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Odd Harmonious Labeling of Complete Bipartite Graphs

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Abstract. A graph $G(p,q)$ is said to be odd harmonious if there exists an injection $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ such that the induced function $f^* : E(G) \to \{1,3,\cdots,2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. In this paper we prove that path union of r copies of $K_{m,n}$, path union of r copies of K_{m_i,n_i}, $1 \leq i \leq r$, $K^t_{m,n}$, $K^t_{(m_1,n_1),(m_2,n_2),\cdots,(m_t,n_t)}$, join sum of graph $\langle K_{m,n}; K_{m,n}; \cdots, K_{m,n}(t\ copies)\rangle$, $\langle K_{m_1,n_1}; K_{m_2,n_2}; \cdots, K_{m_t,n_t}\rangle$, circle formation of r copies of $K_{m,n}$ when $r \equiv 0\ (mod\ 4)$, $S(t.K_{m,n})$ and $P^t_n(t.n.K_{p,q})$ are odd harmonious graphs.

Keywords: Harmonious labeling; Odd harmonious labeling; Path union of graphs; Open star of graphs; Join sum of graphs; One point union of path of graphs.

1. Introduction

Throughout this paper by a graph we mean a finite, simple and undirected one. For standard terminology and notation we follow Harary [4]. A graph

$G = (V, E)$ with p vertices and q edges is called a (p, q) − graph. The graph labeling is an assignment of integers to the set of vertices or edges or both, subject to certain conditions. An extensive survey of various graph labeling problems is available in [2]. Labeled graphs serves as useful mathematical models for many applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encoding of integers. Graham and Sloane [3] introduced harmonious labeling during their study of modular versions of additive bases problems stemming from error correcting codes. A graph G is said to be harmonious if there exists an injection $f : V(G) \rightarrow \mathbb{Z}_q$ such that the induced function $f^* : E(G) \rightarrow \mathbb{Z}_q$ defined by $f^*(uv) = (f(u) + f(v)) \pmod q$ is a bijection and f is called harmonious labeling of G. The concept of an odd harmonious labeling was due to Liang and Bai [17]. A graph G is said to be odd harmonious if there exists an injection $f : V(G) \rightarrow \{0, 1, 2, \cdots, 2q - 1\}$ such that the induced function $f^* : E(G) \rightarrow \{1, 3, \cdots, 2q - 1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. If $f(V(G)) = \{0, 1, 2, ...q\}$ then f is called as strongly odd harmonious labeling and G is called as strongly odd harmonious graph.The odd harmoniousness of graph is useful for the solution of undetermined equations. An interested reader can refer to [1, 17, 18, 19, 20]. The following results have been published in [17].

(1) If G is an odd harmonious graph, then G is a bipartite graph. Hence any graph that contains an odd cycle is not an odd harmonious.
(2) If a (p, q) − graph G is odd harmonious, then $2\sqrt{q} \leq p \leq (2q - 1)$.
(3) If G is an odd harmonious Eulerian graph with q edges, then $q \equiv 0, 2 \pmod 4$.

Motivated by the results in [17], [19] and [20], we have established that several graphs admit odd harmonious labeling (see [5] - [14]).

Definition 1.1. [15] *Let G be a graph and $G_1, G_2, G_3, \cdots, G_n$, $n \geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} $(1 \leq i \leq n - 1)$ is called path union of graph G.*

Let $u_{i,1}, u_{i,2}, \cdots, u_{i,m}$ and $v_{i,1}, v_{i,2}, \cdots, v_{i,n}$, $1 \leq i \leq r$, be the vertices of the i^{th} copy $K_{m,n}$. We join $u_{i,m}$ with $v_{i+1,1}$ by an edge, $1 \leq i \leq r - 1$ to obtain the path union of $K_{m,n}$.

Definition 1.2. [16] *A graph obtained by replacing each vertex of $K_{1,n}$ except the apex vertex by the graphs $G_1, G_2, \cdots, G_n$ is known as open star of graph, denoted by $S(G_1, G_2, G_3, \cdots, G_n)$. If we replace each vertices of $K_{1,n}$ except the apex vertex by the graph G, that is $G_1 = G_2 = \cdots = G_n = G$, such open star of graph denoted by $S(n.G)$.*

Let $v_1^i, v_2^i, \cdots, v_m^i$, $u_1^i, u_2^i, \cdots, u_n^i$, where $1 \leq i \leq t$ be the vertices of the i^{th} copy of $K_{m,n}$. The graph $G = S(t.K_{m,n})$ is obtained by replacing each vertices of $K_{1,t}$ except the apex vertex of $K_{1,t}$ by the graph $K_{m,n}$. Let u_0 be the central vertex of the graph G. We join the central vertex u_0 with the vertices v_1^i, where $1 \leq i \leq t$.

