{X}) \neq \overline{apr}(\widehat{X})$, then $\widehat{X}$ is called 1^{st} L-fuzzy β-covering rough set model.

Definition 4.8. *Let $FIS_L = (U, \widehat{AT}, V, f)$ be a L-fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, where $\widehat{AT} = \left\{\widehat{A}_1, \widehat{A}_2, \cdots, \widehat{A}_m\right\}$ is a L-fuzzy β-covering of U. Then, for each $\widehat{X} \in \widehat{F}(U)$, we define the 2^{nd} L-fuzzy covering lower approximation $\underline{\underline{apr}}(\widehat{X})$ and 2^{nd} L-fuzzy covering upper approximation $\overline{\overline{apr}}(\widehat{X})$ of $\widehat{X}$ as:*

$$underline\underline{apr}(\widehat{X})(y) = \wedge_{x \in f^{-1}(y)}[(\widehat{N}_x^{\beta}(x))' \vee \widehat{X}(x)], y \in V,$$
$$\overline{\overline{apr}}(\widehat{X})(y) = \vee_{x \in f^{-1}(y)}[\widehat{N}_x^{\beta}(x) \wedge \widehat{X}(x)], y \in V.$$

If $\underline{\underline{apr}}(\widehat{X}) \neq \overline{\overline{apr}}(\widehat{X})$, then $\widehat{X}$ is called 2^{nd} L-fuzzy β-covering rough set model.

The properties of the models defined in Definitions 4.7 and 4.8 are similar to those studied in Section 3. Therefore, the researches and proofs of the properties about these two models are omitted here, and L-fuzzy β-covering rough set models is constructed directly.

Definition 4.9. *Let $FIS_L = (U, \widehat{AT}, V, f)$ be a L-fuzzy information system and $f : U \longrightarrow V$ be a surjective mapping from U to V, where $\widehat{AT} = \left\{\widehat{A}_1, \widehat{A}_2, \cdots, \widehat{A}_m\right\}$ is a L-fuzzy β-covering of U. Then, for each $\widehat{X} \in \widehat{F}(U)$, $\underline{apr}(\widehat{X})$ and $\overline{apr}(\widehat{X})$ are*

Table 5: A fuzzy information system $FIS_L = (U, \widehat{AT})$

	$\widehat{A}_1$	$\widehat{A}_2$	$\widehat{A}_3$	$\widehat{A}_4$	$\widehat{A}_5$
x_1	5	6	30	15	3
x_2	10	15	3	6	6
x_3	10	30	6	10	2
x_4	6	6	5	10	30

1^{st} *L-fuzzy covering lower and upper approximation of* $\widehat{X}$ *respectively,* $\underline{\underline{apr}}(\widehat{X})$ *and* $\overline{\overline{apr}}(\widehat{X})$ *are* 2^{nd} *L-fuzzy covering lower and upper approximation of* $\widehat{X}$ *respectively, we call the two-pair of models* $[(\underline{apr}, \overline{apr}), (\underline{\underline{apr}}, \overline{\overline{apr}})]$ *dyad L-fuzzy* β*-covering rough set models.*

Finally, an example is given to illustrate these concepts.

Example 4.10. Let $FIS_L = (U, \widehat{AT}, V, f)$ be a L-fuzzy information system, where $U = \{x_1, x_2, x_3, x_4\}$, $L = \{1, 2, 3, 5, 6, 10, 15, 30\}$ be a set of numbers, and $(L, \wedge, \vee, ', 0, 1)$ be a quasi-complemented lattice in which 0=1, 1=30, $a' = \frac{30}{a}$, $a \vee b$ and $a \wedge b$ are represent the least common multiple and greatest common divisor of a, b respectively. The lattice L is shown in Fig. 2, where the lower number connected by line segments is a factor of the upper numbers. And the $\widehat{AT} = \left\{\widehat{A}_1, \widehat{A}_2, \widehat{A}_3, \widehat{A}_4, \widehat{A}_5\right\}$ is shown as follows (Table 5).

