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$W\hat{g}$-Closed Sets in Ideal Topological Spaces

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Abstract. Characterizations and properties of $\mathcal{I}_{w\hat{g}}$-closed sets and $\mathcal{I}_{w\hat{g}}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{w\hat{g}}$-open sets. Also, it is established that an $\mathcal{I}_{w\hat{g}}$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

Keywords: $W\hat{g}$-closed set; $\mathcal{I}_{w\hat{g}}$-closed set and $\mathcal{I}$-compact space.

1. Introduction and Preliminaries

Throughout this paper, by a space X, we always mean a topological space (X, τ)

with no separation properties assumed. Let H be a subset of X. We denote the interior, the closure and the complement of a set H by $int(H)$, $cl(H)$ and $X\backslash H$ or H^c, respectively.

An ideal $\mathcal{I}$ on a space X is a non-empty collection of subsets of X which satisfies (i) $P \in \mathcal{I}$ and $Q \subseteq P \Rightarrow Q \in \mathcal{I}$ and (ii) $P \in \mathcal{I}$ and $Q \in \mathcal{I} \Rightarrow P \cup Q \in \mathcal{I}$. Given a space X with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^\star : \wp(X) \to \wp(X)$, called a local function [15] of H with respect to τ and $\mathcal{I}$ is defined as follows: for $H \subseteq X$, $H^\star(\mathcal{I}, \tau) = \{x \in X \mid U \cap H \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts about the local functions without mentioning it explicitly (see [14, Theorem 2.3]). A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(\mathcal{I}, \tau)$, called the $\star$-topology, finer than τ is defined by $cl^\star(H) = H \cup H^\star(\mathcal{I}, \tau)$ (see [25]). When there is no chance for confusion, we will simply write $H^\star$ for $H^\star(\mathcal{I}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ). A subset H of an ideal space $(X, \tau, \mathcal{I})$ is called $\star$-closed [14] (resp. $\star$-dense in itself [12]) if $H^\star \subseteq H$ (resp. $H \subseteq H^\star$). A subset H of an ideal space $(X, \tau, \mathcal{I})$ is called $\mathcal{I}_g$-closed if $H^\star \subseteq U$ whenever $H \subseteq U$ and U is open (see [6]).

$Int^\star(H)$ will denote the interior of H in $(X, \tau^\star)$. A subset H of a space (X, τ) is called an α-open [21] (resp. semi-open [17], preopen [18]) set if $H \subseteq int(cl(int(H)))$ (resp. $H \subseteq cl(int(H))$, $H \subseteq int(cl(H))$). The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The closure of H in (X, τ^α) is denoted by $cl_\alpha(H)$. A subset H of a space (X, τ) is said to be g-closed if $cl(H) \subseteq U$ whenever $H \subseteq U$ and U is open (see [16]). The family of all semi-open sets of X is denoted by $SO(X)$.

An ideal $\mathcal{I}$ is said to be codense [7] or τ-boundary [20] if $\tau \cap \mathcal{I} = \{\emptyset\}$. $\mathcal{I}$ is said to be completely codense [7] if $PO(X) \cap \mathcal{I} = \{\emptyset\}$, where $PO(X)$ is the family of all preopen sets in (X, τ). Every completely codense ideal is codense but not converse in [7]. The following Lemmas will be useful in the sequel. In study of the ideal topological space, the concepts of weakly e-I-open sets and r-I-open functions introduced in [3] are useful.

Lemma 1.1. [23, Theorem 5] *Let $(X, \tau, \mathcal{I})$ be an ideal space and $H \subseteq X$. If $H \subseteq H^\star$, then $H^\star = cl(H^\star) = cl(H) = cl^\star(H)$.*

Lemma 1.2. [23, Theorem 3] *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is codense if and only if $G \subseteq G^\star$ for every semi-open set G in X.*

Lemma 1.3. [23, Theorem 6] *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^\star \subseteq \tau^\alpha$.*

Recall that $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{I}}$ ideal space if every $\mathcal{I}_g$-closed set is $\star$-closed.

Definition 1.4. [8] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset G of*

$(X, \tau, \mathcal{I})$ is said to be weakly $\mathcal{I}_{rg}$-closed set if $(Int(G))^\star \subset H$ whenever $G \subset H$ and H is a regular open set in X.

