Southeast Asian Bulletin of Mathematics (2022) 46: 81–90

Southeast Asian
Bulletin of
Mathematics

The Structure of Components of Cayley Conjugate Digraphs

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Received 21 May 2018
Accepted 6 November 2018

Communicated by Yeong-Nan Yeh

AMS Mathematics Subject Classification(2020): 68R10, 05C20, 05C25, 20F65

Abstract. In this paper a new class of Cayley digraphs, namely, Cayley conjugate digraphs $\mathcal{C}(G,S)$ associated with a finite group G and a subset S of G is introduced. The adjacency in $\mathcal{C}(G,S)$ is defined in terms of a conjugacy relation in G by the elements of S. Further, its basic properties as well as the structure of components of $\mathcal{C}(G,S)$ are studied.

Keywords: Cayley digraph; Cayley conjugate digraph; Loop; In-degree; Out-degree; Component of a digraph; Balanced digraph; Regular digraph.

1. Introduction

In 1936 Konig [17] posed the question whether for any abstract group G, there is a graph $\mathcal{G}$ such that the automorphism group of $\mathcal{G}$ is isomorphic to G, or, not? This question was answered by Frucht [10] affirmatively by using the notion of color preserving automorphsims of a Cayley digraph associated with the given group. A directed graph, or, a digraph $\mathcal{D}$ consists of a finite set V of points called vertices and a set E of ordered pairs of distinct elements of V called edges and it is denoted by $\mathcal{D}$ (V,E). Let G be a group and let S be a subset of G.

The *Cayley digraph* $\mathcal{D}$ (G, S) corresponding to G and S has G as vertex set and $E = \{(g, gs)/g \in G, s \in S\}$ as the edge set. The element s is called the color or label of the edge (g, gs). In fact the edges are defined by the right regular representations of G by the elements of S, which are used in Cayley Theorem in establishing that every group G is isomorphic to a permutation group of G. It is well established (see [21, pp. 447, 448]) that the group of color preserving automorphisms of the Cayley digraph $\mathcal{D}(G, S)$ associated with a group G and a subset S of G is isomorphic to G.

Extensive studies on Cayley graphs have been carried out by C.H. Li [7], E. Knill [8], Imrich and Watkins [13], J. Morris [15], Y. Nam [23], Biggs [3] and others. Arithmetic Cayley graphs associated with the Euler totient function, the divisor function and the set of quadratic residues and quartic residues modulo a prime are studied in [4, 5, 18, 19]. In recent times considerable research work has been carried out on Cayley graphs associated with commutative rings [1], groups [2, 16, 22] and semigroups [11, 14, 16].

In this study, a new class of graphs called Cayley conjugate digraphs $\mathcal{C}(G, S)$ associated with a finite group G and a subset S of G is introduced and it is established that these graphs are directed graphs with loops and multiple edges, which are disconnected, balanced and regular. The structure of components of these graphs is also studied. The reader is referred to Bondy and Murty [6], Narsingh Deo [20] and Frank Harary [9] for graph theory and Herstein [12] for group theory terminology and notations that are not explained here.

2. Basic Properties of The Cayley Conjugate Digraph $\mathcal{C}(G, S)$.

Definition 2.1. *Let G be a finite group and S be a subset of G. The Cayley conjugate digraph $\mathcal{C}(G, S)$ has the vertex set $V = G$ and the edge set $E = \{(g, s^{-1}gs)/s \in S\}$. The element s in S is called the color, or, label of the edge $(g, s^{-1}gs)$ in $\mathcal{C}(G, S)$.*

Example 2.2. Let $S_4 = \{a_i \ / \ 0 \le i \le 23\}$ be the permutation group of the set $\{1, 2, 3, 4\}$, where