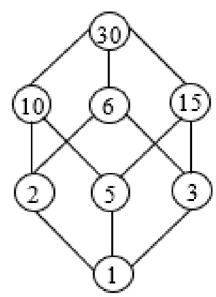

Figure 2: $(L, \wedge, \vee, ', 0, 1)$ in Example 4.10

Then $\widehat{AT} = \left\{\widehat{A}_1, \widehat{A}_2, \widehat{A}_3, \widehat{A}_4, \widehat{A}_5\right\}$ is a L-fuzzy β-covering of U and $\beta \in \{1, 2, 3, 5, 6, 10, 15, 30\}$. Let $\beta = 6$.

Table 6: L-fuzzy 6-neighborhood $\widehat{N}^6_{x_i}$

	x_1	x_2	x_3	x_4
$\widehat{N}^6_{x_1}$	6	3	6	1
$\widehat{N}^6_{x_2}$	3	6	2	10
$\widehat{N}^6_{x_3}$	6	3	6	1
$\widehat{N}^6_{x_4}$	1	1	2	6

Let $V = \{y_1, y_2, y_3\}$, $f : U \longrightarrow V$ be a mapping from U to V, and

$$f(x) = \begin{cases} y_1 & \text{if } \ x = x_1, \\ y_2 & \text{if } \ x \in \{x_2, x_4\}, \\ y_3 & \text{if } \ x = x_3. \end{cases}$$

For

$$\widehat{X} = \frac{6}{x_1} + \frac{2}{x_2} + \frac{15}{x_3} + \frac{10}{x_4}$$

First, we compute the 1^{st} L-fuzzy covering lower and upper approximation of $\widehat{X}$. It is easy to find that

$$\widehat{N}^6_{x_1} = \widehat{A}_2 \sqcap \widehat{A}_3, \widehat{N}^6_{x_2} = \widehat{A}_4 \sqcap \widehat{A}_5, \widehat{N}^6_{x_3} = \widehat{A}_2 \sqcap \widehat{A}_3, \widehat{N}^6_{x_4} = \widehat{A}_1 \sqcap \widehat{A}_2 \sqcap \widehat{A}_5,$$

Hence, we obtain immediately the L-fuzzy 6-neighborhood $\widehat{N}^6_{x_i}$ of $x_i (i = 1, 2, 3, 4)$ as follows (Table 6).

Hence, we obtain that

$$\begin{aligned} \underline{apr}(\widehat{X}) &= \frac{3}{y_1} + \frac{10}{y_2} + \frac{15}{y_3}, \\ \overline{apr}(\widehat{X}) &= \frac{6}{y_1} + \frac{2}{y_2} + \frac{3}{y_3} \end{aligned}$$

Next, we compute the 2^{nd} L-fuzzy covering lower and upper approximation of $\widehat{X}$. It is easy to find that

$$\widehat{M}^6_{x_1} = \widehat{A}'_5, \widehat{M}^6_{x_2} = \widehat{A}'_3, \widehat{M}^6_{x_3} = \widehat{A}'_5, \widehat{M}^6_{x_4} = \widehat{1_U}'$$

Thus, we can calculate the L-fuzzy 6-co-neighborhood $\widehat{M}^6_{x_i}$ of $x_i (i = 1, 2, 3, 4)$ as follows (Table 7).

Finally, we obtain the followings

$$\begin{aligned} \underline{\underline{apr}}(\widehat{X}) &= \frac{6}{y_1} + \frac{2}{y_2} + \frac{30}{y_3}, \\ \overline{\overline{apr}}(\widehat{X}) &= \frac{2}{y_1} + \frac{10}{y_2} + \frac{15}{y_3}. \end{aligned}$$

Table 7: L-fuzzy 6-co-neighborhood $\widehat{M}^6_{x_i}$

	x_1	x_2	x_3	x_4
$\widehat{M}^6_{x_1}$	10	5	15	1
$\widehat{M}^6_{x_2}$	1	10	5	6
$\widehat{M}^6_{x_3}$	10	5	15	1
$\widehat{M}^6_{x_4}$	30	30	30	30

Thus, we obtain the dyad L-fuzzy β-covering rough set models $[(\underline{apr}, \overline{apr}),$ $(\underline{\underline{apr}}, \overline{\overline{apr}})]$ in this example.