Definition 1.5. [9] *A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is called*

(a) *$\mathcal{I}_g^\star$-closed in $(X, \tau, \mathcal{I})$ if $cl(S) \subset N$ whenever $S \subset N$ and N is $\star$-open in $(X, \tau, \mathcal{I})$.*

(b) *$\mathcal{I}_g^\star$-open in $(X, \tau, \mathcal{I})$ if $X \backslash S$ is $\mathcal{I}_g^\star$-closed.*

Remark 1.6. [9] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following diagram holds for a subset S of X. None of the these implications is reversible:

$$\begin{array}{ccccc} \mathcal{I}_g^\star\text{-}open & \rightarrow & g\text{-}open & \rightarrow & \mathcal{I}_g\text{-}open \\ \uparrow & & & \nearrow & \\ open & \rightarrow & \star\text{-}open & & \end{array}$$

Lemma 1.7. [19, Corollary 2.2] *If $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{I}}$ ideal space and H is an $\mathcal{I}_g$-closed set, then H is a $\star$-closed set.*

Lemma 1.8. [6, Theorem 2.1] *Every g-closed set is $\mathcal{I}_g$-closed but not conversely.*

Lemma 1.9. [14] *Let $(X, \tau, \mathcal{I})$ be an ideal space and let M and N be two subsets on X. Then*

(a) $M \subseteq N \Rightarrow M^\star \subseteq N^\star$.

(b) $M^\star = cl(M^\star) \subseteq cl(M)$*($M^\star$ is a closed subset of $cl(M)$).*

(c) $(M^\star)^\star \subseteq M^\star$.

(d) $(M \cup N)^\star = M^\star \cup N^\star$.

(e) $M^\star - N^\star = (M - N)^\star - N^\star \subseteq (M - N)^\star$.

Now, we call $w \subseteq P$ a weak structure (briefly WS) on X if and only if $\emptyset \in w$. Clearly each generalized topology and each minimal structure is a WS (see [5]).

Each member of w is said to be w-open and the complement of a w-open set is called w-closed.

Let w be a weak structure on X and $H \subseteq X$. We define (as in the general case) $i_w(H)$ is the union of all w-open subsets contained in H and $c_w(H)$ is the intersection of all w-closed sets containing H (see [5]).

Let w be a WS on X and $H \subseteq X$. Then $H \in \sigma(w)$ [resp. $H \in \alpha(w)$, $H \in \pi(w)$] if $H \subseteq c_w(i_w(H))$ (resp. $H \subseteq i_w(c_w(i_w(H)))$, $H \subseteq i_w(c_w(H))$) (see [5]).

Let w be a WS on a space X. Then $H \subseteq X$ is called a $\hat{g}w$-closed set if $c_w(H) \subseteq U$ whenever $H \subseteq U \in SO(X)$. The complement of a $\hat{g}w$-closed set is called $\hat{g}w$-open (see [24]).

Remark 1.10. [24] For a weak structure w on a space, every w-closed set is $\hat{g}w$-closed but not conversely.

Remark 1.11. [2] If w is a WS on X, then $i_w(\emptyset) = \emptyset$ and $c_w(X) = X$.

Theorem 1.12. [5] *If w is a WS on X and $A, B \in w$ then*

(a) $i_w(A) \subseteq A \subseteq c_w(A)$,
(b) $A \subseteq B \Rightarrow i_w(A) \subseteq i_w(B)$ *and* $c_w(A) \subseteq c_w(B)$,
(c) $i_w(i_w(A)) = i_w(A)$ *and* $c_w(c_w(A)) = c_w(A)$,
(d) $i_w(X - A) = X - c_w(A)$ *and* $c_w(X - A) = X - i_w(A)$.

Theorem 1.13. [10] *The following properties hold for a WS w on X and $A, B \subset X$:*

(a) $i_w(A \cap B) \subset i_w(A) \cap i_w(B)$.
(b) $c_w(A) \cup c_w(B) \subset c_w(A \cup B)$.

Lemma 1.14. [2] *If w is a WS on X, then*

(a) $x \in i_w(A)$ *if and only if there is a w-open set $G \subseteq A$ such that $x \in G$*,
(b) $x \in c_w(A)$ *if and only if $G \cap A \neq \emptyset$ whenever $x \in G \in w$*,
(c) *If $A \in w$, then $A = i_w(A)$ and if A is w-closed then $A = c_w(A)$.*

2. Properties of $w\hat{g}$-Closed Sets

Definition 2.1. *Let w be a WS on a space X. Then $H \subseteq X$ is called a $w\hat{g}$-closed set if $cl(H) \subseteq U$ whenever $H \subseteq U \in \sigma(w)$.*

Example 2.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$, $w = \{\emptyset, \{a\}, \{b, c\}\}$. Then $\{c\}$ is $w\hat{g}$-closed.

Example 2.3. In Example 2.2, $\{a\}$ is not $w\hat{g}$-closed.

Definition 2.4. *A subset H of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_{w\hat{g}}$-closed if $H^\star \subseteq U$ whenever $H \subseteq U$ and $U \in \sigma(w)$.*

Example 2.5. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\mathcal{I} = \{\emptyset, \{c\}\}$ and $w = \{\emptyset, \{b, c\}\}$. Then $\{a\}$ is $\mathcal{I}_{w\hat{g}}$-closed.

Example 2.6. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$, $w = \{\emptyset, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then $\{a, b\}$ is not $\mathcal{I}_{w\hat{g}}$-closed.

