Southeast Asian Bulletin of Mathematics (2022) 46: 1–14

Southeast Asian
Bulletin of
Mathematics

On Categorical Properties of Topological S-Acts

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Received 26 August 2019
Accepted 1 March 2020

Communicated by Xueming Ren

AMS Mathematics Subject Classification(2020): 20M30, 54H10, 22A30, 08B20

Abstract. The purpose of the paper is to study the product, coproduct, free and projective objects and generators in the category of topological S-acts as well as to revisit some related old results.

Keywords: S-act; Topological monoid; Topological S-act; Projective topological S-act; Indecomposable topological S-act.

1. Introduction

There have been various works on topological semigroups and their structures, a lot of which were initiated by A.D. Wallace in 1953 [15]. Representations of semigroups by transformations of sets give rise to the notion of acts over semigroup. Aspects of topological semigroups as well as topological acts over topological semigroups can be found in [3, 4, 8, 9, 12]. In this paper we concerned about the topological acts over a topological monoid from a categorical point of view. Some studies of the category of topological S-acts were accomplished by Khosravi [5, 6], where (S, τ_S) is a topological monoid. Khosravi introduced the

*author is grateful to CSIR for providing research support

notions of free topological S-acts over a topological space, over a set as well as over an S-act. Then by using the notion of free topological S-acts over S-acts he characterized projective topological S-acts. Then Khosravi [6] considered the category S**-CReg** of Hausdorff completely regular topological S-acts, where S is a completely regular Hausdorff topological monoid and studied the coproduct, free object over completely regular space and characterized the projective object in this category. He also characterized the algebraic and topological structure of a projective topological S-act for an arbitrary topological monoid S. In this paper we identify the product, coproduct in the category of topological S-acts. Then we revisit (cf. Proposition 3.8) the result of Khosravi [5, Prop. 3.9], for the construction of free topological S-act over a set and observe its general structure (cf. Corollary 3.11). We define indecomposable topological S-act which is more general than what is meant by Khosravi [6] and observe that every topological S-act has a unique decomposition into indecomposable topological subacts (cf. Definition 3.18 and Theorem 3.22). Then we study projective topological S-act and revisit (cf. Theorem 3.26) one characterization [6, Theorem 2.2] of it. Finally we define generator in the category of topological S-acts and obtain some of its characterization (cf. Theorem 3.30) which are analogous to [7, Theorem 2.3.16].

We now briefly recall some preliminaries needed in the sequel.

2. Preliminaries

For a monoid S, a set A is a left S-act [7] if there is an action $S \times A \to A$ given by $(s, a) \mapsto sa$ satisfying $(st)a = s(ta)$ and $1_S a = a$ for all $s, t \in S$ and $a \in A$. A nonempty subset B of an S-act A is said to be a subact of A if $SB \subseteq B$. An S-act A is called cyclic if $A = Sa$ for some $a \in A$. For S-acts A and B, a map $f : A \to B$ is an S-map if $f(sa) = sf(a)$ for all $s \in S,\ a \in A$. The category formed by left S-acts together with S-maps is denoted by S**-Act**.

Definition 2.1. [12] *A monoid S with a topology τ_S is a topological monoid if the multiplication $S \times S \to S$ is (jointly) continuous in both the variables, i.e., if $st \in U \in \tau_S$ for some $s, t \in S$, then there exist $V \in \tau_S$ containing s and $W \in \tau_S$ containing t such that $VW \subseteq U$.*

Definition 2.2. [12] *For a topological monoid (S, τ_S), a left S-act A with a topology τ_A is said to be a left topological S-act if the action $S \times A \to A$ is (jointly) continuous, i.e., if $sa \in X \in \tau_A$ for some $s \in S,\ a \in A$ then there exist $U \in \tau_S$ containing s and $Y \in \tau_A$ containing a such that $UY \subseteq X$. Analogously right topological S-act is defined.*

Here we give some usual examples of (left) topological S-acts.

Example 2.3.
(1) (S, τ_S) itself is a topological S-act, where the S-action is given by monoid multiplication.
(2) Any S-act A together with the indiscrete topology is a topological S-act.
(3) Let (A, τ_A) be a topological S-act. Then any subact B of A together with the subspace topology τ_B is also a topological S-act.

Remark 2.4. [5] For a topological monoid (S, τ_S) we denote the category of all left topological S-acts together with continuous S-maps by S-**Top**. Analogously we denote the category of right topological S-acts together with continuous S-maps by **Top**-S.

For further notion and examples of topological S-acts we refer to [12, 5, 6]. For preliminaries on topologies and category theory we refer respectively to [11] and [10, 1].

In the subsequent discussion by S-act we mean left S-act and by topological S-act we mean left topological S-act (cf. Definition 2.2) unless mentioned otherwise.

3. Categorical Properties of Topological S-Acts

We begin this section by producing a canonical example of topological S-act.

Example 3.1. Let (S, τ_S) be a topological monoid. For any non-empty set X, consider $S^X = \{f \mid f : X \to S\}$ with product topology τ together with left S-action defined as

$$\begin{aligned} &S \times S^X \to S^X \\ &(s, f) \mapsto sf \ (x \mapsto sf(x)). \end{aligned}$$

Then a routine verification shows that (S^X, τ) is a topological S-act.

The following result describes the product in the category of topological S-acts.