5. Conclusion.

The twin approximation operator is a concept proposed by L.W. Ma in [28] to make decision-making judgments from various aspects. In this paper, we extend this concept to the information system, and propose a new modeldyad fuzzy β-covering rough set models, which is a combination of covering rough set theory, fuzzy rough set theory and fuzzy information system. The main conclusions of this paper and the work to be done are as follows.

(1) 1^{st} fuzzy β-covering rough set model and 2^{nd} fuzzy β-covering rough set model are defined based on fuzzy information system, properties of these two models are investigated and practical examples are provided.

(2) A new model is defined dyad fuzzy β-covering rough set models, and the practical value of the model is illustrated through an example.

(3) The above model is generalized to fuzzy lattice and construct L-dyad fuzzy β-covering rough set models.

(4) There are still some problems worthy of further study in fuzzy information system decision-making. For example, how to use other mathematical concepts to simplify the model proposed in this paper, so that it is faster and more convenient when dealing with practical big data applications. In addition, the application of dyad fuzzy β-covering rough set models under multi-source information systems is also an area worth exploring. These issues will be studied in our future research.

References

[1] W. Bartol, J. Miro, K. Pioro, F. Rossello, On the coverings by tolerance classes, *Inf. Sci.* **166** (2004) 193–211.

[2] D. Bianucci, G. Cattaneo, D. Ciucci, Entropies and co-entropies of coverings with application to incomplete information systems, *Fundam. Inform.* **75** (2007) 77–105.

[3] Z. Bonikowski, E. Bryniarski, U. Wybraniec-Skardowska, Extensions and intentions in the rough set theory, *Inf. Sci.* **107** (1) (1998) 149–167.

[4] J. Chen, J. Li, Y. Lin, G. Lin, Z. Ma, Relations of reduction between covering generalized rough sets and concept lattices, *Inf. Sci.* **304** (2015) 16–27.

[5] C. Cornelis, R. Jensen, G. Hurtado, D. Slezak, Attribute selection with fuzzy decision reducts, *Inf. Sci.* **180** (2010) 209–224.

[6] J. Dai, S. Gao, G. Zheng, Generalized rough set models determined by multiple neighborhoods generated from a similarity relation, *Soft Comput.* **12** (2017) 1–14.

[7] J. Dai, W. Wang, Q. Xu, H. Tian, Uncertainty measurement for interval-valued decision systems based on extended conditional entropy, *Knowl.-Based Syst.* **27** (2012) 443–450.

[8] L. Deer, C. Cornelis, A comprehensive study of fuzzy covering-based rough set models: definitions, properties and interrelationships, *Fuzzy Sets Syst.* **336** (2018) 1–26.

[9] L. Deer, C. Cornelis, L. Godo, Fuzzy neighborhood operators based on fuzzy coverings, *Fuzzy Sets Syst.* **312** (2017) 17–35.

[10] W.S. Du, B.Q. Hu, Dominance-based rough set approach to incomplete ordered information systems, *Inf. Sci.* **346-347** (2016) 106–129.

[11] W.S. Du, B.Q. Hu, Attribute reduction in ordered decision tables via evidence theory, *Inf. Sci.* **364-365** (2016) 91–110.

[12] Y. Du, Q. Hu, P. Zhu, P. Ma, Rule learning for classification based on neighborhood covering reduction, *Inf. Sci.* **181** (2011) 5457–5467.

[13] T. Feng, S. Zhang, J. Mi, The reduction and fusion of fuzzy covering systems based on the evidence theory, *Int. J. Approx. Reason.* **53** (2012) 87–103.

[14] M.R. Hooshmandasl, A. Shakiba, A.K. Goharshady, A. Karimi, S-approximation: a new approach to algebraic approximation, *J. Discrete Math.* (2014) 909684.

[15] B. Huang, C. Guo, H. Li, G. Feng, X. Zhou, An intuitionistic fuzzy graded covering rough set, *Knowl.-Based Syst.* **107** (2016) 155–178.

[16] G. Lang, Q. Li, M. Cai, T. Yang, Characteristic matrixes-based knowledge reduction in dynamic covering decision information systems, *Knowl.-Based Syst.* **85** (2015) 1–26.

[17] F. Li, Y. Yin, Approaches to knowledge reduction of covering decision systems based on information theory, *Inf. Sci.* **179** (2009) 1694–1704.

