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On Green's Relations Which Are Related to an Algebraic System of Type $((n);(m))$

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Dedicated to the memory of Professor Yuqi Guo (1940–2019)

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Abstract. Green's relations are five equivalence relations which defined on semigroup or monoid. In the present paper, we consider a semigroup with the Cartesian product of the set of terms and the set of atomic formulas as universe. This semigroup extends the idea of the study of semigroups of terms of a given type. We characterize the Green´s relations for that semigroup.

Keywords: Algebraic system; Term; Formula; Relational term; Generalized superposition; Green's relation.

1. Introduction and Basic Concepts

In algebra, the algebraic structure of a given universal algebra is the study an important problem and the semigroup is an important structure. In fact, a semigroup is a pair of a non-empty set and an associative binary operation on this set. In [4, 17], the structure of a semigroup can be characterized by Green's relations. In [8], the authors study the regular elements as well as the Green's relations of linear terms of $\tau = (n, \ldots, n)$, $n \in \mathbb{N}$. In [13], Phusanga and Koppitz extended it for the partial operations [8] and characterized the Green's relations. A hypersubstitutions of type τ is a mapping which takes the operation symbol to the term preserving arities. Hypersubstitutions of type τ are defined in order to define hyperidentities, i.e. identities which are defined for algebras of the corresponding type in the stronger sense that they are valid after substituting the occurring operation symbols by terms [2]. A binary operation $\circ_h$ on the set $Hyp(\tau)$ of all hypersubstitutions of type τ was introduced such that $(Hyp(\tau); \circ_h)$ is a monoid. This monoid was studied intensively for several types τ. For the types $\tau = (n)$, Wismath studied the semigroup properties of $Hyp(n)$, characterize the projection, dual and idempotent elements, and describe the classes of these element under Green's relations (see [19]). In [1], Changphas and Denecke characterize Greens relation $\mathcal{R}$ on the monoid $Hyp(\tau)$ for the types $\tau = (n)$ and $\tau = (n, n)$. In [18], Wismath uses it as a tool to study fundamental-M-Solid and fundamental-M-closed varieties. Leeratanavalee and Denecke extended the concept of hypersubstitution to generalized hypersubstitutions and have used it as a tool to study strong hyperidentities and use strong hyperidentities to classify varieties into collections called strong hypervarieties. Varieties which are closed under arbitrary application of generalized hypersubstitutions are called strongly solid [7]. In [16], Puninagool and Leeratanavalee characterized the Greens relations on $Hyp_G(2)$, set of all generalized hypersubstitutions of type (2).

On the other hand, we can consider algebraic systems in the sense of Mal'cev [9]. An *algebraic system* of type (τ, τ') is a triple $\mathcal{A} := (A; (f_i^{\mathcal{A}})_{i\in I}, (\gamma_j^{\mathcal{A}})_{j\in J})$ consisting of a non-empty set A, a sequence $(f_i^{\mathcal{A}})_{i\in I}$ of operations on A indexed by the index set I, where $f_i^{\mathcal{A}} : A^{n_i} \to A$ is an n_i-ary operation for $i \in I$ and a sequence $(\gamma_j^{\mathcal{A}})_{j\in J}$ of relations on A indexed by the index set J, where $\gamma_j^{\mathcal{A}} \subseteq A^{m_j}$ is an m_j-ary relation for $j \in J$. The pair (τ, τ') with $\tau = (n_i)_{i\in I},\ \tau' = (m_j)_{j\in J}$ of sequences of integers $n_i, m_j \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ is called the *type* of the algebraic system $\mathcal{A}$. Due to Mal'cev, algebraic systems are related to the concepts of term [2] and formula [9, 10, 3]. Let $n \in \mathbb{N}^+$, let $X_n = \{x_1, x_2, \ldots, x_n\}$ be an n-element set of variables, and let $X := \bigcup_{1\leq n} X_n = \{x_1, \ldots, x_n, \ldots\}$ be countably infinite. Then the set $W_\tau(X_n)$ of all n-ary terms of type τ is defined in the usual way by the following conditions :

(i) Every $x_i \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and if f_i is an n_i-ary operation symbol of type τ, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

Let $W_\tau(X) := \bigcup_{n\geq 1} W_\tau(X_n)$ be the set of all terms of type τ.