Definition 2.7. *A subset H of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_{w\hat{g}}$-open if $X - H$ is $\mathcal{I}_{w\hat{g}}$-closed.*

Proposition 2.8. *If $H \in \tau$, then $H \in \sigma(w)$.*

Proof. Let H be any open in X. Since any topology is weak structure, $H \in w$. Then $i_w(H) = H$. Also $c_w(i_w(H)) = c_w(H)$ and $H \subseteq c_w(H) = c_w(i_w(H))$. Therefore $H \in \sigma(w)$. ∎

Example 2.9. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$ and $w = \{\emptyset, \{a\}, \{a, b\}\}$. Then $\sigma(w) = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$. It is clear that $\{a\} \in \sigma(w)$ but it is not an open set.

Theorem 2.10. *If $(X, \tau, \mathcal{I})$ is any ideal space, then every $\mathcal{I}_{w\hat{g}}$-closed set is $\mathcal{I}_g$-closed.*

Example 2.11. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$, $w = \{\emptyset, \{a\}, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then $\mathcal{I}_{w\hat{g}}$-closed sets are $\emptyset$, $X, \{b\}, \{a, b\}, \{b, c\}$ and $\mathcal{I}_g$-closed sets $\emptyset$, $X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}$. It is clear that $\{a\}$ is $\mathcal{I}_g$-closed set but it is not $\mathcal{I}_{w\hat{g}}$-closed.

Example 2.12. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$, $w = \{\emptyset, \{a\}, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $\{b\}$ is a closed set and $\{a\} \in \sigma(w)$. Also $\{b\} \cap (X \backslash \{a\}) = \{b\}$ where $\{b\}^c \notin \sigma(w)$.

Remark 2.13. From the above example, it is proved that $X - (A \cap B) \notin \sigma(w)$ if $A^c \in \tau$ and $B^c \in \sigma(w)$.

Definition 2.14. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to have the property B if $P^c \in \tau$ and $Q^c \in \sigma(w)$ then $X \backslash (P \cap Q) \in \sigma(w)$.*

The following theorem gives characterizations of $\mathcal{I}_{w\hat{g}}$-closed sets.

Theorem 2.15. *If $(X, \tau, \mathcal{I})$ is any ideal space with the property B and $H \subseteq X$, then the following are equivalent:*

(a) *H is $\mathcal{I}_{w\hat{g}}$-closed.*
(b) *$cl^\star(H) \subseteq U$ whenever $H \subseteq U$ and $U \in \sigma(w)$.*
(c) *If $F \subseteq cl^\star(H) - H$ and $F^c \in \sigma(w)$ then $F = \emptyset$.*
(d) *If $F \subseteq H^\star - H$ and $F^c \in \sigma(w)$ then $F = \emptyset$.*

Proof. (a)⇒(b). If H is $\mathcal{I}_{w\hat{g}}$-closed, then $H^\star \subseteq U$ whenever $H \subseteq U$ and $U \in \sigma(w)$ and so $cl^\star(H) = H \cup H^\star \subseteq U$ whenever $H \subseteq U$ and $U \in \sigma(w)$. This proves (b).

(b)⇒(c). Let $F^c \in \sigma(w)$ and $F \subseteq cl^\star(H) - H$. Then $F \subseteq cl^\star(H)$. Also $F \subseteq cl^\star(H) - H \subseteq X - H$ and hence $H \subseteq X - F$. By (b) $cl^\star(H) \subseteq X - F$ and so $F \subseteq X - cl^\star(H)$. Thus $F \subseteq cl^\star(H) \cap (X - cl^\star(H)) = \emptyset$.

(c)⇒(d). We know that $cl^\star(H) - H = (H \cup H^\star) - H = (H \cup H^\star) \cap H^c = (H \cap H^c) \cup (H^\star \cap H^c) = H^\star \cap H^c = H^\star - H$.

(d)⇒(a). Let $H \subseteq U$ where $U \in \sigma(w)$. Therefore $X - U \subseteq X - H$ and so $H^\star \cap (X - U) \subseteq H^\star \cap (X - H) = H^\star - H$. Therefore $H^\star \cap (X - U) \subseteq H^\star - H$. Since $H^\star$ is always closed set and by Definition 2.14 and assumption (d), $H^\star \cap (X - U) = \emptyset$ and hence $H^\star \subseteq U$. Therefore H is $\mathcal{I}_{w\hat{g}}$-closed. ■

Theorem 2.16. *Every $\star$-closed set is $\mathcal{I}_{w\hat{g}}$-closed.*

Proof. Let H be $\star$-closed. Then $H^\star \subseteq H$. Let $H \subseteq U$ where $U \in \sigma(w)$. Hence $H^\star \subseteq U$ whenever $H \subseteq U$ and $U \in \sigma(w)$. Therefore H is $\mathcal{I}_{w\hat{g}}$-closed. ■

Example 2.17. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$, $w = \{\emptyset, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then $\mathcal{I}_{w\hat{g}}$-closed sets are: $P(X)$ and $\star$-closed sets are $\emptyset$, $X, \{a\}, \{c\}, \{a, c\}, \{b, c\}$. It is clear that $\{b\}$ is $\mathcal{I}_{w\hat{g}}$-closed set but it is not $\star$-closed.

