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Some Aspects of Semirings

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Abstract. This article gives a survey of some results on the completely regular semirings, Clifford semirings, Clifford semifields and Leavitt path algebra over Clifford semifield.

Keywords: Completely regular semiring; Clifford semiring; Generalized Clifford semiring; b-lattice; Clifford semifield; Leavitt path algebra.

1. Introduction

Semiring is one of the many concepts of universal algebra which has been eatablished as a very important and interesting area of study. Semirings first appeared implicitly in connection with the study of ideals of a ring. In the year 1934, Vandiver [24] stated the idea of *associative algebra* or, *semiring* as a set of elements forming a semigroup under addition, a semigroup under multiplication, and in which the right and left distributive laws hold. This became the onset of the study of *semirings*. Since then there has been various developments and elaborated algebraic theories regarding the subject, although slowly, but steadily. Semirings inculcate in diverse areas of mathematics such as combinatorics, functional analysis, topology, graph theory, Euclidean geometry, probability theory, commutative and noncommutative ring theory, optimization theory, automata theory, and the mathematical modeling of quantum physics. This article gives a survey on completely regular semiring, Clifford semiring, Clifford semifield and simplicity of Leavitt path algebra over Clifford semifield.

2. Preliminaries and Background Information

A semiring $(S, +, \cdot)$ is a type (2, 2)-algebra whose semigroup reducts $(S, +)$ and $(S, \cdot)$ are connected by ring like distributivity, that is, $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c \in S$. A semiring $(S, +, \cdot)$ is called additively regular if for every element $a \in S$ there exists an element $x \in S$ such that $a + x + a = a$. An additive inverse semiring $(S, +, \cdot)$ is a semiring in which for each $a \in S$, there exists a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$. A semiring $(S, +, \cdot)$ is called a skew-ring [20] if its additive reduct $(S, +)$ is a group.

An ideal I of a semiring S is called a k-ideal if for any $x, y \in S$, $x \in I$ and either $x+y \in I$ or $y+x \in I$ imply $y \in I$ (Golan [7] called such ideals subtractive). Let S and T be two semirings. Then a mapping $f : S \longrightarrow T$ is said to be a homomorphism if $(x+y)f = xf + yf$ and $(xy)f = (xf)(yf)$ for all $x, y \in S$.

A semiring $(S, +, \cdot)$ is a b-lattice [20] if $(S, \cdot)$ is a band and $(S, +)$ is a semilattice. A congruence ζ on a semiring S is called a b-lattice congruence (idempotent semiring congruence) if S/ζ is a b-lattice (respectively, an idempotent semiring). A semiring S is called a b-lattice (idempotent semiring) Y of semirings $S_\alpha\,(\alpha \in Y)$ if S admits a b-lattice congruence (respectively, an idempotent semiring congruence) ζ on S such that $Y = S/\zeta$ and each S_α is a ζ-class.

Recall that a subdirect product T of a family of semirings $S_\alpha\,(\alpha \in \Lambda)$ is a subsemiring of the direct product of $S_\alpha\,(\alpha \in \Lambda)$ such that the projection mappings from T to each S_α is surjective. Let M, T be semirings and H be their common homomorphic image. Let $S = \{(a,b) \in M \times T : a\varphi = b\psi\}$, where $\varphi : M \to H$ and $\psi : T \to H$ are the semiring epimorphisms from M and T onto H, respectively. Then S is called the spined product of semirings M and T with respect to H. The spined product S of the semirings M and T with respect to H is denoted by $S = M \underline{\times}_H T$.

As usual, we denote the Green's relations on the semiring $(S, +, \cdot)$ by $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$, $\mathscr{J}$ and $\mathscr{H}$ and correspondingly, the $\mathscr{L}$-relation, $\mathscr{R}$-relation, $\mathscr{D}$-relation, $\mathscr{J}$-relation and $\mathscr{H}$-relation on $(S, +)$ are denoted by $\mathscr{L}^+$, $\mathscr{R}^+$, $\mathscr{D}^+$, $\mathscr{J}^+$ and $\mathscr{H}^+$, respectively. Throughout this article, we always let $E^+(S)$ be the set of all additive idempotents of the semiring S. Also we denote the set of all additive inverses of a, if it exists, in a semiring S by $V^+(a)$.