Proposition 3.2. *Let $(A_\alpha, \tau_\alpha)_{\alpha\in\Lambda}$ be a collection of topological S-acts. Suppose $\prod_{\alpha\in\Lambda} A_\alpha$ is the product of $(A_\alpha)_{\alpha\in\Lambda}$ in S-**Act** with canonical projections, $p_\beta : \prod_{\alpha\in\Lambda} A_\alpha \to A_\beta$ for $\beta \in \Lambda$. Then $(\prod_{\alpha\in\Lambda} A_\alpha, \prod_{\alpha\in\Lambda} \tau_\alpha)$ is the product of the family $(A_\alpha, \tau_\alpha)_{\alpha\in\Lambda}$ in S-**Top**, where $\prod_{\alpha\in\Lambda} \tau_\alpha$ is the product topology on $\prod_{\alpha\in\Lambda} A_\alpha$.*

Proof. Suppose $\times_{\alpha\in\Lambda} A_\alpha$ is the cartesian product of the family $(A_\alpha)_{\alpha\in\Lambda}$ of S-acts with projections $p_\beta : \times_{\alpha\in\Lambda} A_\alpha \to A_\beta$ defined by $p_\beta((x_\alpha)_{\alpha\in\Lambda}) := x_\beta$, where $\beta \in \Lambda$, $(x_\alpha)_{\alpha\in\Lambda} \in \times_{\alpha\in\Lambda} A_\alpha$. Then we know from [7] that this cartesian product together with the S-action defined on it as componentwise multiplication by elements of S is the product of $(A_\alpha)_{\alpha\in\Lambda}$ in S-**Act** and is denoted by $\prod_{\alpha\in\Lambda} A_\alpha$.

Let $sx \in U \in \prod_{\alpha\in\Lambda} \tau_\alpha$, where $s \in S$, $x = (x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} A_\alpha$. Then there exist $U_{\alpha_i} \in \tau_{\alpha_i}$, where $\alpha_i \in \Lambda$, $i = 1, 2, ..., n$, for some $n \in \mathbb{N}$ such that $sx \in \bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq U$. Therefore we see that for all $i = 1, 2, ..., n$, $p_{\alpha_i}(sx) \in U_{\alpha_i}$ which implies $sx_{\alpha_i} \in U_{\alpha_i}$. Then for each $i = 1, 2, ..., n$, there exist $V_{\alpha_i} \in \tau_S$ and $W_{\alpha_i} \in \tau_{\alpha_i}$ with $s \in V_{\alpha_i}$, $x_{\alpha_i} \in W_{\alpha_i}$ such that $V_{\alpha_i} W_{\alpha_i} \subseteq U_{\alpha_i}$. Thus we have

$$s \in \bigcap_{i=1}^{n} V_{\alpha_i} = V \in \tau_S \quad \text{and} \quad x \in \bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(W_{\alpha_i}) = W \in \prod_{\alpha \in \Lambda} \tau_\alpha.$$

Now since $p_{\alpha_i}(VW) \subseteq VW_{\alpha_i} \subseteq V_{\alpha_i} W_{\alpha_i} \subseteq U_{\alpha_i}$ for all $i = 1, 2, ..., n$, therefore denoting $\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})$ as U we have $VW \subseteq U$, where $s \in V \in \tau_S$, $x \in W \in \prod_{\alpha \in \Lambda} \tau_\alpha$. Hence $(\prod_{\alpha \in \Lambda} A_\alpha, \prod_{\alpha \in \Lambda} \tau_\alpha)$ is a topological S-act.

Let (Q, τ_Q) be a topological S-act and $f_\alpha : Q \to A_\alpha$ be a family of morphisms for all $\alpha \in \Lambda$. Define $f : Q \to \prod_{\alpha \in \Lambda} A_\alpha$ by $f(x) = (f_\alpha(x))_\alpha$. Now for $U_\alpha \in \tau_\alpha$, $x \in f^{-1}(p_\alpha^{-1}(U_\alpha))$ if and only if $f(x)(\alpha) \in U_\alpha$ if and only if $x \in f_\alpha^{-1}(U_\alpha)$. Therefore the continuity of f_α implies that $f^{-1}(p_\alpha^{-1}(U_\alpha)) = f_\alpha^{-1}(U_\alpha) \in \tau_Q$. Hence f is a continuous S-map from (Q, τ_Q) to $(\prod_{\alpha \in \Lambda} A_\alpha, \prod_{\alpha \in \Lambda} \tau_\alpha)$ such that $p_\alpha f = f_\alpha$ for all $\alpha \in \Lambda$.

Again let $g : Q \to \prod_{\alpha \in \Lambda} A_\alpha$ be another continuous S-map such that $p_\alpha g = f_\alpha$ holds for all $\alpha \in \Lambda$. Then for $y \in Q$, $p_\alpha g(y) = f_\alpha(y)$ for all $\alpha \in \Lambda$, which in turn implies that $g(y) = (f_\alpha(y))_\alpha = f(y)$. Therefore $f = g$. This completes the proof. ∎

Notation 3.3. In what follows we write $\prod_{\alpha \in \Lambda}(A_\alpha, \tau_\alpha)$ for $(\prod_{\alpha \in \Lambda} A_\alpha, \prod_{\alpha \in \Lambda} \tau_\alpha)$. If $(A_\alpha, \tau_\alpha) = (A, \tau)$ for all $\alpha \in \Lambda$ then we use the notation $\prod_{\alpha \in \Lambda}(A, \tau)$ for $\prod_{\alpha \in \Lambda}(A_\alpha, \tau_\alpha)$.

The following result describes the coproduct in the category of topological S-acts.

Proposition 3.4. *Let $(A_\alpha, \tau_\alpha)_{\alpha \in \Lambda}$ be a collection of topological S-acts. Suppose $\coprod_{\alpha \in \Lambda} A_\alpha$ is the coproduct of $(A_\alpha)_{\alpha \in \Lambda}$ in S-**Act** with canonical injections $\iota_\beta : A_\beta \to \coprod_{\alpha \in \Lambda} A_\alpha$ for $\beta \in \Lambda$. Then $(\coprod_{\alpha \in \Lambda} A_\alpha, \coprod_{\alpha \in \Lambda} \tau_\alpha)$ is the coproduct of the family $(A_\alpha, \tau_\alpha)_{\alpha \in \Lambda}$ in S-**Top**, where $\coprod_{\alpha \in \Lambda} \tau_\alpha$ is the disjoint union topology, and $\coprod_{\alpha \in \Lambda} \tau_\alpha$ is defined to be the finest topology on $\dot{\bigcup}_{\alpha \in \Lambda} A_\alpha$ such that each $\iota_\beta : A_\beta \to \dot{\bigcup}_{\alpha \in \Lambda} A_\alpha$ is continuous.*