[18] T. Li, Y. Leung, W. Zhang, Generalized fuzzy rough approximation operators based on fuzzy coverings, *Int. J. Approx. Reason.* **48** (2008) 836–856.

[19] T.J. Li, Y. Leung, W.X. Zhang, Generalized fuzzy rough approximation operators based on fuzzy coverings, *Int. J. Approx. Reason.* **48** (2008) 836–856.

[20] T.J. Li, J.M. Ma, Fuzzy approximation operators based on coverings, In: *Proceedings of Joint Rough Set Symposium*, Conference: Rough Sets, Fuzzy Sets, Data Mining and Granular Computing, 11th International Conference, 2007.

[21] X. Li, H. Yi, S. Liu, Rough sets and matroids from a lattice-theoretic viewpoint, *Inf. Sci.* **342** (2016) 37–52.

[22] Z. Li, Y. Zhan, Fuzzy soft -covering based fuzzy rough sets and corresponding decision-making applications, *Int. J. Mach. Learn. Cybern.* **1** (2018) 1–16.

[23] C. Luo, T. Li, Y. Yao, Dynamic probabilistic rough sets with incomplete data, *Inf. Sci.* **417** (2017) 39–54.

[24] Y. Jing, T. Li, H. Fujita, Z. Yu, B. Wang, An incremental attribute reduction approach based on knowledge granularity with a multi-granulation view, *Inf. Sci.* **411** (2017) 23–38.

[25] M. Kryszkiewicz, Rough set approach to incomplete information systems, *Inform. Sci.* **112** (1998) 39–49.

[26] L. Ma, The investigation of covering rough sets by Boolean matrices, *Int. J. Approx. Reason.* **100** (2018) 69–84.

[27] L.W. Ma, Couple fuzzy covering rough set models and their generalizations to CCD lattices, *Int. J. Approx. Reason.* **126** (2020) 48–69.

[28] L.W. Ma, On some types of neighborhood-related covering rough sets, *Int. J. Approx. Reason.* **53** (2012) 901–911.

[29] L.W. Ma, Two fuzzy covering rough set models and their generalizations over fuzzy lattices, *Fuzzy Sets Syst.* **294** (2016) 1–17.

[30] Z. Pawlak, Information systemstheoretical foundations, *Inform. Syst.* **6** (3) (1981) 205-218.

[31] Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.* **11** (1982) 341–356.

[32] Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning About Data, Theory and Decision Library: System Theory, Knowledge Engineering, and Problem Solving*, Kluwer Academic Publishers, 1991.

[33] Z. Pawlak, Vagueness and uncertainty: a rough set perspective, *Comput. Intell.* **11** (2) (1995) 227–232.

[34] Z. Pawlak, A. Skowron, Rough sets and boolean reasoning, *Inf. Sci.* **177** (2007) 41–73.

[35] W. Pedrycz, M. Song, Analytic hierarchy process (ahp) in group decision making and its optimization with an allocation of information granularity, *IEEE Trans. Fuzzy Syst.* **19** (2011) 527–539.

[36] J. Pomykaa, On definability in the nondeterministic information system, *Bull. Pol. Acad. Sci. Math.* **36** (3–4) (1988) 193–210.

[37] Y. Qian, J. Liang, W. Pedrycz, C. Dang, Positive approximation: an accelerator for attribute reduction in rough set theory, *Artif. Intell.* **174** (2010) 597–618.

[38] B. eelja, L-fuzzy covering relation, *Fuzzy Sets Syst.* **158** (2007) 2456–2465.

[39] G. Shafer, *A Mathematical Theory of Evidence*, Vol. **1**, Princeton University Press, Princeton, 1976.

[40] A. Shakiba, M.R. Hooshmandasl, Data volume reduction in covering approximation spaces with respect to twenty-two types of covering based rough sets, *Int. J. Approx. Reason.* **75** (2016) 13–38.

[41] A. Shakiba, M. Hooshmandasl, S-approximation spaces: a three-way decision approach, *Fundam. Inform.* **139** (3) (2015) 307–328.

[42] C. Wang, D. Chen, Q. Hu, Fuzzy information systems and their homomorphisms, *Fuzzy Sets Syst.* **249** (2014) 128–138.