Theorem 2.18. *Let $(X, \tau, \mathcal{I})$ be an ideal space with the property B. For every $H \in \mathcal{I}$, H is $\mathcal{I}_{w\hat{g}}$-closed.*

Proof. Let $H \subseteq U$ where $U \in \sigma(w)$. Since $H^\star = \emptyset$ for every $H \in \mathcal{I}$, $cl^\star(H) = H \cup H^\star = H \subseteq U$. Therefore, by Theorem 2.15, H is $\mathcal{I}_{w\hat{g}}$-closed. ■

Theorem 2.19. *If $(X, \tau, \mathcal{I})$ is an ideal space, then $H^\star$ is always $\mathcal{I}_{w\hat{g}}$-closed for every subset H of X.*

Proof. Let $H^\star \subseteq U$ where $U \in \sigma(w)$. Since $(H^\star)^* \subseteq H^\star$ [14], $(H^\star)^* \subseteq U$ whenever $H^\star \subseteq U$ and $U \in \sigma(w)$. Hence $H^\star$ is $\mathcal{I}_{w\hat{g}}$-closed. ■

Theorem 2.20. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $H \in \sigma(w)$. Then every $\mathcal{I}_{w\hat{g}}$-closed set H is $\star$-closed.*

Proof. Since H is $\mathcal{I}_{w\hat{g}}$-closed, $H^\star \subseteq H$ whenever $H \subseteq H$ and $H \in \sigma(w)$. Hence H is $\star$-closed. ■

Corollary 2.21. *If $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{I}}$ ideal space and H is an $\mathcal{I}_{w\hat{g}}$-closed set, then H is $\star$-closed.*

Proof. Since H is $\mathcal{I}_{w\hat{g}}$-closed, by Theorem 2.10, H is $\mathcal{I}_g$-closed. Since $(X, \tau, \mathcal{I})$ is $T_{\mathcal{I}}$ ideal space, H is $\star$-closed. ■

Proposition 2.22. *Let w be a WS on a space X. If $H \in w$ then $H \in \sigma(w)$.*

Proof. Let $H \in w$. Then $i_w(H) = H$ and $c_w(i_w(H)) = c_w(H)$. Since $H \subseteq c_w(H) = c_w(i_w(H))$, $H \in \sigma(w)$. We have $w \subseteq \sigma(w)$. ■

Example 2.23. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$ and $w = \{\emptyset, \{a\}, \{a, b\}\}$. Then $\sigma(w) = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$. It is clear that $\{c\} \in \sigma(w)$ but it is not a w-open set.

Corollary 2.24. *If $(X, \tau, \mathcal{I})$ be an ideal space with the property B and H be an $\mathcal{I}_{w\hat{g}}$-closed set. Then the following statements are equivalent:*

(a) *H is $\star$-closed.*
(b) *$X - (cl^\star(H) - H) \in \sigma(w)$.*
(c) *$X - (H^\star - H) \in \sigma(w)$.*

Proof. (a)⇒(b). If H is $\star$-closed, then $H^\star \subseteq H$ and so $cl^\star(H) - H = (H \cup H^\star) - H = \emptyset$. $X - (cl^\star(H) - H) = X \in w$ since every $\star$-topology is weak structure. By Proposition 2.22, we get the result.

(b)⇒(c). Since $cl^\star(H) - H = H^\star - H$, $X - (H^\star - H) \in \sigma(w)$.

(c)⇒(a). If $X - (H^\star - H) \in \sigma(w)$, since H is $\mathcal{I}_{w\hat{g}}$-closed set, by Theorem 2.15, $H^\star - H = \emptyset$ and so H is $\star$-closed. ■

Theorem 2.25. *Let (X,τ,$\mathcal{I}$) is an ideal space. Then every $w\hat{g}$-closed set is an $\mathcal{I}_{w\hat{g}}$-closed set.*

Proof. Let H be a $w\hat{g}$-closed set. Then $cl(H) \subseteq U$ whenever $H \subseteq U \in \sigma(w)$. We have $cl^\star(H) \subseteq cl(H) \subseteq U$ whenever $H \subseteq U$ and $U \in \sigma(w)$. Hence H is $\mathcal{I}_{w\hat{g}}$-closed. ■

Example 2.26. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$, $w = \{\emptyset, \{a\}, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $\mathcal{I}_{w\hat{g}}$-closed sets are $\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$ and $w\hat{g}$-closed sets are $\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$. It is clear that $\{a\}$ is $\mathcal{I}_{w\hat{g}}$-closed set but it is not $w\hat{g}$-closed.