Definition 1.3. *A graph G is obtained by replacing each edge of $K_{1,t}$ by a path P_n of length n on $n+1$ vertices is called one point union for t copies of path P_n, denoted by P_n^t.*

Let $v_1^i, v_2^i, \cdots, v_m^i$ and $u_1^i, u_2^i, \cdots, u_n^i$, where $1 \leq i \leq t$ be the vertices of t copies of $K_{m,n}$. The graph $K_{m,n}^t$ is obtained by identifying the vertices v_1^i, $1 \leq i \leq t$ of each copy and consider it as a central vertex u.

Definition 1.4. [15] *Consider t copies of graph G_0. The graph $G = \langle G_0^{(1)}; G_0^{(2)}; \cdots; G_0^{(t)} \rangle$ obtained by joining two copies of the graph $G_0^{(i)}$ and $G_0^{(i+1)}$ by a vertex $1 \leq i \leq t-1$ is called join sum of graphs.*

Definition 1.5. *A graph G is obtained by replacing each vertices of P_n^t except the central vertex by the graphs $G_1, G_2, \cdots, G_n$ is known as one point union for path of graphs, denoted by $P_n^t(G_1, G_2, G_3, \cdots, G_n)$ where P_n^t is the one point union of t copies of path P_n. If we replace each vertices of P_n^t except the central vertex by the graph H, that is $G_1 = G_2 = G_3 = \cdots = G_n = H$, such one point union of path graph denoted by $P_n^t(t.n.H)$.*

Definition 1.6. *Let $u_{i,1}, u_{i,2}, \cdots, u_{i,m}$ and $v_{i,1}, v_{i,2}, \cdots, v_{i,n}$ be the vertices of the i^{th} copy of $K_{m,n}$, where $1 \leq i \leq r$. We join the vertices $u_{i,m}$ to $u_{i+1,1}$, $1 \leq i \leq r-1$ and also join the vertex $u_{r,m}$ to $u_{1,1}$ to construct the circle formation of r copies of $K_{m,n}$.*

2. Main Results

In this section we prove that path union of r copies of $K_{m,n}$, path union of r copies of K_{m_i,n_i}, $1 \leq i \leq r$, $K_{m,n}^t$, $K_{(m_1,n_1),(m_2,n_2),\cdots,(m_t,n_t)}^t$, join sum of graph $\langle K_{m,n}; K_{m,n}; \cdots, K_{m,n}(t\ copies) \rangle$, $\langle K_{m_1,n_1}; K_{m_2,n_2}; \cdots, K_{m_t,n_t} \rangle$, circle formation of r copies of $K_{m,n}$ when $r \equiv 0 (mod\ 4)$, $S(t.K_{m,n})$ and $P_n^t(t.n.K_{p,q})$ are odd harmonious graphs.

Theorem 2.1. *The path union of r copies of $K_{m,n}, r > 1$ is an odd harmonious graph.*

Proof. Let G be a path union of r copies of $K_{m,n}, r > 1$. Let $u_{i,1}, u_{i,2}, \cdots, u_{i,m}$ and $v_{i,1}, v_{i,2}, \cdots, v_{i,n}$, $1 \leq i \leq r$, be the vertices of the i^{th} copy of the graph G. In path union of graph $K_{m,n}$, $|V(G)| = r(m+n)$ and $|E(G)| = rmn + r - 1$. We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(rmn + r - 1) - 1\}$ as follows:

$$
\begin{aligned}
f(u_{1,j}) &= 2n(j-1),\ 1 \leq j \leq m, \\
f(u_{i,j}) &= 2n(m-1) + 2m(i-2) + 2j,\ 2 \leq i \leq r \text{ and } 1 \leq j \leq m, \\
f(v_{1,j}) &= 2j - 1,\ 1 \leq j \leq n, \\
f(v_{i,j}) &= 2n + 2m(i-2)(n-1) + 2m(j-1) + 2i - 3,\ 2 \leq i \leq r, 1 \leq j \leq n.
\end{aligned}
$$

The induced edge labels are

$$\begin{aligned} f^*(u_{1,j}v_{1,k}) &= 2n(j-1)+2k-1,\ 1 \le j \le m \text{ and } 1 \le k \le n, \\ f^*(u_{i,j}v_{i,k}) &= 2n(m-1+2m(i-2)+2j+2n+2m(i-2)(n-1) \\ &\quad +2m(k-1)+2i-3, 2 \le i \le r,\ 1 \le j \le m \text{ and } 1 \le k \le n, \\ f^*(u_{i,m}v_{i+1,1}) &= 2i(nm+1)-1,\ 1 \le i \le r-1. \end{aligned}$$

In view of the above defined labeling pattern, the path union of r copies of $K_{m,n}$ is an odd harmonious graph. ■

An odd harmonious labeling of 3 copies of $K_{2,3}$ is shown in Figure 1.

