Southeast Asian Bulletin of Mathematics (2022) 46: 35–44

Southeast Asian
Bulletin of
Mathematics

Variational Iteration Method (VIM) and Parameter Perturbation Method (PPM) for the Solution of Nonlinear Differential Equation of Beam Elastic Deformation

Alireza Roshan
Department of Civil, Constructional and Environmental Engineeering, Sapienza University of Rome, Italy
Email: roshan.1903579@studenti.uniroma1.it

Masoumeh Ahmadi
Department of Mathematics, University of Birjand, Birjand, Iran
Email: mas.ahmadii@yahoo.com

Davood Domiri Ganji
Department of Mechanical Engineering, Babol University of Technology, Babol, Iran
Email: mirgang@nit.ac.ir

Received 14 October 2019
Accepted 22 January 2020

Communicated by Yongwimon Lenbury

AMS Mathematics Subject Classification(2020): 20E10, 20E34, 20E36

Abstract. In this paper, we apply two approximate analytical methods called parameter perturbation method (PPM) and variational iteration method (VIM) to solve the equation of beam deformation with two fixed end and under uniform distributed load. The presented results in this paper reveal that these two methods are very effective and can be easily extended to other nonlinear systems. Hence, they can be found widely applicable in engineering and other sciences.

Keywords: Beam deformation; Approximate analytical methods; Perturbation method (PPM); Variational iteration method (VIM).

1. Introduction

Nonlinear systems have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion and deformation. The study of nonlinear systems is of interest to many researchers and various methods of solution have been proposed [3, 4, 31, 11, 21, 17, 44, 38, 47]. Surveys of the literature with numerous references, and useful bibliographies, have been given by Nayfeh [37], Mickens [36], Jordan and Smith [34] and more recently by He [10]–[22]. The solving of governing equations due to limitation of existing exact solutions have been one of the most time-consuming and difficult affairs among researchers of nonlinear problems. With the rapid development of nonlinear science, there appears an ever-increasing interest of scientists in the analytical asymptotic techniques for nonlinear problems and several analytical approximate methods have been developed to solve linear and nonlinear ordinary and partial differential equations. The perturbation method (PM) was first proposed in the early 19th century by Poisson to solve nonlinear differential equations was performed later on that century. However, the most significant efforts were focused on celestial mechanics, fluid mechanics, and aerodynamics [5, 32, 33, 36]. However, the classical perturbation methods have many shortcomings, and they are not valid for strongly nonlinear equations. To overcome the shortcomings, many reaserchers introduced new methods [1, 22, 30, 27, 35, 41, 46]. In what follows we briefly review some important of these methods.

In 1992 Liao introduced homotopy perturbation method (HPM) [35], that unlike the classical perturbation methods, does not require a small perturbation parameter in the equation. See [20]–[15] for more details. Moreover, he has proposed some applicable methods for solving nonlinear equations. For example, energy balance method (EBM) is extended for high-order nonlinear oscillators [2, 23, 48], variational approach method (VAM) [6, 25] has been considered to solve the nonlinear systems. Also, two other He's methods known variational iteration method (VIM) and parameter perturbation method (PPM) are considered to be two of powerful methods for handling nonlinear behaviors and can converge to an accurate solution for smooth nonlinear systems. In this paper, we consider variational iteration method (VIM) and parameter perturbation method (PPM) to solve nonlinear differential equation of beam elastic deformation. The variational iteration method was first proposed in 1998 in [12, 26] and was applied to study linear and nonlinear partial differential equations. See [10]–[13] and also [26, 29, 30, 27, 28, 43] for more information and applications of VIM. Also, parameter perturbation method is one of the well-known methods for solving nonlinear vibration equations that was proposed in 1999 [24]. This method is a kind of powerful tool for treating weakly nonlinear problems. See [7, 16, 39, 40, 41, 42, 45, 46] for the main applications and extensions of PPM.