$e = a_0 = (1)$, $a_1 = (1\ 2)$, $a_2 = (1\ 3)$, $a_3 = (1\ 4)$, $a_4 = (2\ 3)$, $a_5 = (2\ 4)$,
$a_6 = (3\ 4)$, $a_7 = (1\ 2\ 3)$, $a_8 = (1\ 3\ 2)$, $a_9 = (1\ 3\ 4)$, $a_{10} = (1\ 4\ 3)$, $a_{11} = (1\ 4\ 2)$,
$a_{12} = (1\ 2\ 4)$, $a_{13} = (2\ 4\ 3)$, $a_{14} = (2\ 3\ 4)$, $a_{15} = (1\ 2\ 3\ 4)$, $a_{16} = (1\ 2\ 4\ 3)$,
$a_{17} = (1\ 3\ 4\ 2)$, $a_{18} = (1\ 3\ 2\ 4)$, $a_{19} = (1\ 4\ 3\ 2)$, $a_{20} = (1\ 4\ 2\ 3)$,
$a_{21} = (1\ 2)(3\ 4)$, $a_{22} = (1\ 3)(2\ 4)$, $a_{23} = (1\ 4)(2\ 3)$.

For the subset $S = \{\beta, \tau\}$, where $\beta = (1\ 2\ 3\ 4)$ and $\tau = (1\ 2)$, let us construct the graph $\mathcal{C}(G, S)$ by using the following calculations:

$\tau^{-1}e\tau = e$	$\tau^{-1}a_{12}\tau = a_{11}$	$\beta^{-1}e\beta = e$	$\beta^{-1}a_{12}\beta = a_9$
$\tau^{-1}a_1\tau = a_1$	$\tau^{-1}a_{13}\tau = a_{10}$	$\beta^{-1}a_1\beta = a_3$	$\beta^{-1}a_{13}\beta = a_8$
$\tau^{-1}a_2\tau = a_4$	$\tau^{-1}a_{14}\tau = a_9$	$\beta^{-1}a_2\beta = a_5$	$\beta^{-1}a_{14}\beta = a_7$
$\tau^{-1}a_3\tau = a_5$	$\tau^{-1}a_{15}\tau = a_{17}$	$\beta^{-1}a_3\beta = a_6$	$\beta^{-1}a_{15}\beta = a_{15}$

$$
\begin{array}{llll}
\tau^{-1}a_4\tau = a_2 & \tau^{-1}a_{16}\tau = a_{19} & \beta^{-1}a_4\beta = a_1 & \beta^{-1}a_{16}\beta = a_{18} \\
\tau^{-1}a_5\tau = a_3 & \tau^{-1}a_{17}\tau = a_{15} & \beta^{-1}a_5\beta = a_2 & \beta^{-1}a_{17}\beta = a_{20} \\
\tau^{-1}a_6\tau = a_6 & \tau^{-1}a_{18}\tau = a_{20} & \beta^{-1}a_6\beta = a_4 & \beta^{-1}a_{18}\beta = a_{17} \\
\tau^{-1}a_7\tau = a_8 & \tau^{-1}a_{19}\tau = a_{16} & \beta^{-1}a_7\beta = a_{12} & \beta^{-1}a_{19}\beta = a_{19} \\
\tau^{-1}a_8\tau = a_7 & \tau^{-1}a_{20}\tau = a_{18} & \beta^{-1}a_8\beta = a_{11} & \beta^{-1}a_{20}\beta = a_{16} \\
\tau^{-1}a_9\tau = a_{14} & \tau^{-1}a_{21}\tau = a_{21} & \beta^{-1}a_9\beta = a_{14} & \beta^{-1}a_{21}\beta = a_{23} \\
\tau^{-1}a_{10}\tau = a_{13} & \tau^{-1}a_{22}\tau = a_{23} & \beta^{-1}a_{10}\beta = a_{13} & \beta^{-1}a_{22}\beta = a_{22} \\
\tau^{-1}a_{11}\tau = a_{12} & \tau^{-1}a_{23}\tau = a_{22} & \beta^{-1}a_{11}\beta = a_{10} & \beta^{-1}a_{23}\beta = a_{21}
\end{array}
$$

Since $\tau^{-1}a_1\tau = a_1$, (a_1, a_1) is an edge with color τ and this is a loop. Further $\tau^{-1}a_2\tau = a_4$ gives an edge (a_2, a_4) with color τ. Similarly, the equation $\beta^{-1}a_4\beta = a_1$ shows that (a_4, a_1) is an edge with color β. In this way the other edges can be found and the conjugate digraph $\mathcal{C}(G,S)$ is given in Fig. 1. Here each edge is denoted by its color. This graph has five components, namely, I, II, III, IV, and V.