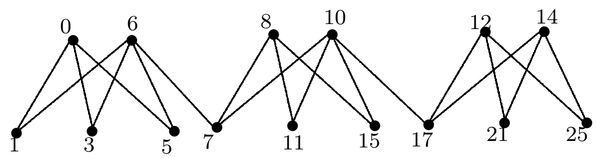

Figure 1: *An odd harmonious labeling of 3 copies of* $K_{2,3}$

Theorem 2.2. *The path union of complete bipartite graphs* $K_{m_1,n_1}, K_{m_2,n_2}, \cdots, K_{m_t,n_t}$ *is an odd harmonious graph, where* $m_1, n_1, m_2, n_2, \cdots, m_t, n_t \in \mathbb{N}$.

Proof. Let G be a path union of complete bipartite graphs $K_{m_1,n_1}, K_{m_2,n_2}, \cdots, K_{m_t,n_t}$, where $m_1, n_1, m_2, n_2, \cdots, m_t, n_t \in \mathbb{N}$. Let $u_{i,j}(1 \le j \le m_i), v_{i,j}(1 \le j \le n_i)$ be the vertices of the complete bipartite graph K_{m_i,n_i}, $i = 1, 2, \cdots, t$. In order to construct the path union of K_{m_i,n_i}, we join u_{i,m_i} with $v_{i+1,1}$, $1 \le i \le t-1$ by an edge. In path union of complete bipartite graphs K_{m_i,n_i}, $i = 1, 2, \cdots, t$,

$$\begin{aligned} |V(G)| &= (m_1+n_1)+(m_2+n_2)+\cdots+(m_t+n_t), \\ |E(G)| &= (m_1n_1+m_2n_2+\cdots+m_tn_t)+t-1. \end{aligned}$$

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2m_1n_1 + \cdots + 2m_tn_t + 2t - 3\}$ as follows:

$$\begin{aligned} f(u_{1,j}) &= 2n_1(j-1),\ 1 \le j \le m_1, \\ f(u_{2,j}) &= 2n_1(m_1-1)+2j,\ 1 \le j \le m_2, \\ f(u_{i,j}) &= 2n_1(m_1-1)+2[m_{i-1}+m_{i-2}+\cdots+m_2]+2j, \\ &\quad 1 \le j \le m_i \text{ and } i = 3, 4, 5, \cdots, t, \\ f(v_{1,j}) &= 2j-1,\ 1 \le j \le n_1, \\ f(v_{2,j}) &= 2n_1+1+2m_2(j-1),\ 1 \le j \le n_2, \\ f(v_{i,j}) &= 2n_1+2(i-2)+1+2[m_{i-1}(n_{i-1}-1)+m_{i-2}(n_{i-2}-1)+\cdots \\ &\quad +m_2(n_2-1)]+2m_i(j-1),\ 1 \le j \le n_i,\ i = 3, 4, \cdots, t. \end{aligned}$$

The induced edge labels are

$$\begin{aligned}
f^*(u_{1,j}v_{1,k}) &= 2n_1(j-1)+2k-1,\ 1\le j\le m_1 \ \text{ and } \ 1\le k\le n_1,\\
f^*(u_{i,j}v_{i,k}) &= 2[m_{i-1}n_{i-1}+m_{i-2}n_{i-2}+\cdots+m_2n_2+m_1n_1]\\
&\quad +2m_i(k-1)+2j+2i-3,\ 1\le k\le n_i,\\
&\quad 1\le j\le m_i \text{ and } i=2,3,\cdots,t,\\
f^*(u_{i,m_i}u_{i+1,1}) &= 2[n_im_i+n_{i-1}m_{i-1}+\cdots+n_2m_2+n_1m_1]+2i-1,\\
&\quad 1\le i\le t-1.
\end{aligned}$$

In view of the above defined labeling pattern, path union of complete bipartite graphs $K_{m_1,n_1}, K_{m_2,n_2},\cdots,K_{m_t,n_t}$ is an odd harmonious graph. ■

An odd harmonious labeling of path union of $K_{2,3}, K_{2,2}, K_{3,5}$ is shown in Figure 2.