Proof. Suppose $\dot{\bigcup}_{\alpha \in \Lambda} A_\alpha$ is the disjoint union of the family $(A_\alpha)_{\alpha \in \Lambda}$ of S-acts with injections $\iota_\beta : A_\beta \to \coprod_{\alpha \in \Lambda} A_\alpha$ defined by $\iota_\beta(a) := (a, \beta)$, where $\beta \in \Lambda$, $a \in A_\beta$. Then we know from [7] that the disjoint union together with the S-action defined on it as

$$\begin{aligned} S \times \dot{\bigcup}_{\alpha \in \Lambda} A_\alpha &\to \dot{\bigcup}_{\alpha \in \Lambda} A_\alpha \\ (s, (a, \beta)) &\mapsto (sa, \beta) \end{aligned}$$

is the coproduct of $(A_\alpha)_{\alpha \in \Lambda}$ in S-**Act** and is denoted by $\coprod_{\alpha \in \Lambda} A_\alpha$.

Let $s(a,\beta) \in U \in \coprod_{\alpha\in\Lambda} \tau_\alpha$ for some $s \in S$, $(a,\beta) \in \coprod_{\alpha\in\Lambda} A_\alpha$. Then $(sa,\beta) \in U$, i.e., $sa \in \iota_\beta^{-1}(U) \in \tau_\beta$. Now (A_β, τ_β) being a topological S-act there exist $V \in \tau_S$ containing s and $W_\beta \in \tau_\beta$ containing a such that $VW_\beta \subseteq \iota_\beta^{-1}(U) = U_\beta$ (say). Denoting $\iota_\beta(W_\beta)$ as W, we have $(a,\beta) \in W \in \coprod_{\alpha\in\Lambda} \tau_\alpha$ such that $VW = \iota_\beta(VW_\beta) \subseteq U$. Hence $(\coprod_{\alpha\in\Lambda} A_\alpha, \coprod_{\alpha\in\Lambda} \tau_\alpha)$ is a topological S-act.

Let (Q, τ_Q) be a topological S-act and $f_\alpha : A_\alpha \to Q$ be a family of morphisms for all $\alpha \in \Lambda$. Define $f : \coprod_{\alpha\in\Lambda} A_\alpha \to Q$ by $f((a,\alpha)) = f_\alpha(a)$, where $\alpha \in \Lambda$, $a \in A_\alpha$. Clearly f is an S-map. Now let $m \in f^{-1}(V) \subseteq \coprod_{\alpha\in\Lambda} A_\alpha$. Therefore $m = (a,\beta)$ for some $\beta \in \Lambda$, $a \in A_\beta$. Now $(a,\beta) \in f^{-1}(V)$ implies that $f_\beta(a) \in V$ whence $a \in f_\beta^{-1}(V)$, i.e., $m \in \iota_\beta(f_\beta^{-1}(V))$. So $f^{-1}(V) \subseteq \bigcup_{\alpha\in\Lambda} \iota_\alpha(f_\alpha^{-1}(V))$. The reverse inclusion follows in a similar manner. Thus $f^{-1}(V) = \bigcup_{\alpha\in\Lambda} \iota_\alpha(f_\alpha^{-1}(V))$, which is clearly open in $\coprod_{\alpha\in\Lambda} A_\alpha$. Thus we have a continuous S-map f such that $f\iota_\alpha = f_\alpha$ for all $\alpha \in \Lambda$.

Let $g : \coprod_{\alpha\in\Lambda} A_\alpha \to Q$ be another continuous S-map such that $g\iota_\alpha = f_\alpha$ holds for all $\alpha \in \Lambda$, i.e., for any $a \in A_\alpha$, $g\iota_\alpha(a) = f_\alpha(a)$ for all $\alpha \in \Lambda$. Therefore $g(a,\alpha) = f_\alpha(a)$ which implies that $g = f$. This completes the proof. ■

Notation 3.5. In what follows we write $\coprod_{\alpha\in\Lambda}(A_\alpha, \tau_\alpha)$ for $(\coprod_{\alpha\in\Lambda} A_\alpha, \coprod_{\alpha\in\Lambda} \tau_\alpha)$. If $(A_\alpha, \tau_\alpha) = (A, \tau)$ for all $\alpha \in \Lambda$ then we use the notation $\coprod_\Lambda (A, \tau)$ for $\coprod_{\alpha\in\Lambda}(A_\alpha, \tau_\alpha)$.

Remark 3.6. The coproduct, described in the above proposition for S**-Top** when restricted to S**-CReg** (the category of completely regular Hausdorff S-acts), is the same as that of Khosravi [6] which is explained below.

Suppose $(A_\alpha, \tau_\alpha)_{\alpha\in\Lambda}$ is a family of topological S-acts in the category [6] S**-CReg** of completely regular Hausdorff S-acts with continuous S-maps between them as morphisms, where S is a Hausdorff completely regular topological monoid and (A, τ) is the coproduct of $(A_\alpha, \tau_\alpha)_{\alpha\in\Lambda}$ in S**-Top**. Let F be a closed subset of A and $(a,\beta) \in A \smallsetminus F$ for some $\beta \in \Lambda$, $a \in A_\beta$. Now since $\iota_\beta^{-1}(F)$ is closed in (A_β, τ_β), there exists a continuous map $f_\beta : A_\beta \to \mathbb{R}$ such that $f_\beta(\iota_\beta^{-1}(F)) = 1$ and $f_\beta(a) = 0$, and for every $\alpha \in \Lambda$, $\alpha \neq \beta$ define $f_\alpha : A_\alpha \to \mathbb{R}$ by $f_\alpha(x) = 1$ for all $x \in A_\alpha$. Now consider the mapping $f : A \to \mathbb{R}$ given by $(y,\alpha) \mapsto f_\alpha(y)$, $\alpha \in \Lambda$, $y \in A_\alpha$. Then clearly f is a continuous real valued function such that $f(F) = 1$, $f((a,\beta)) = 0$. Therefore (A, τ) is completely regular. Now for $(x,\alpha), (y,\gamma) \in A$ with $\alpha \neq \gamma$ in Λ there exist open sets $\iota_\alpha(A_\alpha), \iota_\gamma(A_\gamma) \in \tau$ such that $\iota_\alpha(A_\alpha) \cap \iota_\gamma(A_\gamma) = \phi$. Again for $(m,\alpha), (n,\alpha) \in A$ with $m \neq n$ there exist $U_\alpha, V_\alpha \in \tau_\alpha$ containing m, n respectively such that $U_\alpha \cap V_\alpha = \phi$. Therefore $\iota_\alpha(U_\alpha), \iota_\alpha(V_\alpha) \in \tau$ such that $\iota_\alpha(U_\alpha) \cap \iota_\alpha(V_\alpha) = \phi$. Hence (A, τ) is a completely regular Hausdorff S-act and thus is the coproduct of the family $(A_\alpha, \tau_\alpha)_{\alpha\in\Lambda}$ in S**-CReg**.