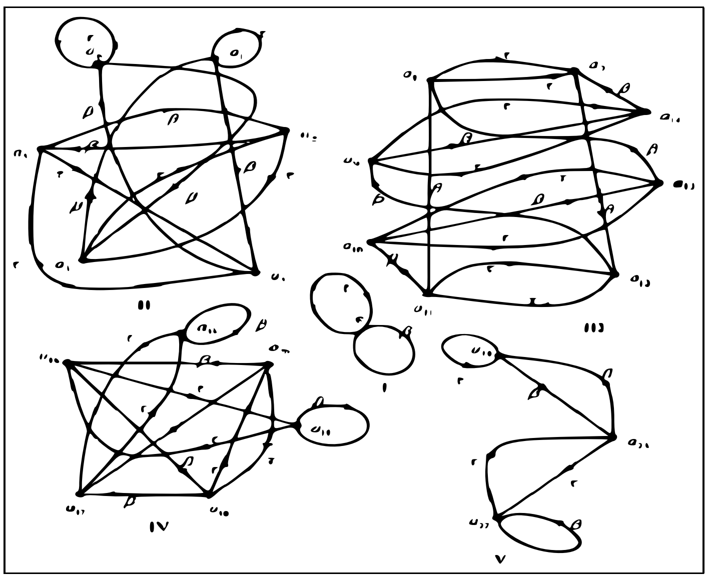

Figure 1: The conjugate digraph $\mathcal{C}(G,S)$

Note 2.3. Observe that S generates S_4 in this example.

Example 2.4. Let G=S_4, and S=$\{\beta\}$, where β=(1 2 3 4). By using the calculations given in Example 2.2, the digraph $\mathcal{C}(G,S)$ is given in Fig. 2.

Note 2.5. Observe that in this example S is not a generating subset of S_4.

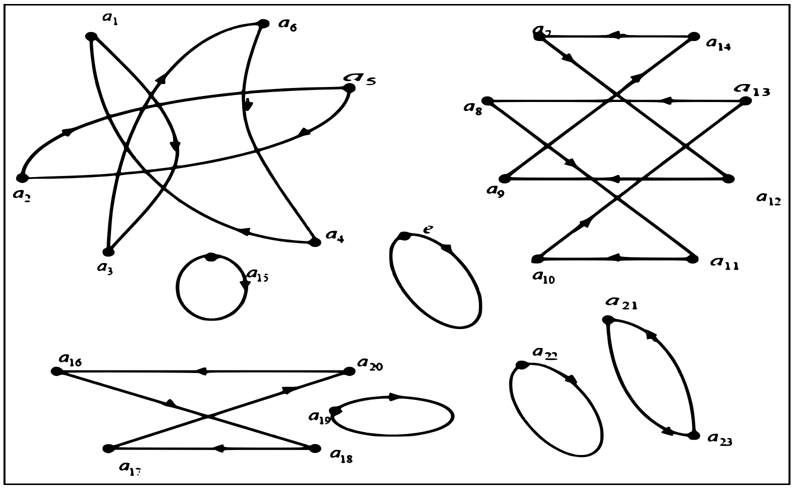

Figure 2: The digraph $\mathcal{C}(G, S)$

Theorem 2.6. *The digraph $\mathcal{C}(G, S)$ is disconnected.*

Proof. Let S be a subset of a group G. Consider the vertex e in the digraph $\mathcal{C}(G, S)$, where e is the identity element in G. Since $s^{-1}es = e$, for every s in S, each s in S induces an edge from e to e, which is a loop at the vertex e. Further, if there is an edge from a vertex g to e, then $s^{-1}gs = e$. This gives, $g = e$, which shows that e is not adjacent to any other vertex of $\mathcal{C}(G, S)$. So e is an isolated vertex of $\mathcal{C}(G, S)$ and thus it is disconnected. ■

Theorem 2.7.