A variety is a class of algebras closed under homomorphic images, taking subalgebras and direct products. A variety $\mathbf{U}$ of semirings is said to be a subvariety of a variety $\mathbf{V}$ of semirings if $\mathbf{U} \subseteq \mathbf{V}$. For a variety $\mathbf{V}$ of semirings, let $\mathscr{L}(\mathbf{V})$ be the collection of all subvarieties of $\mathbf{V}$. Then $(\mathscr{L}(\mathbf{V}), \vee, \wedge)$ forms a lattice, where $\mathbf{U} \vee \mathbf{W} = \{S \in \mathbf{V} : S \,\text{is a subdirect product of some}\, S_1 \in \mathbf{U} \,\text{and}\, S_2 \in \mathbf{W}\}$ and $\mathbf{U} \wedge \mathbf{W} = \mathbf{U} \cap \mathbf{W}$. We denote the variety of all idempotent semirings by $\mathbf{I}$ and the variety of all distributive lattices by $\mathbf{D}$. We also denote the subvariety of $\mathbf{I}$ which satisfies the identity $x + y + x = x$ by $\mathbf{R}^{+}$, whereas the subvariety of $\mathbf{I}$ which satisfies the identity $xyx = x$ is denoted by $\mathbf{R}^{\bullet}$. For any two varieties $\mathbf{V}$ and $\mathbf{W}$ of semirings, the Mal'cev product $\mathbf{V} \circ \mathbf{W}$ of $\mathbf{V}$ and $\mathbf{W}$ is the variety of all semirings S for which there exists a congruence ρ on S such that $S/\rho \in \mathbf{W}$ and the ρ-classes belong to $\mathbf{V}$.

A directed graph $\Gamma = (V, E, r, s)$ consists of two sets, V and E, and two maps $r, s : E \to V$. The elements of V are called vertices and the elements of E are called edges. For any edge e in E, $s(e)$ is called the source of e and $r(e)$ is called the range of e. If $s(e) = v$ and $r(e) = w$, then we say that v emits e and w receives e. In other words, e is imagined having its direction from v to w. If $r(e_1) = s(e_2)$ for some edges $e_1, e_2 \in E$, we say that e_1 and e_2 are adjacent. Throughout this survey article, we refer to directed graphs simply as graphs. From the above definition, it follows that for any vertex v in V, $s^{-1}(v)$ is the set of all edges emitted by v, while $r^{-1}(v)$ is the set of all edges received by v. If v does not emit any edges (i.e., if $s^{-1}(v) = \emptyset$), then v is called a sink whereas a vertex v with $0 < |s^{-1}(v)| < \infty$ is called regular.

A path $p = e_1 e_2 \cdots e_n$ in a graph is a sequence of edges $e_1, e_2, \ldots, e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, 2, \ldots, n-1$. A path consisting of n edges is said to have length n. The functions s, r are naturally extended from edges to paths by considering the source of its initial edge $s(e_1)$ as the source $s(p)$ of p; while (if p has finite length) the range of its final edge $r(e_n)$ is taken as the range of p, denoted by $r(p)$. It is also convenient to think of every vertex $v \in V$ as being a path of length 0, with $s(v) = v = r(v)$. We denote the set of all paths in Γ by $E^{(*)}$. A path p is called a closed path based at v if $s(p) = r(p) = v$. A cycle is a path that begins and ends on the same vertex v and does not pass through any vertex more than once. Hence a path p is a cycle if $s(p) = r(p)$ and $s(e_i) \neq s(e_j)$ for all $i \neq j$. If a graph Γ does not contain any cycles, it is said to be acyclic. Again, an edge e is called an exit to the cycle $p = e_1 e_2 \cdots e_n$ if there exists some $i \in \{1, 2, \ldots, n\}$ such that $s(e_i) = s(e)$ but $e \neq e_i$.