Definition 3.7. [5] *Let (S, τ_S) be a topological monoid. A topological S-act (F, τ_F) together with a map $\iota : X \to F$ is said to be a free topological S-act over a given* *set X if for any topological S-act (A, τ_A) and for any mapping $\sigma : X \to A$, there exists a unique continuous S-map $\overline{\sigma} : (F, \tau_F) \to (A, \tau_A)$ such that $\overline{\sigma}\iota = \sigma$.*

We recall from [7] that for a monoid S, the free S-act over a set X is the S-act $S \times (S \times X) \to S \times X,\ (s, (t, x)) \mapsto (st, x)$ for $t, s \in S$ and $x \in X$ together with the map $\iota : X \to S \times X,\ x \mapsto (1_S, x)$. From now on we denote this act as $F(X)$. Now by providing a direct proof we revisit the following result of Khosravi [5, Proposition 3.9].

Proposition 3.8. [5] *Let (S, τ_S) be a topological monoid and X be a set. Then the free topological S-act on the set X is $F(X)$ with the topology $\tau_{S\times X}$, where τ_X in the definition of $\tau_{S\times X}$ is the discrete topology, and $\tau_{S\times X}$ is the product topology on $S \times X$.*

Proof. Consider the one-one map $\iota : X \to F(X)$ defined by $x \mapsto (1_S, x)$ and for a topological S-act (A, τ_A) consider a function $\sigma : X \to A$. We define $\overline{\sigma} : (F(X), \tau_{S\times X}) \to (A, \tau_A)$ by $\overline{\sigma}((s, x)) = s\sigma(x)$. Clearly $\overline{\sigma}$ is an S-map. Let $s\sigma(x) \in U \in \tau_A$. Then (A, τ_A) being a topological S-act, there exist $V \in \tau_S,\ W \in \tau_A$ such that $s \in V,\ \sigma(x) \in W$ and $VW \subseteq U$. Thus there exist $V \in \tau_S$ containing s and $\sigma^{-1}(W) \in \tau_X$ containing x such that $\overline{\sigma}(V \times \sigma^{-1}(W)) \subseteq VW \subseteq U$. Hence $\overline{\sigma}$ is a continuous S-map such that $\overline{\sigma}\iota(x) = \overline{\sigma}((1_S, x)) = \sigma(x)$, i.e., $\overline{\sigma}\iota = \sigma$. ■

Proposition 3.9. *Let (S, τ_S) be a topological monoid and X be a non-empty set. Then $\coprod_X (S, \tau_S)$ (cf. Notation 3.5) together with the map $f : X \to \coprod_X (S, \tau_S)$ defined by $f(x) := (1_S, x)$, is free over X in **S-Top**.*

Proof. Let (A, τ_A) be a topological S-act and $g : X \to A$ be a mapping. We define $\overline{g} : \coprod_X (S, \tau_S) \to (A, \tau_A)$ by $\overline{g}((s, x)) := sg(x)$. Clearly $\overline{g}$ is an S-map. Let $V \in \tau_A,\ t \in \iota_x^{-1}(\overline{g}^{-1}(V))$ where for $x \in X,\ \iota_x : (S, \tau_S) \to \coprod_X (S, \tau_S)$ is the natural injection given by $s \mapsto (s, x)$. Then $tg(x) \in V$, which implies that there exist $U_t \in \tau_S$ and $W \in \tau_A$ with $t \in U_t,\ g(x) \in W$ such that $U_t W \subseteq V$. Let $s \in U_t$. Then $\overline{g}((s, x)) = sg(x) \in U_t W \subseteq V$ which implies $(s, x) \in \overline{g}^{-1}(V)$, i.e., $s \in \iota_x^{-1}(\overline{g}^{-1}(V))$. Thus for every $t \in \iota_x^{-1}(\overline{g}^{-1}(V))$, there exists $U_t \in \tau_S$ such that $t \in U_t \subseteq \iota_x^{-1}(\overline{g}^{-1}(V))$. Hence $\iota_x^{-1}(\overline{g}^{-1}(V))$ is open in S implying the continuity of the S-map $\overline{g}$ such that for $x \in X,\ \overline{g}f(x) = \overline{g}((1_S, x)) = g(x)$.

Let h be another continuous S-map such that $hf = g$. Then we have, for all $x \in X$,

$$\begin{aligned} & hf(x) = \overline{g}f(x) \\ \text{i.e., } & h((1_S, x)) = \overline{g}((1_S, x)) \\ \text{i.e., } & sh((1_S, x)) = s\overline{g}((1_S, x)) \\ \text{i.e., } & h((s, x)) = \overline{g}((s, x)) \\ \text{i.e., } & h = \overline{g}. \end{aligned}$$

This completes the proof. ■

Remark 3.10. It follows from the above result that any topological monoid (S, τ_S) is a free topological S-act.