Theorem 2.27. *If $(X, \tau, \mathcal{I})$ is an ideal space with the property B and H is a $\star$-dense in itself, $\mathcal{I}_{w\hat{g}}$-closed subset of X, then H is $w\hat{g}$-closed.*

Proof. Suppose H is a $\star$-dense in itself, $\mathcal{I}_{w\hat{g}}$-closed subset of X. Let $H \subseteq U$ where $U \in \sigma(w)$. Then by Theorem 2.15 (b), $cl^\star(H) \subseteq U$ whenever $H \subseteq U$ and $U \in \sigma(w)$. Since H is $\star$-dense in itself, by Lemma 1.1, $cl(H) = cl^\star(H)$. Therefore $cl(H) \subseteq U$ whenever $H \subseteq U$ and $U \in \sigma(w)$. Hence H is $w\hat{g}$-closed. ■

Corollary 2.28. *If $(X, \tau, \mathcal{I})$ is any ideal space with the property B where $\mathcal{I} = \{\emptyset\}$, then H is $\mathcal{I}_{w\hat{g}}$-closed if and only if H is $w\hat{g}$-closed.*

Proof. From the fact that for $\mathcal{I} = \{\emptyset\}$, $H^\star = cl(H) \supseteq H$. Therefore H is $\star$-dense in itself. Since H is $\mathcal{I}_{w\hat{g}}$-closed, by Theorem 2.27, H is $w\hat{g}$-closed. Conversely, by Theorem 2.25, every $w\hat{g}$-closed set is $\mathcal{I}_{w\hat{g}}$-closed. ■

Corollary 2.29. *If $(X, \tau, \mathcal{I})$ is any ideal space with the property B where $\mathcal{I}$ is codense and H is a semi-open, $\mathcal{I}_{w\hat{g}}$-closed subset of X, then H is $w\hat{g}$-closed.*

Proof. By Lemma 1.2, H is $\star$-dense in itself. By Theorem 2.27, H is $w\hat{g}$-closed. ■

Example 2.30. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$, $w = \{\emptyset, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then g-closed sets are $\emptyset, X, \{b\}, \{a, b\}, \{b, c\}$ and $\mathcal{I}_{w\hat{g}}$-closed sets are $\emptyset, X, \{b\}, \{a, c\}, \{b, c\}$. It is clear that $\{a, b\}$ is g-closed set but it is not $\mathcal{I}_{w\hat{g}}$-closed.

Example 2.31. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$, $w = \{\emptyset, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then g-closed sets are $\emptyset, X, \{b\}, \{a, b\}, \{b, c\}$ and $\mathcal{I}_{w\hat{g}}$-closed sets are $\emptyset, X, \{b\}, \{a, c\}, \{b, c\}$. It is clear that $\{a, c\}$ is $\mathcal{I}_{w\hat{g}}$-closed set but it is not g-closed.

Remark 2.32. By Examples 2.30 and 2.31, g-closed sets and $\mathcal{I}_{w\hat{g}}$-closed sets are independent.

Proposition 2.33. *Every closed set is $w\hat{g}$-closed.*

Proof. Let H be a closed set such that $H \subseteq U \in \sigma(w)$. Then $cl(H) = H$ whenever $H \subseteq U \in \sigma(w)$. Thus H is $w\hat{g}$-closed ■

Example 2.34. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$ and $w = \{\emptyset, \{a\}, \{b, c\}\}$. Then $w\hat{g}$-closed sets are $\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$ and closed sets are $\emptyset, X, \{b\}, \{b, c\}$. It is clear that $\{c\}$ is $w\hat{g}$-closed set but it is not closed.

Proposition 2.35. *Every $w\hat{g}$-closed set is g-closed.*

Proof. Let $H \subseteq U \in \tau$. Then $H \subseteq U \in \sigma(w)$ by Proposition 2.8. Also $cl(H) \subseteq U$. Hence H is g-closed. ■

Example 2.36. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$ and $w = \{\emptyset, \{a\}, \{b, c\}\}$. Then $w\hat{g}$-closed sets are $\emptyset, X, \{a, b\}, \{a, c\}$ and g-closed sets are $\emptyset, X, \{a\}, \{b\}, \{a, b\}$, $\{a, c\}, \{b, c\}$. It is clear that $\{b\}$ is g-closed set but it is not $w\hat{g}$-closed.