For notions and terminologies not given in this paper, the readers are referred to Howie [11], Petrich and Reilly [15] for a background on semigroup theory, Hebisch and Weinert [10], Golan [8] for semiring theory and, Abrams and Pino [1] for information concerning Leavitt path algebra.

3. Completely Regular Semiring

In [4], Clifford defined relative inverse of an element in a semigroup. According to him an element a in a semigroup S admits a relative inverse if there exists an element $e \in S$ such that $ae = a = ea$ and there exists an element $a' \in S$ such that $aa' = e = a'a$. If every element of a semigroup S admits a relative inverse, then we say that the semigroup S admits relative inverses. It was established that a semigroup admits relative inverses if and only if it is the union of mutually disjoint groups. This kind of semigroups is presently known as completely regular semigroups.

In 2006, the authors [20] first extended the notion of completely regular semigroups to semirings. They defined completely regular semirings as follows.

Definition 3.1. *A semiring $(S,+,\cdot)$ is said to be completely regular if for each element $a \in S$, there exists an element $x \in S$ such that $a+x+a = a$, $a+x = x+a$ and $a(a+x) = a+x$.*

From the first two conditions, it follows that the additive reduct $(S,+)$ of a semiring $(S,+,\cdot)$ is a completely regular semigroup. The third condition is an extra condition which makes the element a completely regular in a semiring.

Theorem 3.2. [20] *A semiring $(S,+,\cdot)$ is a completely regular semiring if and only if for all $a \in S$ there exists an element $x \in S$ such that the following conditions are satisfied*

(i) $a = a+x+a$,
(ii) $(a+x = x+a$,
(iii)$'$ $(a+x)a = a+x$.

Theorem 3.3. [20] *The following statements on a semiring S are equivalent.*

(i) *a is completely regular.*
(ii) *There exists a unique element $y \in V^+(a)$ such that $a(a+y) = a+y$, $a(y+a) = y+a$, $a+(a+y)a = a, a(y+a)+a = a$ and $a(a+y) = (a+y)a$.*
(iii) *There exists a unique element $y \in V^+(a)$ such that $a+y = y+a$ and $a(a+y) = a+y$.*
(iv) *H_a^+ is a skew-ring, where H_a^+ is the $\mathscr{H}^+$-class containing $a \in S$.*

Notation 3.4. For a completely regular element a in a semiring S, the unique element in $V^+(a)$ satisfying condition of Theorem 3.3 (iii) is denoted by a'. Also, we denote the element $a+a'(= a'+a)$ by 0_a.

Lemma 3.5. [20] *If S is a completely regular semiring, then $E^+(S) = \{0_a : a \in S\}$ and $e^2 = e$ for all $e \in E^+(S)$.*

Lemma 3.6. [20] *Let S be a completely regular semiring. Then for any $a, b \in S$,*

(i) $(a')' = a$,
(ii) $ab' = (ab)' = (a')b$,
(iii) $ab = a'b'$,
(iv) $0_{ab} = 0_a b = a0_b$.

Theorem 3.7. [20] *Let S and T be two semirings such that S is completely regular and $\varphi : S \to T$ be a homomorphism. Then the following statements hold:*

(i) *$S\varphi$ is a completely regular semiring,*

(ii) *for any* $a \in S$, $a'\varphi = (a\varphi)'$ *and* $(0_a)\varphi = 0_{(a\varphi)}$.

Corollary 3.8. [20] *Let* ρ *be a congruence on a completely regular semiring* S. *If for some* $a, b \in S$, $a \ \rho \ b$ *holds, then* $a' \ \rho \ b'$ *and* $0_a \ \rho \ 0_b$.

Theorem 3.9. [20] *Let* T *be a subsemiring of a completely regular semiring* S. *Then* T *is completely regular if and only if* $a' \in T$ *for all* $a \in T$.

Theorem 3.10. *On a completely regular semiring* S, $\mathscr{D}^+ = \mathscr{J}^+$.