To define formulas of type (τ, τ'), we need the logical connectives $\neg$ (for negation), $\vee$ (for disjunction), the equation symbol $\approx$, and the quantifier $\exists$. The classical definition of an n-ary formula is given by Mal'cev [9], pp. 115-116. The variable x_i occurs freely in a formula F means that the quantifier $\exists$ does not occur in front of x_i in F. Otherwise x_i is called bound. We notice that all free or bound variables occur in X_n. Let $n \geq 1$. An n-ary formula of type (τ, τ') is defined in the following inductive way:

(i) If t_1, t_2 are n-ary terms of type τ, then the equation $t_1 \approx t_2$ is an n-ary formula of type (τ, τ'). All variables in $t_1 \approx t_2$ are free.

(ii) If $t_1, \ldots, t_{m_j}$ are n-ary terms of type τ and if γ_j is an m_j-ary relational symbol, then $\gamma_j(t_1, \ldots, t_{m_j})$ is an n-ary formula of type (τ, τ'). All variables in such a formula are free.

(iii) If F is an n-ary formula of type (τ, τ'), then $\neg F$ is an n-ary formula of type (τ, τ'). All free variables in F are also free in $\neg F$. All bound(free, respectively) variables in F are also bound(free, respectively) in $\neg F$.

(iv) If F_1 and F_2 are n-ary formulas of type (τ, τ') such that variables occurring simultaneously in both formulas are free in each of them, then $F_1 \vee F_2$ is an n-ary formula of type (τ, τ'). Variables that are free in at least one of the formulas F_1 or F_2 are also free in $F_1 \vee F_2$. Variables that are bound in either F_1 or F_2 are also bound in $F_1 \vee F_2$.

(v) If F is an n-ary formula of type (τ, τ') and $x_i \in X_n$ occurs freely in F, then $\exists x_i(F)$ is an n-ary formula of type (τ, τ').

Let $\mathcal{F}_{(\tau,\tau')}(X_n)$ be the set of all n-ary formulas of type (τ, τ') and let $\mathcal{F}_{(\tau,\tau')}(X) := \bigcup_{n\geq 1} \mathcal{F}_{(\tau,\tau')}(X_n)$ be the set of all formulas of type (τ, τ'). A formula which is defined by (i) and (ii), we will call *an atomic formula* of type (τ, τ'). A formula having the form $\gamma_j(t_1, \ldots, t_{m_j})$, we will call *an n-ary relational term* of type (τ, τ'). Let ${F^*}_{(\tau,\tau')}(X)$ be the set of all relational terms of type (τ, τ') [12].

First, we recall the concept of a generalized superposition of terms [7]. Let $n \in \mathbb{N}^+$. The operation

$$S^n : W_\tau(X) \times (W_\tau(X))^n \to W_\tau(X)$$

is defined by the following steps:

(i) If $t = x_i; 1 \leq i \leq n$, then $S^n(x_i, t_1, \ldots, t_n) := t_i$.

(ii) If $t = x_i; n < i \in \mathbb{N}^+$, then $S^n(x_i, t_1, \ldots, t_n) := x_i$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then $S^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))$.

Now, we want to extend this generalized superposition to relational terms. If we substitute variables occurring in a relational term by terms, we obtain a new relational term.

Definition 1.1. *Let $t_1, \ldots, t_n \in W_\tau(X)$. The operation*

$$R^n : (W_\tau(X) \cup {F^*}_{(\tau,\tau')}(X)) \times (W_\tau(X))^n \to W_\tau(X) \cup {F^*}_{(\tau,\tau')}(X)$$

is defined by the following inductive steps:

(i) *If $t \in W_\tau(X)$, then $R^n(t, t_1, \ldots, t_n) := S^n(t, t_1, \ldots, t_n)$.*

(ii) *If $j \in J$ and $s_1, \ldots, s_{m_j} \in W_\tau(X)$, then $R^n(\gamma_j(s_1, \ldots, s_{m_j}), t_1, \ldots, t_n) := \gamma_j(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{m_j}, t_1, \ldots, t_n))$.*

In [11], the properties of R^n corresponding to the clone properties (FC1) and (FC3) (see, e.g., [5]) are proved as follows:

Theorem 1.2. [11] *Let $\beta \in W_\tau(X) \cup F^*{}_{(\tau,\tau')}(X)$. The operation R^n satisfies:*

(FC1) $R^n(R^n(\beta, t_1, \ldots, t_n), s_1, \ldots, s_n) = R^n(\beta, R^n(t_1, s_1, \ldots, s_n), \ldots, R^n(t_n, s_1, \ldots, s_n))$, *whenever* $t_1, \ldots, t_n, s_1, \ldots, s_n \in W_\tau(X)$.