Corollary 3.11. *Let (S, τ_S) be a topological monoid. A topological S-act (F, τ_F) is free over a set X if and only if it is isomorphic to $\coprod_X (S, \tau_S)$.*

Proof. In view of Definition 3.7 and Proposition 3.9 the result follows from the categorical fact that free object over a set in a category is unique upto isomorphism. ■

Proposition 3.12. *For any topological S-act (A, τ_A) there exists a free topological S-act (F, τ_F) such that (A, τ_A) is an epimorphic image of (F, τ_F).*

Proof. Let $(F(A), \tau)$ be the free topological S-act over the set A where $\iota : A \to F(A)$ is given by $\iota(a) = (1_S, a)$. Then by Definition 3.7, for the identity map $id_A : A \to A$, there exists a continuous S-map $f : (F(A), \tau) \to (A, \tau_A)$ such that $f\iota = id_A$. Now f being a surjective continuous S-map is an epimorphism. Hence (A, τ_A) is an epimorphic image of a free topological S-act. ■

Definition 3.13. *A topological S-act (P, τ_P) is projective in S-**Top** category, if for any epimorphism $\pi : (A, \tau_A) \to (B, \tau_B)$ between two topological S-acts $(A, \tau_A), (B, \tau_B)$ and any morphism $\varphi : (P, \tau_P) \to (B, \tau_B)$, there exists a morphism $\overline{\varphi} : (P, \tau_P) \to (A, \tau_A)$ such that $\varphi = \pi\overline{\varphi}$.*

Proposition 3.14. *Every free topological S-act is projective.*

Proof. It is well known [7] that in a concrete category if epimorphisms are surjective then every free object is projective. We prove here that in S-**Top** epimorphisms are surjective which in turn proves the result.

Let $f : (A, \tau_A) \to (B, \tau_B)$ be an epimorphism in S-**Top**. Define the relation θ on B by $x\theta y$ if and only if either $x = y$ or $x, y \in Imf$. Then for $x \neq y$ in B, $x\theta y$ implies that there exist $m, n \in A$ such that $x = f(m),\ y = f(n)$. Therefore for $s \in S,\ sx = f(sm),\ sy = f(sn)$, which implies that $sx\theta sy$. Hence θ is a congruence on B and B/θ together with the indiscrete topology τ is a topological S-act where the action is defined as

$$\begin{aligned} S \times B/\theta &\to B/\theta \\ (s, [x]_\theta) &\mapsto [sx]_\theta. \end{aligned}$$

Now define

$$\begin{array}{cccl} g : B \to B/\theta & \text{and} & h : B \to B/\theta & \text{by} \\ x \mapsto [x]_\theta & \text{and} & x \mapsto [f(c)]_\theta & \text{for some fixed } c \in A. \end{array}$$

Since τ is indiscrete both the S-maps are continuous such that $gf(a) = [f(a)]_\theta = [f(c)]_\theta = hf(a)$, for all $a \in A$. Therefore we have $gf = hf$, which implies that

$g = h$, since f is an epimorphism. Thus for any $x \in B$, $[x]_\theta = g(x) = h(x) = [f(c)]_\theta$, which implies $B = Imf$. Hence f is surjective. ■

We recall below one result on projective topological S-acts from [6, Proof of Lemma 2.1] for its immediate use in Example 3.17.

Proposition 3.15. [6] *For any idempotent $e \in S$, Se together with the subspace topology τ_{Se} is a projective topological S-act.*

Remark 3.16. That the converse of Prop. 3.14 is not true is illustrated in the following example.

Example 3.17. Consider the topological monoid $(\mathbb{Z}, \tau_{dis})$, where $\mathbb{Z}$ is the multiplicative monoid and τ_{dis} is the discrete topology. Then in view of Prop. 3.15, $(\{0\}, \tau_{\{0\}})$ is a projective topological $\mathbb{Z}$-act where $\tau_{\{0\}} = \{\phi, \{0\}\}$. But we show below that it is not free. Suppose it is free over a set X with corresponding mapping $\iota : X \to \{0\}$ defined by $x \mapsto 0$ for all $x \in X$. Consider the topological $\mathbb{Z}$-act $(\mathbb{Z}, \tau_{dis})$ and a map $f : X \to \mathbb{Z}$ given by $x \mapsto 1$ for all $x \in X$. Then there exists continuous $\mathbb{Z}$-map $\overline{f} : (\{0\}, \tau_{\{0\}}) \to (\mathbb{Z}, \tau_{dis})$ such that $\overline{f}\iota = f$ which implies that $\overline{f}(0) = 1$ - a contradiction since $\overline{f}$ is a $\mathbb{Z}$-map. Hence $(\{0\}, \tau_{\{0\}})$ is not free.

Definition 3.18. *We call a topological S-act (A, τ_A) decomposable if there is an indexed set Λ of cardinality at least two and non-empty closed proper subacts X_i of A, $i \in \Lambda$ such that $A = \bigcup_{i\in\Lambda} X_i$ and for each pair $i, j \in \Lambda$, with $i \neq j$, $X_i \cap X_j = \phi$. In this case $A = \bigcup_{i\in\Lambda} X_i$ is called a decomposition of (A, τ_A). Otherwise (A, τ_A) is called indecomposable. A subact B of A is said to be indecomposable if (B, τ_B) is an indecomposable topological S-act where τ_B is the induced topology.*

Remark 3.19. Recall that [7] an S-act A is called decomposable in S**-Act** if there exist two subacts $B, C \subseteq A$ such that $A = B \cup C$ and $B \cap C = \phi$. Otherwise A is called indecomposable. We call a topological S-act (A, τ_A) algebraically indecomposable if the underlying S-act A is indecomposable in S**-Act**, whereas this notion is called indecomposable topological S-act by Khosravi [6]. Clearly every algebraically indecomposable topological S-act is indecomposable. But the converse is not true which is evident from the following example.