Theorem 3.11. [17] *For an idempotent semiring* S, *the following conditions are equivalent:*

(i) $\mathscr{D}^+$ *is the least distributive lattice congruence on* S,
(ii) $S \in \mathbf{R}^+ \circ \mathbf{D}$.

Theorem 3.12. [20] *The following conditions on a semiring are equivalent.*

(i) *S is completely regular.*
(ii) *Every* $\mathscr{H}^+$*-class is a skew-ring.*
(iii) *S is a union (disjoint) of skew-rings.*
(iv) *S is a b-lattice of completely simple semirings.*

Theorem 3.13. [14] *If a semiring* S *is a union of rings, then* $\mathscr{H}^+$ *is the least idempotent semiring congruence on* S.

Theorem 3.14. [5] *Let* S *be a semiring. Then the following conditions are equivalent:*

(i) *S is the (disjunctive) union of its subrings;*
(ii) *for every* $x, y \in S$ *there exists unique* $x' \in S$ *such that* $x = x+x'+x, x+x' = x'+x, (x')' = x, x+0_y+0_x+y = 0_x+y+x+0_y$ *and* $x0_x = 0_x$;
(iii) $\mathscr{H}^+$ *is an idempotent semiring congruence on* S *and each* $\mathscr{H}^+$*-class is a ring;*
(iv) *S is an idempotent semiring of rings.*

Theorem 3.15. [17] *For an idempotent semiring* S, *the following conditions are equivalent :*

(i) $S \in (\mathbf{R}^+ \cap \mathbf{R}^\bullet) \circ \mathbf{D}$,
(ii) $S \in \mathbf{R}^+ \circ \mathbf{D}$ *and* $\mathbf{D}^+ = \mathbf{D}^\bullet$.

Theorem 3.16. [17] $(\mathbf{R}^+ \circ \mathbf{D}) \circ \mathbf{D} = \mathbf{R}^+ \circ \mathbf{D}$.

The authors raised the following question in their paper [20].

Question 3.17. Suppose $(S, +, \cdot)$ is a semiring whose reducts $(S, +)$ and $(S, \cdot)$ are both completely regular semigroups. What impact does it have on the nature of the semiring?

In connection with the above question, the authors mentioned the following examples in their paper [20].

Example 3.18. Consider a division ring $(S, +, \cdot)$. Here both the reducts $(S, +)$ and $(S, \cdot)$ are completely regular semigroups. Observe that the semiring here is also completely regular.

Example 3.19. Let $(S, +, \cdot)$ be a distributive lattice. Then one can easily verify that both the reducts $(S, +)$ and $(S, \cdot)$ are completely regular semigroups and the semiring $(S, +, \cdot)$ is also completely regular.

Example 3.20. Let $(S, \cdot)$ be a group with at least two elements. Define $x + y = x$ for all x, $y \in S$. Then $(S, +, \cdot)$ is a semiring such that $(S, +)$ and $(S, \cdot)$ are completely regular semigroups, but one can easily check that the semiring $(S, +, \cdot)$ is not completely regular.

4. Clifford Semiring and Clifford Semifield

As a special case of completely regular semigroup, we recall that a semigroup S is a Clifford semigroup if for each $a \in S$, there exists an element $x \in S$ such that $axa = a$ and $ae = ea$, for all idempotents e of S. Clearly, a semigroup S is a Clifford semigroup if S is completely regular and its idempotents commmute with all elements of S. Similar to the result of Clifford semigroups, Bandelt and Petrich [3] have shown that a semiring S whose additive reduct $(S, +)$ is a regular semigroup can be expressed as a subdirect product of a distributive lattice and a ring if and only if $(S, +)$ is commutative and the following conditions hold

(4.1) $a + a' = a' + a$,

(4.2) $a(a + a') = a + a'$,

(4.3) $a(b + b') = (b + b')a$,

(4.4) $a + (a + a')b = a$, for all $a, b \in S$,

(4.5) If $a \in S$ and $b + a = b$ for some $b \in S$, then $a + a = a$.