(FC3) $R^n(\beta, x_1, \ldots, x_n) = \beta$.

A relational hypersubstitution for algebraic systems of type (τ, τ') is a mapping which maps any n_i-ary operation symbol to an n_i-ary term and maps any m_j-ary relational symbol to an m_j-ary relational term. Some algebraic properties of the monoid of all relational hypersubstitutions for algebraic systems of a special type. In [6], the authors study the Greens relations on the regular part of this monoid of a particular type $(\tau, \tau') = ((n), (m))$, where $m, n \geq 2$. In present paper, we study a semigroup, whose universe is the Cartesian product of the set of all terms of type τ and a particular subset of the set of all relational terms of type (τ, τ').

Let S be a semigroup and $1 \notin S$. We extended the binary operation from S to $S \cup 1$ by define $x1 = 1x = x$ for all $x \in S \cup 1$. Then $S^1 = S \cup 1$ is a semigroup with identity 1. We define the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ on S as follow:

$$\begin{aligned}
a\mathcal{L}b &\Leftrightarrow a = xb \text{ and } b = ya \text{ for some } x, y \in S^1,\\
a\mathcal{R}b &\Leftrightarrow a = bx \text{ and } b = ay \text{ for some } x, y \in S^1,\\
a\mathcal{H}b &\Leftrightarrow a\mathcal{L}b \text{ and } a\mathcal{R}b,\\
a\mathcal{D}b &\Leftrightarrow a\mathcal{L}c \text{ and } c\mathcal{R}b \text{ for some } c \in S^1,\\
a\mathcal{J}b &\Leftrightarrow a = xby \text{ and } b = zau \text{ for some } x, y, u, z \in S^1.
\end{aligned}$$

We call the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ the Greens relations on S [14, 15].

The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ on S are equivalent relations. We have $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$.

Let (τ, τ') be any type and let

$$W \times F^*_{(\tau,\tau')}(X) := \{(t, F) : t \in W_\tau(X), F \in F^*_{(\tau,\tau')}(X)\}.$$

For any $n \in \mathbb{N}^+$. For $t \in W_\tau(X)$, let

$$\begin{aligned}
W^n_\tau(t, X) &:= \{S^n(s, t, \ldots, t) : s \in W_\tau(X)\},\\
F^{*,n}_{(\tau,\tau')}(t, X) &:= \{R^n(G, t, \ldots, t) : G \in F^*_{(\tau,\tau')}(X)\}.
\end{aligned}$$

We define a mapping

$$+_n : (W \times F^*_{(\tau,\tau')}(X))^2 \to W \times F^*_{(\tau,\tau')}(X)$$

by $(t, F) +_n (t', F') := (S^n(t', t, \ldots, t), R^n(F', t, \ldots, t))$. Because of Theorem 1.2 (FC1) in [11], it is easy to verify that $+_n$ is an associative operation on $W \times F^*_{(\tau,\tau')}(X)$.

2. On Green's Relations Which Are Related to an Algebraic System of Type $((n);(m))$

We will study the Green's relations on $(W \times F^*_{(\tau,\tau')}(X), +_n)^{\mathbf{1}}$, where $\mathbf{1}$ is the admitted identity element, such that $(W \times F^*_{(\tau,\tau')}(X), +_n)^{\mathbf{1}}$ is a monoid. For an n-ary term t and an m-ary relational term F, we introduce the following notations:

$var(t) :=$ the set of all variables occurring in t,

$var(F) :=$ the set of all variables occurring in F,

$l(t) :=$ the length of t, i.e. the number the variable occurring in t.

We start with the $\mathcal{R}$-relation.