Remark 2.37. We have the following implications for the subsets stated above.

$$\begin{tikzcd} \textit{closed} \arrow[r] \arrow[d] & w\hat{g}\textit{-closed} \arrow[r] \arrow[d] & g\textit{-closed} \arrow[d] \\ \star\textit{-closed} \arrow[r] & \mathcal{I}_{w\hat{g}}\textit{-closed} \arrow[r] & \mathcal{I}_{g}\textit{-closed} \end{tikzcd}$$

Theorem 2.38. *Let $(X, \tau, \mathcal{I})$ be an ideal space with the property B and $H \subseteq X$. Then H is $\mathcal{I}_{w\hat{g}}$-closed if and only if $H = F - N$ where F is $\star$-closed and $B \subseteq N$, where $B^c \in \sigma(w)$ and $B = \emptyset$.*

Proof. If H is $\mathcal{I}_{w\hat{g}}$-closed, then by Theorem 2.15 (d), $B \subseteq N = H^\star - H$ where $B^c \in \sigma(w)$ and $B = \emptyset$. If $F = cl^\star(H)$, then F is $\star$-closed such that $F - N = (H \cup H^\star) - (H^\star - H) = (H \cup H^\star) \cap (H^\star \cap H^c)^c = (H \cup H^\star) \cap ((H^\star)^c \cup H) = (H \cup H^\star) \cap (H \cup (H^\star)^c) = H \cup (H^\star \cap (H^\star)^c) = H$.

Conversely, suppose $H = F - N$ where F is $\star$-closed and $B \subseteq N$, where $B^c \in \sigma(w)$ and $B = \emptyset$. Let $U \in \sigma(w)$ such that $H \subseteq U$. Then $F - N \subseteq U \Rightarrow F \cap (X - U) \subseteq N$. Now $H \subseteq F$ and $F^\star \subseteq F then H^\star \subseteq F^\star$ and so $H^\star \cap (X - U) \subseteq F^\star \cap (X - U) \subseteq F \cap (X - U) \subseteq N$. By hypothesis, since $X - (H^\star \cap (X - U)) \in \sigma(w)$, $H^\star \cap (X - U) = \emptyset$ and so $H^\star \subseteq U$. Hence H is $\mathcal{I}_{w\hat{g}}$-closed. ■

Theorem 2.39. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $H \subseteq X$. If $M \subseteq N \subseteq M^\star$, then $M^\star = N^\star$ and N is $\star$-dense in itself.*

Proof. Since $M \subseteq N$, $M^\star \subseteq N^\star$ and since $N \subseteq M^\star$, $N^\star \subseteq (M^\star)^\star \subseteq M^\star$. Therefore $M^\star = N^\star$ and $N \subseteq M^\star \subseteq N^\star$. Hence proved. ■

Theorem 2.40. *Let $(X, \tau, \mathcal{I})$ be an ideal space with the property B. If M and N are subsets of X such that $M \subseteq N \subseteq cl^\star(M)$ and M is $\mathcal{I}_{w\hat{g}}$-closed, then N is $\mathcal{I}_{w\hat{g}}$-closed.*

Proof. Since M is $\mathcal{I}_{w\hat{g}}$-closed, by Theorem 2.15 (c), $F \subseteq cl^\star(M) - M$ where $F^c \in \sigma(w)$ and $F = \emptyset$. Since $cl^\star(N) - N \subseteq cl^\star(M) - M$ and so $G \subseteq cl^\star(N) - N$ where $G^c \in \sigma(w)$ and $G = \emptyset$. Hence N is $\mathcal{I}_{w\hat{g}}$-closed. ■

Corollary 2.41. *Let $(X, \tau, \mathcal{I})$ be an ideal space with the property B. If M and N are subsets of X such that $M \subseteq N \subseteq M^\star$ and M is $\mathcal{I}_{w\hat{g}}$-closed, then M and N are $w\hat{g}$-closed sets.*

Proof. Let M and N be subsets of X such that$M \subseteq N \subseteq M^\star \Rightarrow M \subseteq N \subseteq M^\star \subseteq cl^\star(M)$ and M is $\mathcal{I}_{w\hat{g}}$-closed. By the above Theorem, N is $\mathcal{I}_{w\hat{g}}$-closed. Since $M \subseteq N \subseteq M^\star$, $M^\star = N^\star$ and so M and N are $\star$-dense in itself. By Theorem 2.27, M and N are $w\hat{g}$-closed sets. ■

The following theorem gives a characterization of $\mathcal{I}_{w\hat{g}}$-open sets.

Theorem 2.42. *Let $(X, \tau, \mathcal{I})$ be an ideal space with the property B and $H \subseteq X$. Then H is $\mathcal{I}_{w\hat{g}}$-open if and only if $F \subseteq int^{\star}(H)$ whenever $F^c \in \sigma(w)$ and $F \subseteq H$.*

Proof. Suppose H is $\mathcal{I}_{w\hat{g}}$-open. If $F^c \in \sigma(w)$ and $F \subseteq H$, then $X - H \subseteq X - F$ and so $cl^{\star}(X-H) \subseteq X-F$ by Theorem 2.15(b). Therefore $F \subseteq X - cl^{\star}(X-H) = int^{\star}(H)$. Hence $F \subseteq int^{\star}(H)$.