In view of above result, Ghosh [6] has further given a characterization for semirings whose additive reduct $(S, +)$ is commutative and he has consequently defined Clifford semirings, by assuming that the additive reduct is commutative. According to Ghosh [6], a Clifford semiring S is an additively commutative inverse semiring such that $E^+(S)$ is a distributive sublattice as well as a k-ideal of S. Later on, Sen, Ghosh and Mukhopadhyay [16] establisheded that an additive commutative inverse semiring S satisfies the above conditions (4.1), (4.2), (4.3) and (4.4) if and only if $E^+(S)$ is a distributive lattice of S and the semiring S satisfies the condition (4.5) if and only if $E^+(S)$ is a k-ideal of S. Thus, we can see that $E^+(S)$ of a semiring S plays an important role in studying the structure of semirings.

Definition 4.1. [19] *A completely regular semiring $(S, +, \cdot)$ is said to be a generalized Clifford semiring if S is an additive inverse semiring and $E^+(S)$ is a k-ideal of S.*

Theorem 4.2. [19] *An additive inverse semiring $(S, +, \cdot)$ is a generalized Clifford semiring if and only if S satisfies conditions (4.1), (4.2) and (4.5).*

Definition 4.3. [19] *Let T be b-lattice and $\{S_\alpha : \alpha \in T\}$ be a family of pairwise disjoint semirings which are indexed by the elements of T. For each $\alpha \leq \beta$ in T, we embed S_α in S_β via a semiring monomorphism $\varphi_{\alpha,\beta}$ satisfying the following conditions*

(i) $\varphi_{\alpha,\alpha} = I_{S_\alpha}$, *the identity mapping on* S_α,

(ii) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$, *if* $\alpha \leq \beta \leq \gamma$,

(iii) $S_\alpha\varphi_{\alpha,\gamma}S_\beta\varphi_{\beta,\gamma} \subseteq S_{\alpha\beta}\varphi_{\alpha\beta,\gamma}$, *if* $\alpha + \beta \leq \gamma$, *i.e.,* $\alpha + \beta + \alpha\beta \leq \gamma$.

On $S = \bigcup_{\alpha \in T} S_\alpha$, we define addition $\oplus$ and multiplication $\odot$ for $a \in S_\alpha, b \in S_\beta$, as follows :

$$a \oplus b = a\varphi_{\alpha,\alpha+\beta} + b\varphi_{\beta,\alpha+\beta}$$

and $a \odot b = c \in S_{\alpha\beta}$ such that

$$c\varphi_{\alpha\beta,\alpha+\beta} = a\varphi_{\alpha,\alpha+\beta}b\varphi_{\beta,\alpha+\beta}.$$

The above system is denoted by $S = \langle T, S_\alpha, \phi_{\alpha,\beta}\rangle$ and is called the strong b-lattice T of the semirings $S_\alpha, \alpha \in T$.

Theorem 4.4. [19] *With the above notation in Definition 4.3, the system $S = \langle T, S_\alpha, \phi_{\alpha,\beta}\rangle$ is a semiring.*

Theorem 4.5. [19] *A semiring S is a generalized Clifford semiring if and only if it is a strong b-lattice of skew-rings.*

Lemma 4.6. [19] *Let $S = \langle T, S_\alpha, \varphi_{\alpha,\beta}\rangle$ be a strong b-lattice T of semirings $S_\alpha (\alpha \in T)$ and θ a binary relation on S defined by $a\,\theta\, b$ if and only if $a\varphi_{\alpha,\alpha+\beta} = b\varphi_{\beta,\alpha+\beta}\,(a \in S_\alpha, b \in S_\beta)$. Then θ is a congruence on S and S is a subdirect product of T and S/θ.*

Theorem 4.7. [19] *A semiring S is an additive inverse semiring and is a subdirect product of a b-lattice and a skew-ring if and only if $S = \langle T, R_\alpha, \varphi_{\alpha,\beta}\rangle$.*

Definition 4.8. [19] *A completely regular semiring $(S, +, \cdot)$ is said to be a Clifford semiring if S is an additive inverse semiring and $E^+(S)$ is a distributive sublattice as well as a k-ideal of S.*