(1) *A vertex v in the digraph $\mathcal{C}(G, S)$ has a loop if, and only if, $S \cap N(v)$ is nonempty, where $N(v)$ is the normalizer of v in G.*
(2) *The number of loops at a vertex v is $|S \cap N(v)|$.*
(3) *The digraph $\mathcal{C}(G, S)$ has at least $2|S|$ loops.*

Proof. (1). $\mathcal{C}(G, S)$ has a loop at the vertex $v \iff s^{-1}vs = v$, for some $s \in S \iff vs = sv$, for some $s \in S$, $\iff s \in N(v)$ fore some $s \in S \iff S \cap N(v) \neq \emptyset$.

(2). Let v be a vertex in $\mathcal{C}(G, S)$. By (i) each element in $S \cap N(v)$ induces a loop at v. Further each element s in $S - (S \cap N(v))$ induces an edge $(v, s^{-1}vs)$ at v and this edge is not a loop, since $s \notin N(v)$ implies that $s^{-1}vs \neq v$. Therefore the total number of loops at a vertex v is $|S \cap N(v)|$.

(3). From the proof of Theorem 2.6, it is evident that there are $|S|$ loops at the vertex e. Further, for each s in S, $s^{-1}ss = s$, so that each s in S induces a loop at the vertex s. Hence the digraph $\mathcal{C}(G, S)$ has at least one loop at each $s \in S$, whose number is $|S|$. Hence the digraph $\mathcal{C}(G, S)$ has at least $2|S|$ loops. ■

The following theorem gives a graphical representation of the centre of a group.

Theorem 2.8. *Let S be a generating subset of G. The set of all vertices in $\mathcal{C}(G, S)$ having loops only, forms the centre $Z(G)$ of the group G.*

Proof. Let A be the set of all vertices in $\mathcal{C}(G, S)$ having loops only. For any element v in $Z(G)$, $s^{-1}vs = vs^{-1}s = v$, which shows that there is a loop at v, so that $Z(G) \subseteq A$.

On the other hand, let $v \in A$ and $g \in G$. Since S is a generating set of G, $g = s_1s_2...s_n$, for some $s_1, s_2, ..., s_n \in S$, so that

$$g^{-1}vg = (s_1s_2...s_n)^{-1}v(s_1s_2...s_n) = s_n^{-1}s_{n-1}^{-1}s_{n-2}^{-1}...s_2^{-1}s_1^{-1}vs_1s_2...s_n.$$

As A consists of vertices having only loops, $s_1^{-1}vs_1 = s_2^{-1}vs_2 = ... = s_n^{-1}vs_n = v$, which gives $g^{-1}vg = v$, so that $v \in Z(G)$ and hence $Z(G) = A$. ■

Remark 2.9. Theorem 2.8 need not be true if S is not a generating subset of G. In the example 2.4, the subset $S = \{(1\ 2\ 3\ 4)\}$ is not a generating set of S_4. The subset $\{e, a_{15}, a_{19}, a_{22}\}$ of vertices, which have loops only, is not the centre of S_4.

Theorem 2.10. *The digraph $\mathcal{C}(G, S)$ has only loops at each vertex if, and only if, $S \subseteq Z(G)$, where $Z(G)$ is the centre of G.*

Proof. $\mathcal{C}(G, S)$ has only loops at each of its vertex v

$\iff$ $s^{-1}vs = v$, for all $v \in G$ and for all $s \in S$

$\iff$ $vs = sv$, for all $v \in G$ and for all $s \in S$

$\iff$ $s \in Z(G)$, the center of G, for all $s \in S$. ■

The following corollary is immediate.

Corollary 2.11. *If S generates G and the digraph $\mathcal{C}(G, S)$ has only loops, then the group G is abelian.*

Definition 2.12. *A digraph $\mathcal{D}(V, E)$ can alternatively be viewed as the finite set V of points together with a relation R on V, where $R = E$. In the digraph $\mathcal{D}(V, R)$ if R is a reflexive relation on V, then $\mathcal{D}(V, R)$ is called a reflexive digraph; if R is a symmetric relation on V, then $\mathcal{D}(V, R)$ is called a symmetric digraph and if R is a transitive relation on V, then $\mathcal{D}(V, R)$ is called a transitive digraph. A reflexive, symmetric and transitive digraph is called an equivalence digraph.*

The following theorems give the conditions under which the Cayley conjugate digraph $\mathcal{C}(G, S)$ becomes an equivalence digraph.