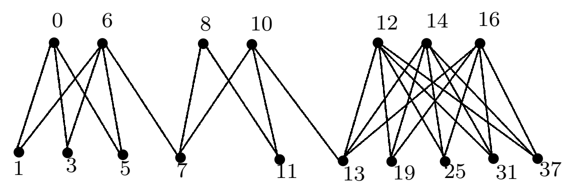

Figure 2: *An odd harmonious labeling of* $K_{2,3}, K_{2,2}, K_{3,5}$

Theorem 2.3. *The join sum of graph* $G=\langle K_{m,n};K_{m,n};\cdots;K_{m,n}(t\ \textit{copies})\rangle$ *is odd harmonious,* $t>1$.

Proof. Let G be a join sum of complete bipartite graph $K_{m,n}$ (t copies). Let $u_{i,j}$ $(1\le j\le n)$ and $v_{i,j}$ $(1\le j\le n)$ be the vertices of i^{th} copy of $K_{m,n}$, $i=1,2,\cdots,t$. Let $w_1,w_2,\cdots,w_{i-1}$ be the vertices of join sum of complete bipartite graphs. We join the vertices $(u_{i,m},w_i)$ and $(w_iu_{i+1,1})$, $i=1,2,3,\cdots,t-1$ by an edge. The join sum of complete bipartite graph is having $|V(G)|=t(m+n)+t-1$ and $|E(G)|=mnt+2(t-1)$. We define a labeling $f:V(G)\to\{0,1,2,\cdots,2[mnt+2(t-1)]-1\}$ as follows:

$$\begin{aligned}
f(u_{i,j}) &= 2n(i-1)(m-1)+2(i-1)+2n(j-1),\ 1\le i\le t,\ 1\le j\le m,\\
f(v_{i,j}) &= 2n(i-1)+2i-3+2j,\ 1\le i\le t, 1\le j\le n,\\
f(w_i) &= 2i(n+1)-1,\ 1\le i\le t-1.
\end{aligned}$$

The induced edge labels are

$$\begin{aligned}
f^*(u_{i,j}v_{i,k}) &= 2n(i-1)(m-1)+2(i-1)+2n(j-1)+2n(i-1)+2i\\
&\quad +2k-3, 1\le i\le t,\ 1\le j\le m \text{ and } 1\le k\le n,
\end{aligned}$$

$$f^*(u_{i,m}w_i) = 2i(nm+2)-3,\ 1 \le i \le t-1,$$
$$f^*(w_iu_{i+1,1}) = 2i(nm+2)-1,\ 1 \le i \le t-1.$$

In view of the above defined labeling pattern, the join sum of t copies of complete bipartite graph $K_{m,n}$ is an odd harmonious graph. ■

An odd harmonious labeling of $\langle K_{3,5}; K_{3,5}; K_{3,5}\rangle$ is shown in Figure 3.

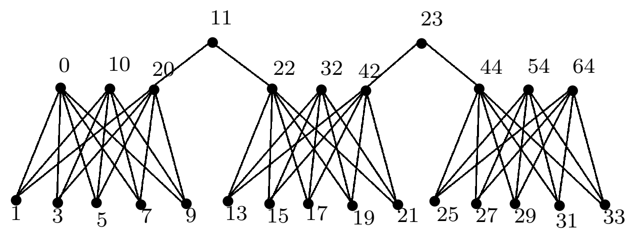

Figure 3: *An odd harmonious labeling of join sum of 3 copies of* $K_{3,5}$

Theorem 2.4. *The join sum of complete bipartite graphs* $G = \langle K_{m_1,n_1}; K_{m_2,n_2}; \cdots ; K_{m_t,n_t}\rangle$ *is odd harmonious, where* $m_1, n_1, m_2, n_2, \cdots, m_t, n_t \in \mathbb{N}$.

Proof. Let G be a join sum of complete bipartite graphs $K_{m_1,n_1}; K_{m_2,n_2}; \cdots ; K_{m_t,n_t}\rangle$, where $m_1, n_1, \cdots, m_t, n_t \in \mathbb{N}$. Let $u_{i,j}$ $(1 \le j \le m_i)$ and $v_{i,j}$ $(1 \le j \le n_i)$ be vertices of the complete bipartite graphs K_{m_i,n_i}, $i = 1, 2, \cdots, t$. Let $w_1, w_2, \cdots, w_{t-1}$ be the vertices of join sum of the complete bipartite graphs. We join the vertices (u_{i,m_i}, w_i), $(w_i, u_{i+1,1})$, $i = 1, 2, 3, \cdots, t-1$ by an edge to construct the join sum of graphs $G = \langle K_{m_1,n_1}; K_{m_2,n_2}; \cdots ; K_{m_t,n_t}\rangle$. In graph G, $|V(G)| = (m_1+n_1)+(m_2+n_2)+\cdots+(m_t+n_t)+t-1$ and $|E(G)| = (m_1n_1+m_2n_2+\cdots+m_tn_t)+2(t-1)$. We define a labeling $f : V(G) \to \{0, 1, \cdots, 2[(m_1n_1+\cdots+m_tn_t)+2(t-1)]-1\}$ as follows:

$$\begin{aligned} f(u_{i,j}) &= 2[n_{i-1}m_{i-1}+n_{i-2}m_{i-2}+\cdots+n_1m_1]-2[n_i+n_{i-1}+\cdots+n_1] \\ &\quad +2n_ij+2(i-1),\ 1 \le i \le t \text{ and } 1 \le j \le m_i, \\ f(v_{i,j}) &= 2[n_{i-1}+n_{i-2}+\cdots+n_1]+2j+2i-3,\ 1 \le i \le t \text{ and } 1 \le j \le n_i, \\ f(w_i) &= 2[n_i+n_{i-1}+\cdots+n_1]+2(i-1)+1,\ 1 \le i \le t-1. \end{aligned}$$

The induced edge labels are

$$\begin{aligned} f^*(u_{i,j}v_{i,k}) &= 2[n_{i-1}m_{i-1}+n_{i-2}m_{i-2}+\cdots+n_1m_1]+2n_i(j-1) \\ &\quad +2k+4i-5, 1 \le i \le t,\ 1 \le k \le n_i \text{ and } 1 \le j \le m_i, \\ f^*(u_{i,m_i}w_i) &= 2[n_im_i+n_{i-1}m_{i-1}+\cdots+n_1m_1]+4(i-1)+1,\ 1 \le i \le t, \\ f^*(w_iu_{i+1,1}) &= 2[n_im_i+n_{i-1}m_{i-1}+\cdots+n_1m_1]-2n_{i+1}+2n_i+4i-1, \\ &\quad 1 \le i \le t-1. \end{aligned}$$

In view of the above defined labeling pattern, the join sum of the complete bipartite graphs $\langle K_{m_1,n_1}; K_{m_2,n_2}; \cdots ; K_{m_t,n_t}\rangle$ is an odd harmonious graph. ■

An odd harmonious labeling of $\langle K_{2,3}; K_{2,2}; K_{3,5}\rangle$ is shown in Figure 4.

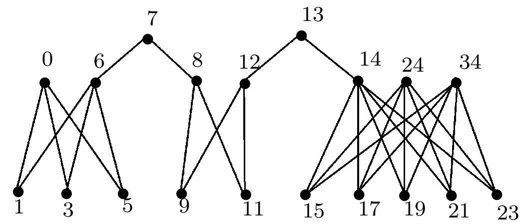

Figure 4: *An odd harmonious labeling of* $\langle K_{2,3}; K_{2,2}; K_{3,5}\rangle$

Theorem 2.5. *The graph* $K^t_{m,n}$, $t > 1$ *is odd harmonious.*

Proof. Let the graph G be $K^t_{m,n}$. Let $v^i_1, v^i_2, \cdots , v^i_m$ and $u^i_1, u^i_2, \cdots , u^i_n$, where $1 \leq i \leq t$, be the vertices of the i^{th} copy of G. Let v be the central vertex of G. In graph G, $|V(G)| = t(m + n - 1)$ and $|E(G)| = mnt$. We define a labeling $f : V(G) \to \{0, 1, 2, \cdots , 2mnt - 1\}$ as follows:

$$\begin{aligned} f(v) &= 0, \\ f(v^j_i) &= 2mn(t-j) + 2n(i-1),\ 2 \leq i \leq m \text{ and } 1 \leq j \leq t, \\ f(u^j_i) &= 2n(j-1) + 2i - 1,\ 1 \leq i \leq n \text{ and } 1 \leq j \leq t. \end{aligned}$$

The induced edge labels are

$$\begin{aligned} f^*(vu^j_i) &= 2n(j-1) + 2i - 1,\ 1 \leq i \leq n \text{ and } 1 \leq j \leq t, \\ f^*(v^j_i u^j_s) &= 2mn(t-j) + 2n(i-1) + 2n(j-1) + 2s - 1, \\ &\quad 2 \leq i \leq m, 1 \leq s \leq n \text{ and } 1 \leq j, k \leq t. \end{aligned}$$

In view of the above defined labeling pattern, $K^t_{m,n}$, $t > 1$ is an odd harmonious graph. ■

An odd harmonious labeling of $K^3_{3,4}$ is shown in Figure 5.