Theorem 4.9. [19] *An additive inverse semiring S is a Clifford semiring if and only if S satisfies conditions (4.1), (4.2), (4.3), (4.4) and (4.5).*

Theorem 4.10. [19] *A semiring S is a Clifford semiring if and only if it is a strong distributive lattice of skew-rings.*

Corollary 4.11. [19] *Let $(S,+,\cdot)$ be a semiring such that $(S,+)$ is commutative. Then S is a Clifford semiring if and only if it is a strong distributive lattice of rings.*

Definition 4.12. [9] *A semiring S is said to be a left Clifford semiring if S is a distributive lattice of left rings.*

Theorem 4.13. [9] *A semiring S is a left Clifford semiring if and only if the additive reduct $(S,+)$ of S is a left Clifford semigroup in which each maximal subgroup is abelian, $E^+(S) \subseteq E^\bullet(S)$ and S satisfies the following conditions*

(i) $V^+(a)+a \supseteq a(V^+(a)+a)$ *for all* $a \in S$,
(ii) $V^+(ab)+ab \supseteq (V^+(b)+b)a$ *for all* $a,b \in S$,
(iii) $V^+(a)+a \supseteq V^+(ab)+(V^+(a)+a)+ab$ *for all* $a,b \in S$.

Theorem 4.14. [9] *The spined product $L \underline{\times}_D S = \bigcup_{\alpha\in D}(L_\alpha \times R_\alpha)$ of left regular band semiring L and Clifford semiring S with respect to the same distributive lattice skeleton D is a left Clifford semiring. Conversely, every left Clifford semiring can be expressed by such a spined product.*

Corollary 4.15. [9] *A semiring S is a left Clifford semiring if and only if $\mathscr{H}^+$ is a left regular band semiring congruence on S and every $\mathscr{H}^+$-class is a ring.*

Theorem 4.16. [9] *Every strong distributive lattice $S = \langle D, S_\alpha, \varphi_{\alpha,\beta}\rangle$ of left rings S_α is a left Clifford semiring in which $E^+(S)$ is a left normal band semiring and is left unitary in $(S,+)$. Conversely, every left Clifford semiring S in which $E^+(S)$ is a left unitary (in $(S,+)$) left normal band semiring is a strong distributive lattice of left rings.*

Corollary 4.17. [9] *A Clifford semiring is a strong distributive lattice of rings if and only if $E^+(S)$ is unitary in the commutative regular additive reduct of S.*

Definition 4.18. [18] *Let S be a Clifford semiring with 1 such that $1 \notin E^+(S)$. An non additive idempotent element $a \in S$ is said to be left invertible if there exists an element $r \in S$ such that $ra+1+1' = 1$. In this case, r is called the left inverse of a. Similarly, we can define right invertible element in a Clifford semiring. An element is said to be invertible if it is left invertible as well as right invertible. If a is invertible, we say that a is a unit of S.*

Definition 4.19. [18] *A Clifford semiring S is called a Clifford semifield if*

(i) $1 \in S$ *such that* $1 \notin E^+(S)$,
(ii) S *is multiplicative commutative,*
(iii) *every non additive idempotent element of* S *is a unit.*

Theorem 4.20. [18] *Let* S *be a Clifford semiring with* 1 *such that* $(S, \cdot)$ *is commutative. Then a* k*-ideal* M *of* S *is maximal if and only if* S/M *is a Clifford semifield.*

Theorem 4.21. [18] *If* S *is a Clifford semifield, then* S *is a subdirect product of a field and a distributive lattice with a greatest element.*

Theorem 4.22. [18] *A multiplicative commutative Clifford semiring* S *with* 1 *is a Clifford semifield if and only if* S *is full ideal free.*

Theorem 4.23. [18] *An additive commutative and multiplicative commutative Clifford semiring* S *with* 0 *and* 1 *is* k*-ideal free if and only if* S *is a field or* $S = \{0, 1\}$.