Example 3.20. Let us consider the topological multiplicative monoid $(\mathbb{N}, \eta)$, where η is the discrete topology and the topological $\mathbb{N}$-act $(\mathbb{Z}, \tau)$ with τ as the indiscrete topology and the action given by

$$\begin{aligned} \mathbb{N} \times \mathbb{Z} &\to \mathbb{Z} \\ (n, a) &\mapsto na. \end{aligned}$$

Here $(\mathbb{Z},\tau)$ is indecomposable since it has no non-empty closed proper subact. But there are subacts $\mathbb{Z}^+\cup\{0\},\mathbb{Z}^-$ such that $\mathbb{Z}=(\mathbb{Z}^+\cup\{0\})\cup\mathbb{Z}^-$. Hence $(\mathbb{Z},\tau)$ is algebraically decomposable.

Lemma 3.21. *For topological S-act (A,τ_A), let $(A_i)_{i\in I}$ be subacts of A such that (A_i,τ_i) (τ_i's are subspace topologies) are indecomposable topological S-acts. Then $\bigcup_{i\in I}A_i$ equipped with the subspace topology τ^* is an indecomposable topological S-act whenever $\bigcap_{i\in I}A_i\neq\phi$.*

Proof. Clearly $(\bigcup_{i\in I}A_i,\tau^*)$ is a topological S-act. Let $\bigcup_{i\in I}A_i=\bigcup_{\alpha\in\Lambda}X_\alpha$ be a decomposition of $(\bigcup_{i\in I}A_i,\tau^*)$, where X_α's are non-empty closed proper subacts in $\bigcup_{i\in I}A_i$. Take $x\in\bigcap_{i\in I}A_i$ with $x\in X_\beta$ for some $\beta\in\Lambda$. Then for $k\in I,\ A_k=\bigcup_{\alpha\in\Lambda}(A_k\cap X_\alpha)$, where $(A_k\cap X_\alpha)$ is a closed subact of A_k for all $\alpha\in\Lambda$. But since (A_k,τ_k) is indecomposable, it follows that $A_k\cap X_\alpha=\phi$ for all $\alpha\in\Lambda,\ \alpha\neq\beta$. This is true for all $k\in I$. Therefore $\bigcup_{i\in I}A_i=X_\beta$ - a contradiction. Hence the proof. ■

Theorem 3.22. *Every topological S-act (A,τ_A) has a unique decomposition into indecomposable subacts.*

Proof. Take $a\in A$. Since the cyclic S-act Sa is indecomposable in S-**Act**[7], Sa equipped with subspace topology τ_{Sa} induced by τ_A is indecomposable in S-**Top**. Let $Sub(A)$ be the collection of all subacts of A. Then by Lemma 3.21, we get that $U_a=\cup\{V\in Sub(A)\mid (V,\tau_V)$ is indecomposable and $a\in V\}$ (where τ_V is the subspace topology on V) together with the subspace topology τ_a induced by τ_A is indecomposable topological S-act.

Let $\overline{U_a}$ denote the closure of U_a in (A,τ_A). We claim to prove that $\overline{U_a}$ is an indecomposable subact of (A,τ_A). For this, let $s\in S,\ b\in\overline{U_a}$ and U be an open set in A containing sb. Then (A,τ_A) being a topological S-act there exist $W\in\tau_A$ containing b such that $sW\subseteq U$. Now $b\in W\in\tau_A$ implies that there exists some $y\in W\cap U_a$ such that $sy\in U_a\cap sW\subseteq U_a\cap U$, i.e., $U_a\cap U\neq\phi$. Hence $sb\in\overline{U_a}$. Now if $\overline{U_a}=\bigcup_{i\in I}X_i$, where X_i's are closed proper subacts of $\overline{U_a}$, then $U_a=\bigcup_{i\in I}(X_i\cap U_a)$. But since U_a is indecomposable we must have $U_a=X_k\cap U_a$ for some $k\in I$, which in turn implies that $\overline{U_a}=X_k$ - a contradiction. Thus $\overline{U_a}$ together with the induced topology is an indecomposable topological S-act containing a. Therefore $U_a=\overline{U_a}$, i.e., U_a is closed.

For $x,y\in A$, we get that $U_x=U_y$ or $U_x\cap U_y=\phi$. Indeed, $z\in U_x\cap U_y$ implies $U_x,U_y\subseteq U_z$. Thus $x\in U_x\subseteq U_z,\ y\in U_y\subseteq U_z$, i.e., $U_z\subseteq U_x\cap U_y$. Therefore $U_x=U_y=U_z$. Denote by A' a representative subset of elements $x\in A$ with respect to the equivalence relation $\sim$ defined by $x\sim y$ if and only if $U_x=U_y$. Then $A=\underset{x\in A'}{\cup}U_x$ is a decomposition of A in indecomposable subacts.

Now for uniqueness, let $A=\underset{\alpha\in B}{\cup}V_\alpha$ be another decomposition of (A,τ_A) into indecomposable subacts. Then there exists at least one U_y for some $y\in A'$, such that $U_y\neq V_\alpha$ for all $\alpha\in B$. Now $U_y=A\cap U_y=\underset{\alpha\in B}{\cup}(V_\alpha\cap U_y)$. For $a\in V_\beta\cap U_y$

for some $\beta \in B$ implies $V_\beta \subseteq U_a = U_y$. By hypothesis we have $U_y \neq V_\beta$ therefore for $\alpha \in B$, either $V_\alpha \cap U_y = \phi$ or $V_\alpha \subsetneq U_y$. Let $J = \{\alpha \in B \mid V_\alpha \subsetneq U_y\}$. It is evident that J is a non-empty, non-singleton set such that $U_y = \underset{\alpha \in J}{\cup} V_\alpha$, where V_α is indecomposable subact for all $\alpha \in J$. Thus we have a decomposition of the topological S-act (U_y, τ_y) - a contradiction. Hence $A = \underset{x \in A'}{\cup} U_x$ is the unique decomposition of A in indecomposable subacts. ■

Theorem 3.23. *For any indecomposable projective topological S-act (P, τ) there exists an idempotent $e \in S$ such that (P, τ) is isomorphic to (Se, τ_{Se}), where τ_{Se} is the subspace topology.*