Theorem 2.6. *The graph* $K^t_{(m_1,n_1),(m_2,n_2),\cdots,(m_t,n_t)}$, $t > 1$ *is odd harmonious.*

Proof. Let G be a graph $K^t_{(m_1,n_1),(m_2,n_2),\cdots,(m_t,n_t)}$. Let $v^i_1, v^i_2, \cdots , v_{m^i_i}$ and $u^i_1, u^i_2, \cdots , u_{n^i_i}$ where $1 \leq i \leq t$ be the vertices of the i^{th} copy of G. Identify the first vertex v^i_1, $1 \leq i \leq t$ of each copy and consider that as a central vertex v.

In graph G, $|V(G)| = (m_1+n_1)+(m_2+n_2-1)+(m_3+n_3-1)+\cdots+(m_t+n_t-1)$ and $|E(G)| = m_1n_1 + m_2n_2 + \cdots + m_tn_t$. We define a labeling $f : V(G) \to$

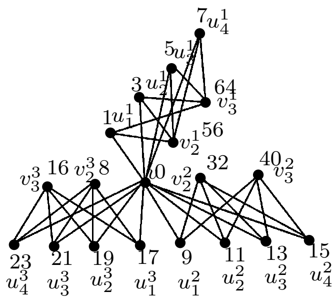

Figure 5: *An odd harmonious labeling of* $K^3_{3,4}$.

$\{0, 1, 2, \cdots, 2(m_1n_1 + m_2n_2 + \cdots + m_tn_t) - 1\}$ as follows:

$$\begin{aligned}
f(v) &= 0,\\
f(v_i^t) &= 2n_t(i-1),\ 1 \le i \le m_t,\\
f(u_i^j) &= 2i - 1 + 2[n_1 + n_2 + \cdots + n_{j-1}],\ 1 \le i \le n_j \text{ and } 1 \le j \le t,\\
f(v_i^j) &= f(v_{m_{j+1}}^{j+1}) + 2(n_j + n_{j+1}) + 2n_j(i-2),\ \ 1 \le i \le m_j,\\
&\quad j = t-1, t-2, \cdots, 1.
\end{aligned}$$

The induced edge labels are

$$\begin{aligned}
f^*(vu_i^j) &= 2i - 1 + 2(n_1 + n_2 + \cdots + n_{j-1}),\ 1 \le i \le n_j,\ 1 \le j \le t,\\
f^*(v_i^j u_s^k) &= f(v_{m_{j+1}}^{j+1}) + 2(n_j + n_{j+1}) + 2n_j(i-2) + 2s - 1 + 2(n_1 + n_2 + \cdots\\
&\quad + n_{k-1}), 2 \le i \le m_j,\ 1 \le j, k \le t-1 \text{ and } 1 \le s \le n_k,\\
f^*(v_i^t u_s^t) &= 2n_t(i-1) + 2s - 1 + 2(n_1 + n_2 + \cdots + n_{t-1}),\\
&\quad 2 \le i \le m_t \text{ and } 1 \le s \le n_t.
\end{aligned}$$

In view of the above defined labeling pattern, $K^t_{(m_1,n_1),(m_2,n_2),\cdots,(m_t,n_t)}$ is an odd harmonious graph. ■

An odd harmonious labeling of $K^3_{(2,2),(2,3),(3,4)}$ is shown in Figure 6.

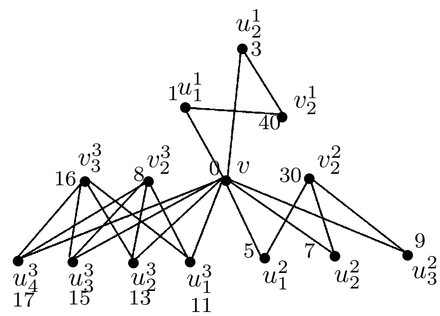

Figure 6: *An odd harmonious labeling of* $K^3_{(2,2),(2,3),(3,4)}$

Theorem 2.7. *The circle formation of r copies of $K_{m,n}$ when $r \equiv 0 (mod\ 4)$ is odd harmonious.*

Proof. Let G be a graph of circle formation of r copies of $K_{m,n}$. Let $u_{i,1}, u_{i,2}, \cdots, u_{i,m}$ and $v_{i,1}, v_{i,2}, \cdots, v_{i,n}$ be the vertices of the i^{th} copy of $K_{m,n}$, where $1 \leq i \leq r$.