One of the responsibilities of the structural design engineer is to devise arrangements and proportions of members that can withstand, economically and efficiently, the conditions anticipated during the lifetime of a structure. A central aspect of this function is the calculation of the beam deformation, which has very wide applications in structural engineering.

The main objective of this paper is to approximately solve nonlinear differential equation of beam elastic deformation with two fixed end and under uniform distributed load (Fig. 1), by applying the variational iteration method (VIM) and parameter perturbation method (PPM) and to compare the approximate results with formula in mechanics of materials for beams with two fixed end and under uniform distributed load. The results presented in this paper reveal that the methods are very effective for solution of nonlinear differential equations of beam elastic deformation and can be easily extended to other nonlinear systems and can therefore be found widely applicable in engineering and other sciences.

The equation of beam elastic deformation with two fixed end and under uniform distributed load is in the following form:

$$\left(\frac{d^2}{dx^2}y(x)\right) - \left(\frac{M(x)}{EI}\right) * \left(1 + \left(\frac{d}{dx}y(x)\right)^2\right)^{1.5} = 0 \tag{1}$$

where in Eq. (1):

$$M(x) = \frac{W}{12} * (6Lx - L^2 - 6x^2). \tag{2}$$

In this equation, M is bending moment, E is the elastic modulus and I is the second moment of area. It must be calculated with respect to axis perpendicular to the applied load. With the boundary conditions:

$$y(0) = y(L) = 0, \quad y'(0) = y'(L) = 0 \tag{3}$$

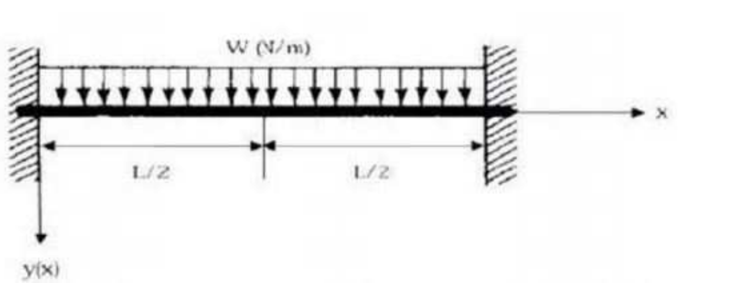

Figure 1: Beam with two fixed end and under uniform distributed load

2. Computational Method

2.1. The Basic Idea of Variational Method (VIM)

Consider governing equation of beam elastic deformation with two fixed end and under uniform distributed load in the following form:

$$\left(\frac{d^2}{dx^2}y(x)\right) - \left(\frac{M(x)}{EI}\right) * \left(1 + \left(\frac{d}{dx}y(x)\right)^2\right)^{1.5} = 0. \tag{4}$$

To clarify the basic ideas of He's VIM, we consider the following differential equation:

$$Lu + Nu = g(x) \tag{5}$$

where L is a linear operator, N a nonlinear operator and $g(t)$ an inhomogeneous term. According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(\tau) + Nu_n(\tau) - g(\tau)\}d\tau \tag{6}$$

where λ is a general Lagrange multiplier that can be identified optimally via the variational theorem.

$$\lambda(t) = x - \tau \tag{7}$$

2.2. The Basic Idea of Parameter Perturbation Method

Parameter perturbation method (PPM) is perturbation technique where the coefficients in an equation are also expressed in power of artificial parameter which can be used to nonlinear systems.

Comparisons between the parameter perturbation method and the probabilistic approach from mathematical proofs and numerical simulations were performed. The numerical results are in agreement with the mathematical proofs. The response range given by the parameter perturbation method encloses that obtained by the probabilistic approach. The results also show good robustness of the proposed method.