Proposition 2.1. *Let* $(t_1, F_1), (t_2, F_2) \in W \times F^*_{(\tau,\tau')}(X)$. *Then* $(t_1, F_1)\mathcal{R}(t_2, F_2)$ *if and only if* (1) *or* (2) *or* (3) *is satisfied:*

(1) $t_1 = t_2$ *with* $var(t_1) \cap X_n \neq \emptyset$ *and* $F_i \in F^{*,n}_{(\tau,\tau')}(t_1, X)$, *for* $i = 1, 2$.

(2) $(var(t_1) \cup var(t_2) \cup var(F_1) \cup var(F_2)) \cap X_n = \emptyset$.

(3) $(t_1, F_1) = (t_2, F_2)$

Proof. Suppose that $(t_1, F_1)\mathcal{R}(t_2, F_2)$ and $(t_1, F_1) \neq (t_2, F_2)$. Then there are $(t, F), (t', F') \in W \times F^*_{(\tau,\tau')}(X)$ such that $(t_1, F_1) +_n (t, F) = (t_2, F_2)$ and $(t_2, F_2) +_n (t', F') = (t_1, F_1)$. This implies $t_2 = S^n(t, t_1, \ldots, t_1)$, $F_2 = R^n(F, t_1, \ldots, t_1)$, $t_1 = S^n(t', t_2, \ldots, t_2)$, and $F_1 = R^n(F', t_2, \ldots, t_2)$. This provides $var(t_1) \cap X_n \neq \emptyset$ if and only if $var(t_2) \cap X_n \neq \emptyset$. Then $var(t_1) \cap X_n \neq \emptyset$ and $t_1 = S^n(t, t_2, \ldots, t_2)$ implies $l(t_2) < l(t_1)$ or $t_1 = t_2$. On the other hand, $var(t_2) \cap X_n \neq \emptyset$ and $t_2 = S^n(t', t_1, \ldots, t_1)$ implies $l(t_1) < l(t_2)$ or $t_1 = t_2$. Since $l(t_2) < l(t_1)$ and $l(t_1) < l(t_2)$ is not possible, we obtain either $t_1 = t_2$ with $var(t_1) \cap X_n \neq \emptyset$ or $(var(t_1) \cup var(t_2)) \cap X_n = \emptyset$. If $(var(t_1) \cup var(t_2)) \cap X_n = \emptyset$ then $F_2 = R^n(F, t_1, \ldots, t_1)$ and $F_1 = R^n(F', t_2, \ldots, t_2)$ implies $(var(F_1) \cup var(F_2)) \cap X_n = \emptyset$. Suppose that $(var(t_1) \cup var(t_2)) \cap X_n \neq \emptyset$ and $t_1 = t_2$. Then $F_2 = R^n(F, t_1, \ldots, t_1)$ and $var(F_2) \cap X_n \neq \emptyset$ implies $F_2 \in F^{*,n}_{(\tau,\tau')}(t_1, X)$. Dually, we obtain $F_1 \in F^{*,n}_{(\tau,\tau')}(t_1, X)$.

The converse direction can be proved by straightforward calculations. ∎

Now we study the $\mathcal{L}$-relation.

Notation 2.2. For $t \in W_\tau(X)$ and $F \in F^{*,n}_{(\tau,\tau')}(X)$, we put

$$(t, F)^{X_n} := \{(S^n(t, a, \ldots, a), R^n(F, a, \ldots, a)) \mid a \in X_n\}.$$

Note that $(s, G) \in (t, F)^{X_n}$ and $(t, F) \in (s, G)^{X_n}$ implies $(s, G), (t, F) \in (t, F)^{X_n} = (s, G)^{X_n}$.

Proposition 2.3. *Let* $(t_1, F_1), (t_2, F_2) \in W \times F^*_{(\tau,\tau')}(X)$. *Then* $(t_1, F_1)\mathcal{L}(t_2, F_2)$ *if and only if* $(t_1, F_1) = (t_2, F_2)$ *or* $(t_1, F_1), (t_2, F_2) \in (t_1, F_1)^{X_n} = (t_2, F_2)^{X_n}$.