In graph G, $|V(G)| = r(m+n)$ and $|E(G)| = r(mn+1)$. We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2r(mn+1)-1\}$ as follows:

$$\begin{aligned}
f(u_{i,j}) &= (i-1)(nm+1) + 2nj - 2n,\ i = 1, 3, 5, \cdots, \frac{r}{2} - 1 \text{ and } 1 \leq j \leq m,\\
f(u_{i,j}) &= (i-1)nm + 2nj - 2n + i + 1,\ i = \frac{r}{2} + 1, \frac{r}{2} + 3, \cdots, (r-1),\\
&\quad 1 \leq j \leq m,\\
f(u_{i,j}) &= (i-2)nm + 2nj + i - 1,\ i = 2, 4, 6, \cdots, r \text{ and } 1 \leq j \leq m,\\
f(v_{i,j}) &= (i-1)nm + 2j + i - 2,\ i = 1, 3, 5, \cdots, r-1 \text{ and } 1 \leq j \leq n,\\
f(v_{i,j}) &= inm - 2n + 2j + i - 2,\ i = 2, 4, 6, \cdots, \frac{r}{2} \text{ and } 1 \leq j \leq n,\\
f(v_{i,j}) &= inm - 2n + 2j + i,\ i = \frac{r}{2} + 2, \frac{r}{2} + 4, \cdots, r \text{ and } 1 \leq j \leq n.
\end{aligned}$$

The induced edge labels are

$$\begin{aligned}
f^*(u_{i,j}v_{i,k}) &= 2(i-1)nm + 2i - 3 + 2nj - 2n + 2k, i = 1, 2, 3, \cdots, \frac{r}{2},\\
f^*(u_{i,j}v_{i,k}) &= 2(i-1)nm + 2i - 1 + 2nj - 2n + 2k,\\
&\quad i = \frac{r}{2} + 1, \frac{r}{2} + 2, \frac{r}{2} + 3, \cdots, r,
\end{aligned}$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$.

In view of the above defined labeling pattern, the circle formation of r copies of $K_{m,n}$ is an odd harmonious graph. ∎

An odd harmonious labeling of circle formation of 4 copies of $K_{2,3}$ is shown in Figure 7.

Theorem 2.8. *An open star of complete bipartite graph $S(t.K_{m,n})$, $t > 1$ is odd harmonious.*

Proof. Let $G = S(t.K_{m,n})$ be a graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex of $K_{1,t}$ by the graph $K_{m,n}$. Let u_0 be the apex vertex of $K_{1,t}$. That is u_0 is the central vertex of the graph G. Let $v_1^i, v_2^i, \cdots, v_m^i$, $u_1^i, u_2^i, \cdots, u_n^i$ where $1 \leq i \leq t$ be the vertices of the i^{th} copy of $S(t.K_{m,n})$. We join the central vertex u_0 with the vertices v_1^i, where $1 \leq i \leq t$. In $G = S(t.K_{m,n})$, $|V(G)| = t(m+n+1)$ and $|E(G)| = t(mn+1)$. We define a labeling

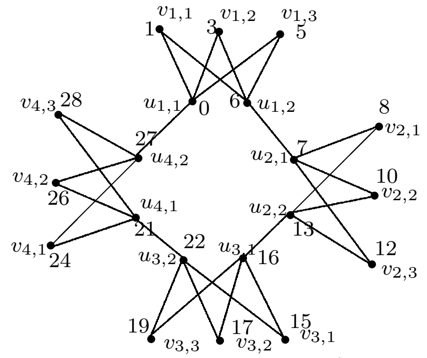

Figure 7: *An odd harmonious labeling circle formation of 4 copies of* $K_{2,3}$

$f : V(G) \to \{0, 1, 2, \cdots, 2t(mn+1)-1\}$ as follows:

$$
\begin{aligned}
f(u_0) &= 0,\\
f(v_k^i) &= 2nt(k-1) + 2i - 1,\ 1 \le i \le t \text{ and } 1 \le k \le m,\\
f(u_k^i) &= 2k + 2(n+1)(t-i),\ 1 \le i \le t \text{ and } 1 \le k \le n.
\end{aligned}
$$

The induced edge labels are

$$
\begin{aligned}
f^*(u_0 v_1^i) &= 2i - 1,\ 1 \le i \le t,\\
f^*(v_k^i u_z^i) &= 2ntk - 2ni + 2z + 2t - 1,\ 1 \le i \le t,\ 1 \le z \le n \text{ and } 1 \le k \le m.
\end{aligned}
$$

In view of the above defined labeling pattern $S(t.K_{m,n})$ is an odd harmonious graph. ■

An odd harmonious labeling of $S(6.K_{2,3})$ is shown in Figure 8.