Theorem 2.13. *The digraph $\mathcal{C}(G,S)$ is reflexive if $S \cap Z(G) \neq \emptyset$.*

Proof. Let $S \cap Z(G) \neq \emptyset$. Then there is an element $s \in S$ and $s \in Z(G)$, so that $sv = vs$, for every $v \in G$, or, $s^{-1}vs = v$ for every $v \in G$. This shows that for every $v \in G, (v,v)$ is an edge of $\mathcal{C}(G,S)$, which is a loop at v and the digraph $\mathcal{C}(G,S)$ is reflexive. ■

Theorem 2.14. *The digraph $\mathcal{C}(G,S)$ is transitive if, and only if, S is a subgroup of G.*

Proof. Let S be a subgroup of G and let (a,b) and (b,c) be any two edges of $\mathcal{C}(G,S)$. Then, for some $s_1, s_2 \in S, b = s_1^{-1}as_1$ and $c = s_2^{-1}bs_2$, so that $c = (s_1s_2)^{-1}a(s_1s_2)$. Since S is a subgroup of G, $s_1, s_2 \in S$ implies that $s_1s_2 \in S$ and hence (c,a) is also an edge of $\mathcal{C}(G,S)$, so that the digraph is transitive.

Conversely, let the digraph $\mathcal{C}(G,S)$ be transitive. Let $s_1, s_2 \in S$ and $a \in G$. Then $(a, s_1^{-1}as_1)$ and $(s_1^{-1}as_1, s_2^{-1}(s_1^{-1}as_1)s_2)$ are edges in $\mathcal{C}(G,S)$. This shows that $(a, s_1^{-1}as_1)$ and $(s_1^{-1}as_1, (s_1s_2)^{-1}a(s_1s_2))$ are edges in $\mathcal{C}(G,S)$. Since $\mathcal{C}(G,S)$ is transitive, $(a, (s_1s_2)^{-1}a(s_1s_2))$ is also an edge in $\mathcal{C}(G,S)$. This gives $s_1s_2 \in S$, so that S is a closed subset of G. Since G is finite, S is a subgroup of G. ■

Theorem 2.15. *The digraph $\mathcal{C}(G,S)$ is an equivalence digraph if, and only if, S is a subgroup of G.*

Proof. Let the digraph $\mathcal{C}(G,S)$ be an equivalence digraph. Then $\mathcal{C}(G,S)$ is a transitive digraph, and hence by Theorem 2.14, S is a subgroup of G. On the other hand, if S is a subgroup of G, then by Theorem 2.14, the digraph $\mathcal{C}(G,S)$ is transitive. Further $S \cap Z(G) \neq \emptyset$, as the identity element $e \in S \cap Z(G)$. So by Theorem 2.13, the digraph $\mathcal{C}(G,S)$ is reflexive. Let (a,b) be an edge of $\mathcal{C}(G,S)$. Then $b = s^{-1}as$, or, $a = (s^{-1})^{-1}bs^{-1}$ for some $s \in S$. Since S is a subgroup of G, $s^{-1} \in S$ and (b,a) is also an edge of $\mathcal{C}(G,S)$. Therefore the digraph $\mathcal{C}(G,S)$ is symmetric and hence it is an equivalence digraph. ■

Definition 2.16. [20, p. 195] *The number of edges incident into a vertex v of a digraph $\mathcal{D}$ is called the in-degree of v and it is denoted by $d^-(v)$ .The number of edges incident out of a vertex v of a digraph $\mathcal{D}$ is called the out-degree of v and it is denoted by $d^+(v)$.*

Definition 2.17. [20, p. 195] *A digraph $\mathcal{D}$ is said to be balanced if for every vertex v of $\mathcal{D}$ the in-degree equals the out-degree, that is, $d^-(v) = d^+(v)$.*

Definition 2.18. [20, p. 197] *A balanced digraph is said to be regular if every vertex has the same in-degree and out-degree as every other vertex.*

Theorem 2.19. *The Cayley conjugate digraph* $\mathcal{C}(G, S)$ *is balanced,* $2|S|$ *-regular and the number of edges in* $\mathcal{C}(G, S)$ *is* $|G||S|$.