Conversely, suppose the condition holds. Let $U \in \sigma(w)$ such that $X - H \subseteq U$. Then $X - U \subseteq H$ and so $X - U \subseteq int^{\star}(H)$. Therefore $cl^{\star}(X - H) \subseteq U$. By Theorem 2.15 (b), $X - H$ is $\mathcal{I}_{w\hat{g}}$-closed. Hence H is $\mathcal{I}_{w\hat{g}}$-open. ∎

Corollary 2.43. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $H \subseteq X$. If H is $\mathcal{I}_{w\hat{g}}$-open, then $F \subseteq int^{\star}(H)$ whenever $F^c \in \tau$ and $F \subseteq H$.*

The following theorem gives a property of $\mathcal{I}_{w\hat{g}}$-closed.

Theorem 2.44. *Let $(X, \tau, \mathcal{I})$ be an ideal space with the property B and $M \subseteq X$. If M is $\mathcal{I}_{w\hat{g}}$-open and $int^{\star}(M) \subseteq N \subseteq M$, then N is $\mathcal{I}_{w\hat{g}}$-open.*

Proof. Since M is $\mathcal{I}_{w\hat{g}}$-open, $X - M$ is $\mathcal{I}_{w\hat{g}}$-closed. By Theorem 2.15 (c), $F \subseteq cl^{\star}(X - M) - (X - M)$ where $F^c \in \sigma(w)$ and $F = \emptyset$. Since $int^{\star}(M) \subseteq int^{\star}(N)$ which implies that $cl^{\star}(X - N) \subseteq cl^{\star}(X - M)$ and so $cl^{\star}(X - N) - (X - N) \subseteq cl^{\star}(X - M) - (X - M)$. Hence M is $\mathcal{I}_{w\hat{g}}$-open. ∎

The following theorem gives a characterization of $\mathcal{I}_{w\hat{g}}$-closed sets in terms of $\mathcal{I}_{w\hat{g}}$-open sets.

Theorem 2.45. *Let $(X, \tau, \mathcal{I})$ be an ideal space with the property B and $H \subseteq X$. Then the following statements are equivalent:*

(a) *H is $\mathcal{I}_{w\hat{g}}$-closed.*
(b) *$H \cup (X - H^{\star})$ is $\mathcal{I}_{w\hat{g}}$-closed.*
(c) *$H^{\star} - H$ is $\mathcal{I}_{w\hat{g}}$-open.*

Proof. (a)⇒(b). Suppose H is $\mathcal{I}_{w\hat{g}}$-closed. If $U \in \sigma(w)$ is such that $H \cup (X - H^{\star}) \subseteq U$, then $X - U \subseteq X - (H \cup (X - H^{\star})) = X \cap (H \cup H^{\star})^c)^c = H^{\star} \cap H^c = H^{\star} - H$. Since H is $\mathcal{I}_{w\hat{g}}$-closed, by Theorem 2.15 (d), it follows that X−U=∅ and so X=U. Therefore H∪(X−H*)⊆U⇒ H∪(X−H*)⊆X and so (H∪(X−H*))* ⊆X* ⊆X=U. Hence H∪(X−H*) is $\mathcal{I}_{w\hat{g}}$-closed.

(b)⇒(a). Suppose $H \cup (X - H^{\star})$ is $\mathcal{I}_{w\hat{g}}$-closed. If $F^c \in \sigma(w)$ is such that $F \subseteq H^{\star} - H$, then $F \subseteq H^{\star}$ and $F \nsubseteq H \Rightarrow X - H^{\star} \subseteq X - F$ and $H \subseteq X - F$. Therefore $H \cup (X - H^{\star}) \subseteq H \cup (X - F) = X - F$ and $X - F \in \sigma(w)$. Since $(H \cup (X - H^{\star}))^{\star} \subseteq X - F \Rightarrow H^{\star} \cup (X - H^{\star})^{\star} \subseteq X - F$ and so $H^{\star} \subseteq X - F \Rightarrow F \subseteq X - H^{\star}$. Since $F \subseteq H^{\star}$, it follows that $F = \emptyset$. Hence H is $\mathcal{I}_{w\hat{g}}$-closed.

(b)⇔(c). $X - (H^{\star} - H) = X \cap (H^{\star} \cap H^c)^c = X \cap ((H^{\star})^c \cup H) = (X \cap (H^{\star})^c) \cup (X \cap H) = H \cup (X - H^{\star})$. ∎

Theorem 2.46. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then every subset of X is $\mathcal{I}_{w\hat{g}}$-closed if and only if every subset of $\sigma(w)$ is $\star$-closed.*

Proof. Suppose every subset of X is $\mathcal{I}_{w\hat{g}}$-closed. If U$\subseteq$X and U$\in \sigma(w)$, then U is $\mathcal{I}_{w\hat{g}}$-closed and so U$^\star \subseteq$U. Hence U is $\star$-closed. Conversely, suppose that every subset of $\sigma(w)$ is $\star$-closed. If U$\in \sigma(w)$ is such that H$\subseteq$U$\subseteq$X, then H$^\star \subseteq$U$^\star \subseteq$U and so H is $\mathcal{I}_{w\hat{g}}$-closed. ∎

The following theorem gives a characterization of normal spaces in terms of $\mathcal{I}_{w\hat{g}}$-open sets.