Proof. For any $p \in P$, consider the continuous S-map $\sigma_p : (S, \tau_S) \to (P, \tau)$ defined by $s \mapsto sp$. Then there exists a continuous S-map

$$\sigma = \coprod_{p \in P} \sigma_p : \coprod_{p \in P} (S_p, \tau_p) \to (P, \tau) \qquad ((S_p, \tau_p) = (S, \tau_S))$$
$$(s, p) \mapsto \sigma_p(s)$$

such that $Im\sigma = P$. Therefore (P, τ) being projective there exists a continuous S-map $\gamma : (P, \tau) \to \coprod_{p \in P} (S_p, \tau_p)$ such that $\sigma\gamma = id_P$. Consider $(\gamma(P), \tau^*)$, where τ^* is the subspace topology, i.e., $\tau^* = \{U \cap \gamma(P) \mid U \in \coprod_{p \in P} \tau_p\}$. Then $V \in \tau^*$ implies that $V = V' \cap \gamma(P)$ for some $V' \in \coprod_{p \in P} \tau_p$, which implies that $\gamma^{-1}(V) = \gamma^{-1}(V') \in \tau$. Hence $\gamma : (P, \tau) \to (\gamma(P), \tau^*)$ is continuous and also $\sigma^* = \sigma|_{\gamma(P)} : (\gamma(P), \tau^*) \to (P, \tau)$ is continuous such that $\sigma^*\gamma = \sigma\gamma = id_P$ and $\gamma\sigma^* = id_{\gamma(P)}$. Hence $(\gamma(P), \tau^*)$ is isomorphic to (P, τ) and thus is indecomposable. Now consider the injections $\iota_p : S_p \to \coprod_{x \in P} S_x$ defined by $s \mapsto (s, p)$. Then we have an algebraic decomposition of $\gamma(P)$ as follows :

$$\gamma(P) = \underset{x \in P}{\cup} (\gamma(P) \cap \iota_x(S)) = \underset{x \in P}{\cup} A_x. \tag{1}$$

Then for any $p \in P$,

$$\gamma(P) \setminus A_p = \underset{x \in P \setminus \{p\}}{\cup} A_x = \gamma(P) \cap \left(\underset{x \in P \setminus \{p\}}{\cup} \iota_x(S) \right) \in \tau^*.$$

Also $SA_p = S(\gamma(P) \cap \iota_p(S)) \subseteq (\gamma(P) \cap \iota_p(S)) = A_p$. Therefore A_p is a closed subact of $\gamma(P)$ for all $p \in P$. Now since $(\gamma(P), \tau^*)$ is indecomposable, therefore $\gamma(P) \subseteq \iota_m(S)$ for a unique $m \in P$. So we have, $P = id_P(P) = \sigma\gamma(P) \subseteq \sigma\iota_m(S) = \sigma_m(S) = Sm \subseteq P$, i.e., $P = Sm$.

Now for the epimorphism, $\sigma_m : (S, \tau_S) \to (P, \tau)$, there exists a continuous S-map $\varphi : (P, \tau) \to (S, \tau_S)$ such that $\sigma_m\varphi = id_P$. Denote $\varphi(m) = e \in S$. Since $m = id_P(m) = \sigma_m\varphi(m) = \sigma_m(e) = em$, we have $e = \varphi(m) = \varphi(em) = e\varphi(m) = e^2$. Again $\varphi(P) = \varphi(Sm) = S\varphi(m) = Se$.

Also (P,τ) is isomorphic to $\varphi(P)$ together with subspace topology. Therefore (P,τ) is isomorphic to (Se,τ_{Se}). ■

Remark 3.24. As mentioned earlier that by indecomposable topological S-acts Khosravi [6] meant the topological S-acts which are algebraically indecomposable and obtained a characterization [6, Lemma 2.1] similar to as that of Theorem 3.23, which we recall below.

Theorem 3.25. [6] *Any indecomposable projective S-space P is cyclic and there exists $e^2 = e \in S$ such that P is topologically isomorphic to Se.*

Khosravi [6, Theorem 2.2] proved the following result using Theorem 3.25. But it can be proved by using our result given in Theorem 3.23.

Theorem 3.26. *A topological S-act (P,τ_P) is projective if and only if $(P,\tau_P) = \coprod_{i\in I}(P_i,\tau_i)$ where each (P_i,τ_i) is isomorphic to (Se_i,τ_{Se_i}) for some idempotent $e_i \in S$ together with subspace topology $\tau_{Se_i},\ i \in I$.*

To conclude the paper we introduce the notion of generator in the category S-**Top** and characterize it (cf. Theorem 3.30) which is a partial analogue of [7, Theorem 2.3.16].

Definition 3.27. *A topological S-act (G,τ_G) is said to be a generator in S-**Top** if for $f,\ g : (X,\tau_X) \to (Y,\tau_Y)$ in S-**Top** with $f \neq g$ there exists a continuous S-map $\alpha : (G,\tau_G) \to (X,\tau_X)$ such that $f\alpha \neq g\alpha$.*

Remark 3.28. Suppose (S,τ_S) is a topological monoid and $(X,\tau_X),(Y,\tau_Y)$ are topological S-acts. Then for notational convenience we denote the set of all continuous S-maps from (X,τ_X) to (Y,τ_Y) by $C(X,Y)$ when there is no ambiguity regarding the topology of X and Y.

Before giving a characterization of generators in S-**Top** we recall the following Lemma from [7].

Lemma 3.29. [7] *Suppose C is an arbitrary category and $G \in C$ is a generator in C. If for every $X \in C$ there exists $X \coprod X$ in C such that the injections $u_1, u_2 : X \to X \coprod X$ are different, then $Mor_C(G,X) \neq \phi$ for all $X \in C$, where $Mor_C(G,X)$ denotes the set of all morphisms from G to X in C.*

Theorem 3.30. *Suppose (S,τ_S) is a topological monoid. For $(G,\tau_G) \in S$-**Top**, the following conditions are equivalent:*

(i) *(G,τ_G) is a generator in S-**Top**.*

(ii) *Every $(X,\tau_X) \in S$-**Top** is an epimorphic image of $\coprod\limits_{C(G,X)} (G,\tau_G)$.*

(iii) *For every* $(X, \tau_X) \in S$-***Top*** *there exists a set* I *such that* (X, τ_X) *is an epimorphic image of* $\coprod\limits_I (G, \tau_G)$.