Proof. Suppose that $(t_1, F_1)\mathcal{L}(t_2, F_2)$ with $(t_1, F_1) \neq (t_2, F_2)$. Then there are $(t, F), (t', F') \in W \times F^*_{(\tau,\tau')}(X)$ such that $(t, F) +_n (t_1, F_1) = (t_2, F_2)$ and $(t', F') +_n (t_2, F_2) = (t_1, F_1)$. This provides

(1) $t_2 = S^n(t_1, t, \ldots, t)$ and $t_1 = S^n(t_2, t', \ldots, t')$ and

(2) $F_2 = R^n(F_1, t, \ldots, t)$ and $F_1 = R^n(F_2, t', \ldots, t')$.

From (1), it follows $l(t_1) \leq l(t_2)$ and $l(t_2) \leq l(t_1)$, i.e. $l(t_2) = l(t_1)$. Then again by (1), we obtain $t, t' \in X_n$ or $t_1 = t_2$ with $(var(t_1) \cup var(t_2)) \cap X_n = \emptyset$. From (2), it follows $l(F_1) \leq l(F_2)$ and $l(F_2) \leq l(F_1)$ and again by (1), we obtain $t, t' \in X_n$ or $F_1 = F_2$ with $(var(F_1) \cup var(F_2)) \cap X_n = \emptyset$. If $t, t' \in X_n$ then we have $(t_1, F_1), (t_2, F_2) \in (t_1, F_1)^{X_n} = (t_2, F_2)^{X_n}$. In the case $(var(t_1) \cup var(t_2) \cup var(F_1) \cup var(F_2)) \cap X_n = \emptyset$, we can calculate that $t_1 = t_2$ and $F_1 = F_2$ and thus, $(t_1, F_1) = (t_2, F_2) \in (t_1, F_1)^{X_n} = \{(t_1, F_1)\}$.

The converse direction can be proved by straightforward calculations. ∎

Now we characterize the relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

Proposition 2.4. *Let* $(t_1, F_1), (t_2, F_2) \in W \times F^*_{(\tau,\tau')}(X)$. *Then* $(t_1, F_1)\mathcal{H}(t_2, F_2)$ *if and only if* $t_1 = t_2$ *and* $F_1 = F_2$.

Proof. If $F_1 = F_2$ and $t_1 = t_2$ then clearly $(t_1, F_1)\mathcal{H}(t_2, F_2)$.

Suppose that $(t_1, F_1)\mathcal{H}(t_2, F_2)$. Then $(t_1, F_1)\mathcal{L}(t_2, F_2)$ and by Proposition 2.3, we have $(t_1, F_1), (t_2, F_2) \in (t_1, F_1)^{X_n} = (t_2, F_2)^{X_n}$. By Proposition 2.1, from $(t_1, F_1)\mathcal{R}(t_2, F_2)$, it follows (1) or (2) or (3):

(1) $t_1 = t_2$ with $var(t_1) \cap X_n \neq \emptyset$ and $F_i \in F^{*,n}_{(\tau,\tau')}(t_1, X)$, for $i = 1, 2$.

(2) $(var(t_1) \cup var(t_2) \cup var(F_1) \cup var(F_2)) \cap X_n = \emptyset$.

Suppose that (1) is satisfied. Assume that $F_1 \neq F_2$. Then $(t_1, F_1), (t_2, F_2) \in (t_1, F_1)^{X_n}$ implies that there are different variables $x, y \in X_n$ such that $F_1 = R^n(F_1, x, \ldots, x)$ and $F_2 = R^n(F_2, y, \ldots, y)$, i.e. $var(F_1) \cap X_n = \{x\}$ and $var(F_2) \cap X_n = \{y\}$. By (1), we have $F_i \in F^{*,n}_{(\tau,\tau')}(t_1, X)$ with $t_1 = t_2$ and $var(t_1) \cap X_n \neq \emptyset$, for $i = 1, 2$. This provides $var(F_1) \cap X_n = var(F_2) \cap X_n = var(t_1) \cap X_n$, a contradiction to $var(F_1) \cap X_n = \{x\} \neq \{y\} = var(F_2) \cap X_n$. Hence, $F_1 = F_2$. On the other hand, (2) implies $\{(t_2, F_2)\} = (t_2, F_2)^{X_n} = (t_1, F_1)^{X_n} = \{(t_1, F_1)\}$, i.e. $t_1 = t_2$ and $F_1 = F_2$.