[43] C. Wang, Y. Shi, X. Fan, M. Shao, Attribute reduction based on -nearest neighborhood rough sets, *Int. J. Approx. Reason.* **106** (2019) 18–31.

[44] J. Wang, X. Zhang, Matrix approaches for some issues about minimal and maximal descriptions in covering-based rough sets, *Int. J. Approx. Reason.* **104** (2019) 126–143.

[45] J. Wang, X. Zhang, Multigranulation single valued neutrosophic covering-based rough sets and their applications to multi-criteria group decision making, *Iran. J. Fuzzy Syst.* **17** (2020) 109–126.

[46] J. Wang, W. Zhu, Applications of bipartite graphs and their adjacency matrices to covering-based rough sets, *Fundam. Inform.* **156** (2017) 237–254.

[47] J. Wang, W. Zhu, F. Wang, G. Liu, Conditions for coverings to induce matroids, *Int. J. Mach. Learn. Cybern.* **5** (2014) 947–954.

[48] W. Wu, Attribute reduction based on evidence theory in incomplete decision systems, *Inf. Sci.* **178** (2008) 1355–1371.

[49] B. Yang, B.Q. Hu, A fuzzy covering-based rough set model and its generalization over fuzzy lattice, *Inf. Sci.* **367-368** (2016) 463–486.

[50] B. Yang, B.Q. Hu, Communication between fuzzy information systems using fuzzy covering-based rough sets, *Int. J. Approx. Reason.* **103** (2018) 414–436.

[51] B. Yang, B.Q. Hu, Fuzzy neighborhood operators and derived fuzzy coverings, *Fuzzy Sets Syst.* **370** (2019) 1-33.

[52] B. Yang, B.Q. Hu, On some types of fuzzy covering-based rough sets, *Fuzzy Sets Syst.* **312** (2017) 36–65.

[53] J. Yao, A. Vasilakos, W. Pedrycz, Granular computing: perspectives and challenges, *IEEE Trans. Cybern.* **43** (6) (2013) 1977–1989.

[54] Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, *Inf. Sci.* **111** (1) (1998) 239–259.

[55] Y. Yao, On generalizing rough set theory, In: *Rough Sets, Fuzzy Sets, Data Mining and Granular Computing*, Ed. by G. Wang, Q. Liu, Y. Yao, A. Skowron, Vol. **2639**, Springer, Berlin, Heidelberg, 2003.

[56] Y. Yao, B. Yao, Covering based rough set approximations, *Inf. Sci.* **200** (2012) 91–107.

[57] Y. Yao, Y. Zhao, Attribute reduction in decision-theoretic rough set models, *Inf. Sci.* **178** (2008) 3356–3373.

[58] Z. Yun, X. Ge, X. Bai, Axiomatization and conditions for neighborhoods in a covering to form a partition, *Inf. Sci.* **181** (2011) 1735–1740.

[59] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, *Inf. Sci.* **8** (1975) 199–249.

[60] L. Zadeh, Fuzzy sets, *Inf. Control* **8** (3) (1965) 338–353.

[61] L. Zadeh, The role of fuzzy logic in the management of uncertainty in expert systems, *Fuzzy Sets Syst.* **11** (1) (1983) 197–198.

[62] L. Zadeh, Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic, *Fuzzy Sets Syst.* **90** (2) (1997) 111–127.

[63] W. Zakowski, Axiomatization in the space (U,), *Demonstr. Math.* **XVI** (1983) 761–769.

[64] J. Zhan, W. Xu, Two types of coverings based multigranulation rough fuzzy sets and applications to decision making, *Artif. Intell. Rev.* **1** (2018) 1–32.

[65] X. Zhang, J. Dai, Y. Yu, On the union and intersection operations of rough sets based on various approximation spaces, *Inf. Sci.* **292** (2015) 214–229.

[66] X. Zhang, D. Miao, C. Liu, M. Le, Constructive methods of rough approximation operators and multigranulation rough sets, *Knowl.-Based Syst.* **91** (2016) 114–125.

[67] W. Zhu, F. Wang, Reduction and axiomatization of covering generalized rough sets, *Inf. Sci.* **152** (2003) 217–230.