Theorem 2.47. *Let $(X,\tau,\mathcal{I})$ be an ideal space where $\mathcal{I}$ is completely codense. Then, the following statements are equivalent:*

(a) *X is normal.*
(b) *For any disjoint closed sets M and N, there exist disjoint $\mathcal{I}_{w\hat{g}}$-open sets P and Q such that M$\subseteq$P and N$\subseteq$Q.*
(c) *For any closed set M and open set Q containing M, there exists an $\mathcal{I}_{w\hat{g}}$-open set P such that M$\subseteq$P$\subseteq$cl*(P)$\subseteq$Q.*

Proof. (a)$\Rightarrow$(b). The proof follows from the fact that every open set is $\mathcal{I}_{w\hat{g}}$-open.

(b)$\Rightarrow$(c). Suppose that M is closed and Q is an open set containing M. Since M and X$-$Q are disjoint closed sets, there exist disjoint $\mathcal{I}_{w\hat{g}}$-open sets P and R such that M$\subseteq$P and X$-$Q$\subseteq$R. Since (X$-$Q)$^c \in \sigma(w)$ and R is $\mathcal{I}_{w\hat{g}}$-open, X$-$Q$\subseteq$int*(R) and so X$-$int*(R)$\subseteq$Q. Again P$\cap$R$=\emptyset \Rightarrow$P$\cap$ int*(R)$=\emptyset$ and so P$\subseteq$X$-$int*(R)$\Rightarrow$cl*(P)$\subseteq$X$-$int*(R) $\subseteq$Q. P is the required $\mathcal{I}_{w\hat{g}}$-open sets with M$\subseteq$P$\subseteq$cl*(P)$\subseteq$Q.

(c)$\Rightarrow$(a). Let M and N be two disjoint closed subsets of X. Then, by hypothesis there exists an $\mathcal{I}_{w\hat{g}}$-open set P such that M$\subseteq$P$\subseteq$cl*(P)$\subseteq$X$-$N. Since P is $\mathcal{I}_{w\hat{g}}$-open, M$\subseteq$int*(P). Since $\mathcal{I}$ is completely codense, by Lemma 1.3, $\tau^\star \subseteq \tau^\alpha$ and so int*(P) and X$-$cl*(P) in τ^α. Hence M$\subseteq$int*(P)$\subseteq$int(cl(int(int*(P))))=S and N$\subseteq$X$-$cl*(P)$\subseteq$ int(cl(int(X$-$cl*(P)))) =T. S and T are the required disjoint open sets containing M and N respectively, which proves (a). ∎

A subset H of an ideal space (X,τ,$\mathcal{I}$) is said to be an αgs$w$-closed set if cl$_\alpha$(H)$\subseteq$U whenever H$\subseteq$U and U$\in \sigma(w)$. The complement of an αgs$w$-closed set is said to be an αgs$w$-open set. If $\mathcal{I} = \mathcal{N}$, then $\mathcal{I}_{w\hat{g}}$-closed sets coincide with αgs$w$-closed sets and so we have the following Corollary.

Corollary 2.48. *Let $(X,\tau,\mathcal{I})$ be an ideal space where $\mathcal{I} = \mathcal{N}$. Then the following are equivalent:*

(a) *X is normal.*
(b) *For any disjoint closed sets M and N, there exist disjoint αgs$w$-open sets P and Q such that M$\subseteq$P and N$\subseteq$Q.*

(c) *For any closed set M and open set Q containing M, there exists an αgsw-open set P such that $M \subseteq P \subseteq cl_\alpha(P) \subseteq Q$.*

A subset H of an ideal space is said to be $\mathcal{I}$-compact [7] or compact modulo $\mathcal{I}$ [18] if for every open cover $\{U_\alpha \mid \alpha \in \Delta\}$ of H, there exists a finite subset Δ_0 of Δ such that $H- \cup \{U_\alpha \mid \alpha \in \Delta_0\} \in \mathcal{I}$. The space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-compact if X is $\mathcal{I}$-compact as a subset. In closing this paper, we state the following theorem.

Theorem 2.49. [16, Theorem 2.17] *Let $(X,\tau,\mathcal{I})$ be an ideal space. If H is an $\mathcal{I}_g$-closed subset of X, then H is $\mathcal{I}$-compact.*

Corollary 2.50. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If H is an $\mathcal{I}_{w\hat{g}}$-closed subset of X, then H is $\mathcal{I}$-compact.*

Proof. The proof follows from the fact that every $\mathcal{I}_{w\hat{g}}$-closed set is $\mathcal{I}_g$-closed. ∎

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