(3) $(t_1, F_1) = (t_2, F_2)$. ∎

Notation 2.5. For $n \in \mathbb{N}^+$ and $t \in W_\tau(X)$ let $t^{X_n} := \{S^n(t, , a, \ldots, a) : a \in X_n\}$. Note that $t^{X_n} = \{t\}$, whenever $var(t) \cap X_n = \emptyset$.

Finally, we characterize the $\mathcal{J}$-relation.

Proposition 2.6. *Let* $(t_1, F_1), (t_2, F_2) \in W \times F^*_{(\tau,\tau')}(X)$. *Then* $(t_1, F_1)\mathcal{J}(t_2, F_2)$ *if and only if* $t_1 \in t_2^{X_n} \cup \{t_2\}$, $t_2 \in t_1^{X_n} \cup \{t_1\}$, *and* $F_i \in F^{*,n}_{(\tau,\tau')}(t_i, X)$ *for* $i = 1, 2$ *or* $(t_1, F_1)\mathcal{L}(t_2, F_2)$.

Proof. Suppose that $(t_1,F_1)\mathcal{J}(t_2,F_2)$, where (t_1,F_1) and (t_2,F_2) are not $\mathcal{L}$-related. Then there are $\overline{g},\overline{h},\overline{g'}\overline{h'} \in W \times F^*_{(\tau,\tau')}(X) \cup \{\mathbf{1}\}$ such that $\overline{g} +_n (t_1,F_1) +_n \overline{h} = (t_2,F_2)$ and $\overline{g'} +_n (t_2,F_2) +_n \overline{h'} = (t_1,F_1)$. We have $\overline{h} = (h,G) \in W \times F^*_{(\tau,\tau')}(X)$ whenever $\overline{h} \neq 1$ and $\overline{h'} = (h',G') \in W \times F^*_{(\tau,\tau')}(X)$ whenever $\overline{h'} \neq 1$. Then $t_1 = S^n(h', S^n(t_2,g'_1,\dots,g'_n),\dots,S^n(t_2,g'_1,\dots,g'_n))$ or $t_1 = S^n(t_2,g'_1,\dots,g'_n)$ (if $\overline{h'}=1$)

$t_2 = S^n(h, S^n(t_1,g_1,\dots,g_n),\dots,S^n(t_1,g_1,\dots,g_n))$ or $t_2 = S^n(t_1,g_1,\dots,g_n)$ (if $\overline{h}=1$)

$F_1 = R^n(G', S^n(t_2,g'_1,\dots,g'_n),\dots,S^n(t_2,g'_1,\dots,g'_n))$ or $F_1 = R^n(F_2, g'_1,\dots,g'_n)$ (if $\overline{h'}=1$).

$F_2 = R^n(G, S^n(t_1,g_1,\dots,g_n),\dots,S^n(t_1,g_1,\dots,g_n))$ or $F_2 = R^n(F_1, g_1,\dots,g_n)$ (if $\overline{h}=1$), where $g_1 = \cdots = g_n = g \in W_\tau(X)$, whenever $\overline{g} \neq \mathbf{1}$ and $g'_i = x_i$ for $1 \le i \le n$ otherwise. The same for $g'_1,\dots,g'_n$. By simple calculations, we obtain that $l(t_1) = l(t_2)$. This provides $h,h' \in X_n$ and $g_1,\dots,g_n,g'_1,\dots,g'_n \in X_n$ or $(var(t_1) \cup var(t_2)) \cap X_n = \emptyset$ whenever $\overline{h}=1$ and $\overline{h'}=1$, respectively.

Hence $t_1 = S^n(t_2,g'_1,\dots,g'_n)$ and $t_2 = S^n(t_1,g_1,\dots,g_n)$. This implies $t_1 = t_2$ or $t_1 \in t_2^{X_n}$ and $t_2 \in t_1^{X_n}$. On the other hand, we have $F_1 = R^n(G', t_1,\dots,t_1)$ if $\overline{h'} \neq 1$, $F_1 = R^n(F_2,g'_1,\dots,g'_n)$ if $\overline{h'}=1$, $F_2 = S^n(G,t_2,\dots,t_2)$ if $\overline{h} \neq 1$, and $F_2 = R^n(F_1,g'_1,\dots,g'_n)$ if $\overline{h}=1$.