Theorem 4.24. [21] *Let* S *be an additive commutative and a multiplicative commutative Clifford semiring with identity. Then for every full* k*-ideal* I *of* S, $M_n(I)$ *is a full* k*-ideal of* $M_n(S)$. *On the other hand, for each full* k*-ideal* J *of* $M_n(S)$ *there exists a unique full* k*-ideal* T *of* S *such that* $J = M_n(T)$.

Theorem 4.25. [21] *If* S *is a Clifford semifield, then* $M_n(S)$ *is full* k*-ideal simple for any* $n \in \mathbb{N}$.

5. Leavitt Path Algebra with Coefficients in a Clifford Semifield

In recent times, Leavitt path algebra has emerged as one of the most engaging fields of study. Ever since it was introduced by Abrams and pino [1], several mathematicians have worked in this new topic. Abrams and Pino first introduced the Leavitt path algebra $L_K(\Gamma)$ of a directed graph Γ with coefficients in a field K. Thus Leavitt path algebra associates algebraic structures with graphs and so, involves both graph theory and algebra. Later on, Tomforde [22] defined Leavitt path algebra over rings and Katsov et al. [13] defined Leavitt path algebra over commutative semiring. In this section, we gives a survey of some results on the Leavitt path algebra $L_S(\Gamma)$ of a directed graph Γ with coefficients in a Clifford semifield S.

Definition 5.1. [21] *Let* $\Gamma = (V, E, s, r)$ *be a directed graph and* S *be a Clifford semiring. The Leavitt path algebra* $L_S(\Gamma)$ *of the graph* Γ *with coefficients in* S *is the* S*-algebra given by the set of generators* $V \cup E \cup E^*$ *(where* $e \mapsto e^*$ *is a bijection* $E \to E^*$, *and* V, E, E^* *are pairwise disjoint sets) satisfying the following relations :*
(i) $vu = \delta_{v,u} v$ *for all* $v, u \in V$*;*

(ii) $s(e)e = e = er(e)$, $r(e)e^* = e* = e^*s(e)$ *for all* $e \in E$*;*
(iii) $e^*f = \delta_{e,f}r(e)$ *for all* $e, f \in E$*;*
(iv) $v = \sum_{e \in s^{-1}(v)} ee^*$ *whenever* $v \in V$ *is a regular vertex.*

Proposition 5.2. [21] *If* Γ *is a graph and* S *is a Clifford semiring with identity, then the Leavitt path algebra* $L_S(\Gamma)$ *has the following properties:*

(i) *all elements of the set* $\{v, e, e^* : v \in V, e \in E\}$ *are nonadditive idempotents;*
(ii) *if* a, b *are distinct elements in* S*, then* $av \neq bv$ *for all* $v \in V$*.*

Proposition 5.3. [21] *Let* Γ *be a graph with the property that every cycle in* Γ *has an exit, and let* S *be a Clifford semifield. If* $\alpha \in L_S(\Gamma)$ *is a polynomial in only real edges whose coefficients are all in* $S \setminus E^+(S)$*, then there exist* $a, b \in L_S(\Gamma)$ *such that* $a\alpha b = \lambda v$ *for some* $\lambda \in S \setminus E^+(S)$ *and* $v \in V$*.*

Corollary 5.4. [21] *Let* Γ *be a graph with the property that every cycle in* Γ *has an exit. Also, let* S *be a Clifford semifield. If a full* k*-ideal* J *in* $L_S(\Gamma)$ *contains a nonadditive idempotent polynomial* α *in only real edges, then* J *contains a vertex.*

Lemma 5.5. [21] *If* S *is a Clifford semifield, then the product of two nonadditive idempotents cannot be an additive idempotent.*

Proposition 5.6. [21] *The product of two nonadditive idempotent polynomials in* $L_S(\Gamma)$ *(where* S *is a Clifford semifield and* Γ *is a graph) cannot be an additive idempotent polynomial.*