This method was first proposed in 1999. According to an expanding parameter is introduced by a linear transformations:

$$u(t) = \varepsilon v(t) + b \tag{8}$$

where ε is the perturbation parameter, by substituting Eq. (1) into an original equation in order to have no secular term in the equation we can obtain the unknown constant parameter b. Then, the solution is expanded in the form:

$$v = \sum_{i=0}^{n} \varepsilon^i v_i = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots, \tag{9}$$

here ε is an artificial book keeping parameter. Unlike traditional methods, we keep $v_0(0) = v(0)$ and $\sum_i = v_i(0) = 0$.

All analytical expressions gained by PPM are in very good agreement with numerical results and can be used in many calculations related to industries.

2.3. The Application of Variational Iteration Method (VIM)

In this section, variational iteration method is developed for solving beam deformation equation. Consider beam deformation equation Eq. (1). To solve Eq. (1) via VIM at first consider the beam equation such as Eq. (10):

$$\left(\frac{d^2}{dx^2}y(x)\right)-\left(\frac{M(x)}{EI}\right)*\left(1+\frac{3}{2}\left(\frac{d}{dx}y(x)\right)^2\right)=0 \tag{10}$$

and the variational iteration formula is obtained in the form:

$$y_{n+1}(x)=y_{n(x)}+\left\{\int_0^x(x-\tau)\left(\frac{d^2}{d\tau^2}y_n(\tau)-\left(1+\frac{3}{2}\left[\frac{d}{d\tau}y_n(\tau)\right]^2\right)\right)*M(x)\right\}d\tau \tag{11}$$

After solving Eq. (11), $y_1(x)$ is obtained

$y_1(x)=\frac{5*10^{-12}(2*10^{11}*AEI-8.33*10^9*Wx^4+1.66*10^{10}*Wx^3L-8.33*10^9*x^2WL^2)}{EI}$.

Now substituting ($EI=1000$, $W=100$, $L=1$ and $A=0$), we have

$$y_1(x)=-0.004166x^4+0.00833x^3-0.004166x^2$$

The curve of beam elastic deformation solving by VIM is shown in Fig. 2.

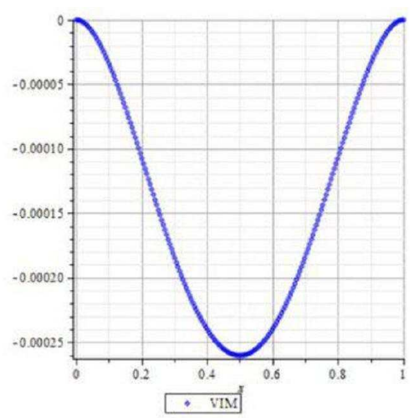

Figure 2: Beam elastic deformation for ($EI=1000$, $W=100$, $L=1$ and $A=0$)

2.4. The Application of Parameter Perturbation Method (PPM)

To solve Eq. eqrefeq1 by means of parameter method (PPM), we consider the following process. First we change Eq. (1) to the following form:

$$\left(\frac{d^2}{dx^2}y(x)\right)-\left(\frac{M(x)}{EI}\right)*\left(1+\frac{3}{2}\left(\frac{d}{dx}y(x)\right)^2\right)=0 \tag{12}$$

substituting Eq. (8) in to Eq. (12) and rearranging the resultant equation based on powers of ε-terms, we get

$$\varepsilon_0 = \left(\frac{d^2}{dx^2} v_0(x) + \frac{\frac{-0.5*WLx}{EI} + \frac{0.0833*WL^2}{EI} + \frac{0.5*Wx^2}{EI}}{\varepsilon} \right) \tag{13}$$

$v_0(x)$ might be written as follows by solving the Eq. (13)

$$v_0(x) = -\frac{1}{10^{11}} * \frac{W\Big(-\frac{25*10^{10}*Lx^3}{3} + \frac{8333333333*L^2x^2}{2} + \frac{125*10^8*x^4}{3}}{EI\varepsilon} + b. \tag{14}$$