(iv) *There exists an epimorphism* $\pi : (G, \tau_G) \to (S, \tau_S)$.

(v) (S, τ_S) *is a retract of* (G, τ_G).

(vi) *There exists* $\psi^2 = \psi \in C(G, G)$ *such that* $\psi(G)$ *is topologically isomorphic to* (S, τ_S).

Proof. (i)⇒(ii). Suppose $(X, \tau_X), (Y, \tau_Y) \in S$-**Top** and $f, g : (X, \tau_X) \to (Y, \tau_Y)$ are continuous S-maps such that $f \neq g$. We already have from Lemma 3.29 that $C(G, X) \neq \phi$. Now consider the diagram in S-**Top** (see Fig. 1).

$$\begin{tikzcd}
(G,\tau_G) \arrow[r, "\alpha"] \arrow[d, "\iota_\alpha"'] & (X,\tau_X) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,\tau_Y) \\
\coprod\limits_{\alpha \in C(G,X)} (G,\tau_G) \arrow[ur, "{[(\alpha)]}"'] & &
\end{tikzcd}$$

Fig. 1

where ι_α's are the canonical injections into $\coprod\limits_{C(G,X)} (G, \tau_G)$ and $[(\alpha)]$ is coproduct induced. By (i) there exists $\beta \in C(G, X)$ such that $f\beta \neq g\beta$. Therefore if we assume that $f[(\alpha)] = g[(\alpha)]$ then we have $f[(\alpha)]\iota_\beta = g[(\alpha)]\iota_\beta$ which implies that $f\beta = g\beta$ - a contradiction. This proves that $[(\alpha)]$ is an epimorphism.

(iii) follows trivially from (ii).

(iii)⇒(iv). Let $f : \coprod_{i \in I}(G_i, \tau_i) \to (S, \tau_S)$ be an epimorphism, where $(G_i, \tau_i) = (G, \tau_G)$ for all $i \in I$. Since epimorphisms are surjective in S-**Top** (cf. proof of Prop. 3.14) there exists $(g, k) \in \coprod_{i \in I}(G_i, \tau_i)$ such that $k \in I, g \in G_k$ and $f((g, k)) = 1_S$. Therefore for any $s \in S$, $s = s.1_S = s.f((g, k)) = f((sg, k)) = f\iota_k(sg)$, where $\iota_k : (G_k, \tau_k) \to \coprod_{i \in I}(G_i, \tau_i)$ denotes the canonical injection. Then $\pi = f\iota_k : (G_k, \tau_k) \to (S, \tau_S)$ is a surjection and also being the composition of two continuous S-maps is a continuous S-map. Thus $\pi : (G, \tau_G) \to (S, \tau_S)$ is an epimorphism in S-**Top**.

(iv)⇒(v). Consider the diagram in S-**Top** (see Fig. 2).

In view of Remark 3.10 and Prop. 3.14 (S, τ_S) is projective so there exists a continuous S-map $\gamma : (S, \tau_S) \to (G, \tau_G)$ such that $\pi\gamma = id_S$. Hence the proof.

(v)⇒(vi). Let $\pi : (G, \tau_G) \to (S, \tau_S)$ be a retraction in S-**Top**. Then there exists a continuous S-map $\gamma : (S, \tau_S) \to (G, \tau_G)$ such that $\pi\gamma = id_S$. Then clearly $\psi = \gamma\pi \in C(G, G)$ is an idempotent and since $\gamma(1_S) \in G$ we get that

$\gamma(1_S) = \gamma(\pi\gamma(1_S)) = (\gamma\pi)\gamma(1_S) \in \psi(G)$ i.e., $S\gamma(1_S) \subseteq \psi(G) = \gamma\pi(G) = \gamma(S) = S\gamma(1_S)$. Thus $\gamma(S) = \psi(G)$. Also since γ is a coretraction, (S, τ_S) is isomorphic to $(\gamma(S), \tau_{\gamma(S)})$, where $\tau_{\gamma(S)}$ is the subspace topology induced from τ_G. Hence $\psi(G)$ is topologically isomorphic to (S, τ_S).

Since $\psi(G)$ is topologically isomorphic to (S, τ_S), (iv) follows from (vi).

(iv)$\Rightarrow$(i). Consider $f, g : (X, \tau_X) \to (Y, \tau_Y)$ in S-**Top** with $f \neq g$. Then there exists $x \in X$ such that $f(x) \neq g(x)$. In view of Prop. 3.9 (S, τ_S) is a free topological S-act over any singleton set $\{a\}$ so we consider the diagram (see Fig. 3)

$$\begin{tikzcd} & (S,\tau_S) \arrow[d, "id_S"] \\ (G,\tau_G) \arrow[r, "\pi"'] & (S,\tau_S) \end{tikzcd}$$

Fig. 2

$$\begin{tikzcd} \{a\} \arrow[r, "\phi"] \arrow[d, "\psi"'] & (X,\tau_X) \\ (S,\tau_S) & \end{tikzcd}$$

Fig. 3

where $\phi(a) = x,\ \psi(a) = 1_S$. Then there exists $\overline{\phi} : (S, \tau_S) \to (X, \tau_X)$ in S-**Top** such that $\overline{\phi}\psi = \phi$ i.e., $\overline{\phi}(1_S) = x$. Then we have $\overline{\phi}\pi : (G, \tau_G) \to (X, \tau_X)$ such that $f(\overline{\phi}\pi) \neq g(\overline{\phi}\pi)$, since π is an epimorphism. Hence the proof. ■

4. Concluding Remark

The results obtained in the paper may be considered to be some of the necessary tools required to initiate the study of Morita equivalence of topological monoids whose counterpart for monoids and semigroups has been a topic of sustained research interest which is evident from various works mentioned in [7] and [14, 2, 13].

Acknowledgement. The authors are grateful to the learned referees for their valuable suggestions.

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