Theorem 5.7. [21] *Let* Γ *be a graph and* S *be a Clifford semifield. Also, let* $E^+(L_S(\Gamma))$ *be a* k*-ideal. If* $x \in L_S(\Gamma)$ *and* $x \notin E^+(L_S(\Gamma))$*, then there exists* $\gamma \in E^{(*)}$ *such that* $x\gamma \notin E^+(L_S(\Gamma))$ *and* $x\gamma$ *is a polynomial in only real edges.*

Corollary 5.8. [21] *Let* Γ *be a graph and* S *be a Clifford semifield such that* $E^+(L_S(\Gamma))$ *is a* k*-ideal. If* J *is a nontrivial full* k*-ideal of* $L_S(\Gamma)$*, then* J *contains a nonadditive idempotent polynomial in only real edges.*

Theorem 5.9. [21] *Let* $\Gamma = (V, E, s, r)$ *be a graph such that every cycle in* Γ *has an exit. Let* S *be a Clifford semifield. If* J *is a nontrivial full* k*-ideal of* $L_S(\Gamma)$*, then* $J \cap V \neq \emptyset$ *and* $J \cap V$ *is a hereditary and saturated subset of* V*.*

Theorem 5.10. [21] *Let* $\Gamma = (V, E, s, r)$ *be a graph. Let* S *be a Clifford semifield such that* $E^+(L_S(\Gamma))$ *is a* k*-ideal. Then* $L_S(\Gamma)$ *is full* k*-ideal free if and only if both the following conditions are satisfied:*

(i) *The only hereditary and saturated subsets of* V *are* $\emptyset$ *and* V*,*
(ii) *Every cycle in* Γ *has an exit.*

Regarding Leavitt path algebras, an important aspect is the Uniqueness theorem (also known as *Cuntz-Krieger Uniqueness theorem*). In recent time, the Uniqueness theorems for Leavitt path algebra are established over fields and commutative rings with 1. In our paper [21], we established it over Clifford semifields. In this connection, first, we state the Uniqueness theorems for Leavitt path algebras defined respectively over fields and commutative rings with 1.

Theorem 5.11. [22, Cuntz-Krieger Uniqueness theorem] *Let Γ be a graph such that every cycle in Γ has an exit, and let K be a field. If S is a ring and $f : L_K(\Gamma) \to S$ is a ring homomorphism with the property that $f(v) \neq 0$ for all $v \in V$, then f is injective.*

Theorem 5.12. [22, Cuntz-Krieger Uniqueness theorem] *Let every cycle in a graph Γ have an exit, and let R be a commutative ring with 1. If S is a ring and $f : L_R(\Gamma) \to S$ is a ring homomorphism with the property that $f(rv) \neq 0$ for all $v \in V$ and for all $r \in R\backslash\{0\}$, then f is injective.*

Definition 5.13. *Let S and T be two Clifford semirings. Let $f : S \to T$ be a mapping. f is called a c-homomorphism if $f(a + b) = f(a) + f(b), f(ab) = f(a)f(b), f(0) = 0, f(1) = 1$ and f maps additive idempotent elements into additive idempotent elements.*

Definition 5.14. [21] *Let S and T be two Clifford semirings. Let $f : S \to T$ be a c-homomorphism. The c-kernel of f is the subset of S defined as : $cker(f) = \{x \in S : f(x)$ is an additive idempotent of $T\}$.*

Definition 5.15. [21] *Let S and T be two Clifford semirings. Let $f : S \to T$ be a c-homomorphism. f is called c-injective if for any $a, b \in S$, $f(a) = f(b)$ implies that $a + e = b + e$ for some additive idempotent $e \in S$.*

The next result for c-homomorphisms is similar to the Cuntz-Krieger Uniqueness theorems.

Theorem 5.16. [21] *Let $\Gamma = (V, E, r, s)$ be a graph such that every cycle in Γ has an exit. Suppose S is a Clifford semifield such that $E^+(L_S(\Gamma))$ is a k- ideal. Let T be a Clifford semiring. If f is a c-homomorphism from $L_S(\Gamma)$ to T with the property that $f(v)$ is not an additive idempotent for any $v \in V$, then f is c-injective.*

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