Consider

$$y(x) = \varepsilon v(t) + b \tag{15}$$

$y_0(0)$ might be written as follows by solving Eqs. (14) and (15) and substituting ($W = 100$, $EI = 1000$, $L = 1$, $\varepsilon = 0$ and $b = 0$):

$$y_0(x) = 0.00833x^3 - 0.004166x^2 - 0.004166x^4. \tag{16}$$

The curve of beam elastic deformation solving by PPM is shown in Fig. 3:

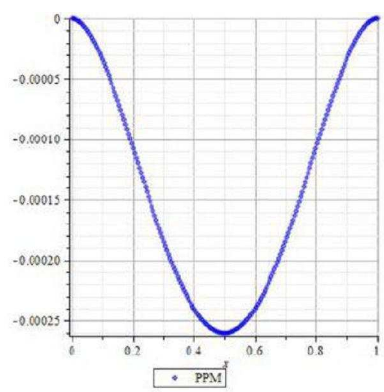

Figure 3: Beam elastic deformation for ($EI = 1000$, $W = 100$, $L = 1$ and $\varepsilon = 0$)

3. Results

In this section, we compare the results variational iteration method (VIM) and of parameter perturbation method (PPM) with formula in mechanics of materials for beam with two fixed end and under uniform distributed load. In mechanics of materials for beams with two fixed end and under uniform distributed load the deformation is computed by following formula

$$y(x) = \frac{1}{24} \frac{Wx^2(L-x)^2}{EI}. \tag{17}$$

The approximate analytical results are in good agreement with the results obtained by the formula in mechanics of materials for beams with two fixed end and under uniform distributed load. The results of comparison between variational iteration method (VIM) with formula in mechanics of materials for beams with two fixed end and under uniform distributed load are given in Table 1.

Table 1: Comparison variational iteration method (VIM) with formulain mechanics of materials for beams with two fixedend and under uniform distributed load for ($EI = 1000$, $W = 100$, $L = 1$)

X (displacement from left support)	variational iteration method (VIM)	formula in mechanics of materials
0.10	0.0000337	0.000034
0.20	0.0001066	0.000107
0.30	0.0001837	0.000184
0.40	0.0002399	0.000240
0.50	0.0002604	0.000260

The results of comparison between parameter perturbation method (PPM) with formula in mechanics of materials for beams with two fixed end and under uniform distributed load are given in Table 2.

Table 2: Comparison parameter perturbation method (PPM) with formulain mechanics of materials for beams with two fixedend and under uniform distributed load for ($EI = 1000$, $W = 100$, $L = 1$)

X (displacement from left support)	parameter perturbation method (PPM)	formula in mechanics of materials
0.10	0.0000337	0.000034
0.20	0.0001066	0.000107
0.30	0.0001837	0.000184
0.40	0.0002400	0.000240
0.50	0.0002604	0.000260

Comparison between different results with curve is shown in Fig. 4.

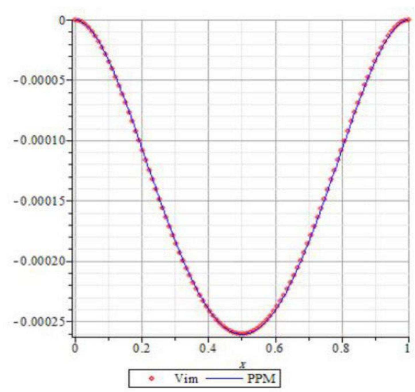

Figure 4: comparison between different results

4. Conclusion

In this paper, the variational iteration method (VIM) and parameter perturbation method (PPM) have been successfully applied to the nonlinear differential equation of beam deformation with two fixed end and under uniform distributed load. These methods enable to convert a difficult problem into a simple problem which can easily be solved. Comparisons of the results obtained here provide more realistic solutions, reinforcing the conclusions pointed out by many researchers about the efficiency of these two methods. Therefore, the homotopy perturbation method and perturbation method are powerful mathematical tools that can be widely applied to structural engineering such as beam problems.

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