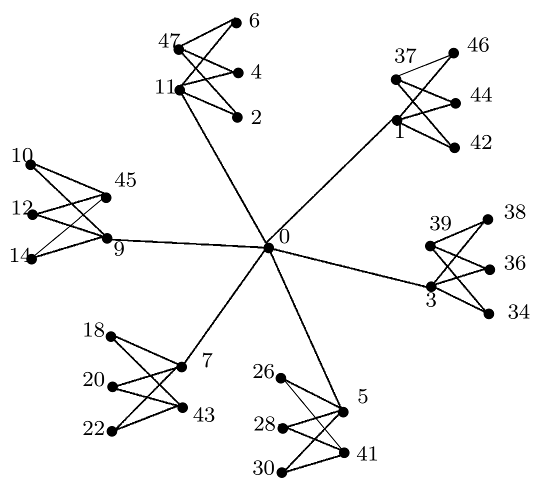

Figure 8: *An odd harmonious labeling of* $S(6.K_{2,3})$.

Theorem 2.9. *An one point union of path of graph $P_n^t(t.n.K_{p,q})$ is odd harmonious if t is odd.*

Proof. Let $G = P_n^t(t.n.K_{p,q})$. Let $u_{l,k,i}$ $(1 \le l \le t,\ 1 \le k \le n$ and $1 \le i \le p)$ and $v_{l,k,j}$ $(1 \le l \le t, 1 \le k \le n$ and $1 \le j \le q)$ be the vertices of k^{th} copy of path union of n copies of $K_{m,n}$ lies in the l^{th} branch of the graph G, $l = 1, 2, \cdots, t$. We join the vertices of $u_{l,1,1}$ with u_0 by an edge, $l = 1, 2, \cdots, t$. Also we join the vertices $v_{l,k,m}$ to $u_{l,k+1,1}$ for $k = 1, 2, \cdots, n-1$ and $l = 1, 2, \cdots, t$ by an edge. In $G = P_n^t(t.n.K_{p,q})$, $|V(G)| = tn(p+q)+1$ and $|E(G)| = tn(pq+1)$.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2tn(pq+1)-1\}$ as follows:

$$\begin{aligned}
f(u_0) &= 0,\\
f(u_{l,1,1}) &= 2l-1,\ 1 \le l \le t,\\
f(v_{l,1,1}) &= 2+4(t-l),\ 1 \le l \le t,\\
f(u_{l,k,i}) &= 2l-1+2tq(i-1)+tp(p+q-1)(k-1),\ 1 \le l \le t,\\
&\quad 1 \le k \le n, 1 \le i \le p,\\
f(v_{l,k,j}) &= 2+4(t-l)+2t(j-1)+2tq(k-1),\ 1 \le l \le t,\\
&\quad 1 \le k \le n \text{ and } 1 \le j \le q.
\end{aligned}$$

The induced edge labels are

$$\begin{aligned}
f^*(u_0u_{l,1,1}) &= 2l-1,\ 1 \le l \le t,\\
f^*(u_{l,k,i}v_{l,k,j}) &= 2l-1+2tq(i-1)+tp(p+q-1)(k-1)+2\\
&\quad +4(t-l)+2t(j-1)+2tq(k-1),\ 1 \le l \le t,\ 1 \le k \le n,\\
&\quad 1 \le i \le p \text{ and } 1 \le j \le q,\\
f^*(v_{l,k,q}u_{l,k+1,1}) &= 4(t-l)+2t(q-1)+ktp(p+q-1+2tq(k-1)\\
&\quad +2l+1,\quad 1 \le l \le t \text{ and } 1 \le k \le n-1.
\end{aligned}$$

In view of the above defined labeling pattern, $P_n^t(t.n.K_{p,q})$ is an odd harmonious graph. ■

An odd harmonious labeling of $P_2^3(3.2.K_{2,3})$ is shown in Figure 9.

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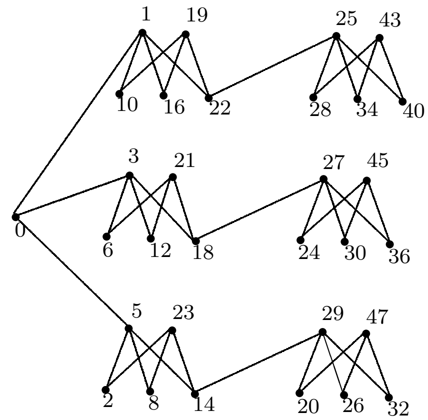

Figure 9: *An odd harmonious labeling of* $P_2^3(3.2.K_{2,3})$

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