If $\overline{h} \neq 1$ and $\overline{h'} = 1$ then $F_1 = R^n(F_2,g'_1,\dots,g'_n) = R^n(S^n(G,t_2,\dots,t_2),g'_1,\dots,g'_n) = R^n(G, R^n(t_2,g'_1,\dots,g'_n),\dots,R^n(t_2,g'_1,\dots,g'_n)) = R^n(G,t_1,\dots,t_1)$. Hence, $F_1 \in F^{*,n}_{(\tau,\tau')}(t_1,X)$ and $F_2 \in F^{*,n}_{(\tau,\tau')}(t_2,X)$.

If $\overline{h}=1$ and $\overline{h'} \neq 1$ then we have dually $F_i \in F^{*,n}_{(\tau,\tau')}(t_i,X)$ for $i=1,2$.

Clearly, It holds whenever $\overline{h} \neq 1$ and $\overline{h'} \neq 1$.

If $\overline{h} = \overline{h'} = 1$ then $(t_1,F_1) = (t_2,F_2)$ if $\overline{g}=1$ or $\overline{g'}=1$ and $(t_1,F_1),(t_2,F_2) \in (t_1,F_1)^{X_n} = (t_2,F_2)^{X_n}$ otherwise. This shows that $(t_1,F_1)\mathcal{L}(t_2,F_2)$.

We consider now the converse direction. If $(t_1,F_1)\mathcal{L}(t_2,F_2)$ then all is clear. Suppose now that (t_1,F_1) and (t_2,F_2) are not $\mathcal{L}$-related. Suppose that $t_1 = t_2$. Using Proposition 2.1, we can calculate that $(t_1,F_1)\mathcal{R}(t_2,F_2)$, where $\mathcal{R} \subset \mathcal{J}$. Suppose now that $t_1 \in t_2^{X_n}$ and $t_2 \in t_1^{X_n}$. Then there are $a,b \in X_n$ such that $t_1 = S^n(x_1, S^n(t_2,a,\dots,a),\dots,S^n(t_2,a,\dots,a))$ and $t_2 = S^n(x_1, S^n(t_1,b,\dots,b)\dots,S^n(t_2,b,\dots,b))$. From $F_i \in F^{*,n}_{(\tau,\tau')}(t_i,X)$ for $i=1,2$, it follows that there are $F,G \in F^*_{(\tau,\tau')}(X)$ with $F_1 = R^n(F,t_1,\dots,t_1)$ and $F_2 = R^n(G,t_2,\dots,t_2)$. Therefore, we have $F_1 = R^n(F, S^n(t_2,a,\dots,a),\dots,S^n(t_2,a,\dots,a)) = R^n(R^n(F,t_2,\dots,t_2),a,\dots,a)$ and dually $F_1 = R^n(R^n(G,t_1,\dots,t_1),b,\dots,b)$. This implies $(t_1,F_1) = (a,G) +_n (t_2,F_2) +_n (x_1,G)$ and $(t_2,F_2) = (b,F) +_n (t_1,F_1) +_n (x_1,F)$, i.e. $(t_1,F_1)\mathcal{J}(t_2,F_2)$. ■

If we replace $(t_1,F_1)\mathcal{L}(t_2,F_2)$ in Proposition 2.6 with Proposition 2.3, we get that the following corollary holds true:

Corollary 2.7. *Let* $(t_1, F_1), (t_2, F_2) \in W \times F^*_{(\tau,\tau')}(X)$. *Then* $(t_1, F_1)\mathcal{J}(t_2, F_2)$ *if and only if* $t_1 \in t_2^{X_n} \cup \{t_2\}$, $t_2 \in t_1^{X_n} \cup \{t_1\}$, *and* $F_i \in F^{*,n}_{(\tau,\tau')}(t_i, X)$ *for* $i = 1, 2$ *or* $(t_1, F_1) \in (t_2, F_2)^{X_n} \cup \{(t_2, F_2)\}$ *and* $(t_2, F_2) \in (t_1, F_1)^{X_n} \cup \{(t_1, F_1)\}$.

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