Proof. Let v be any vertex of $\mathcal{C}(G, S)$. Clearly each element s in S induces exactly one out-edge $(v, s^{-1}vs)$ and exactly one in-edge (svs^{-1}, v) at v. Hence

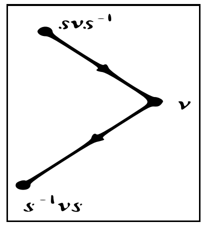

Figure 3: One out-edge and one in-edge

$d^-(v) = |S| = d^+(v)$, where $d^-(v)$ is the in-degree and $d^+(v)$ is the out-degree of the vertex v, so that the digraph $\mathcal{C}(G, S)$ is balanced. Also the degree of any vertex v is $d^-(v) + d^+(v) = 2|S|$. So the digraph $\mathcal{C}(G, S)$ is $2|S|$ - regular. Since the digraph $\mathcal{C}(G, S)$ is $2|S|$ -regular and the number of vertices in $\mathcal{C}(G, S)$ is $|G|$, the sum of the degrees of the vertices in $\mathcal{C}(G, S)$ is $2|S||G|$. Since each edge induces two degrees, the total number of edges in $\mathcal{C}(G, S)$ is $|G||S|$. ∎

3. Structure of the Components of $\mathcal{C}(G, S)$

In Theorem 2.6, it is proved that the digraph $\mathcal{C}(G, S)$ is disconnected, so that it is decomposed into the disjoint union of its components. In this section we study the nature of the *components* in the digraph $\mathcal{C}(G, S)$. Conditions under which parallel edges exist in $\mathcal{C}(G, S)$ are also discussed at the end of this section.

Theorem 3.1. *The components of the digraph* $\mathcal{C}(G, S)$ *are of the form* $\mathcal{C}(G', S)$, *for some subset* G' *of* G.

Proof. Let W be a component of $\mathcal{C}(G, S)$. Then the vertex set of W is a subset of G, say G'. We shall show that $W = \mathcal{C}(G', S)$.

Let f be an edge of $\mathcal{C}(G', S)$. Then $f = (v, s^{-1}vs)$ for some $v \in G'$ and $s \in S$. So $v \in G'$ and hence $(v, s^{-1}vs)$ is an edge in $\mathcal{C}(G, S)$. Since any two components of a digraph are edge disjoint as well as vertex disjoint, $v \in G'$ implies that the vertex $s^{-1}vs$ must lie in the vertex set G' of W. So $f = (v, s^{-1}vs)$ is an edge of W, so that $\mathcal{C}(G', S) \subseteq W$.

Also, if (a, b) is any edge in W, then it is also an edge in $\mathcal{C}(G, S)$, so that $b = s^{-1}as$, for some $s \in S$. Since $a \in G'$, the vertex set of W, the edge $(a, s^{-1}as) \in \mathcal{C}(G', S) \implies (a, b) \in \mathcal{C}(G', S)$. This shows that $W \subseteq \mathcal{C}(G', S)$ and $W = \mathcal{C}(G', S)$. ∎

Theorem 3.2. *Every component of* $\mathcal{C}(G, S)$ *is strongly connected.*

Proof. Let W be any component of $\mathcal{C}(G, S)$. Then by Theorem 3.1, $W = \mathcal{C}(G^{'}, S)$, for some subset $G^{'}$ of G. As in Theorem 2.19, one can see that W is balanced and $2|S|$-regular. That is, every component of $\mathcal{C}(G, S)$ is connected and balanced, so that it is an Euler digraph [20, Theorem 9.1, p. 204]. Since every Euler digraph is strongly connected, every component of $\mathcal{C}(G, S)$ is also strongly connected [20]. ■

Theorem 3.3. *Let* S *be a set of generators of* G*. Then any two vertices of* $\mathcal{C}(G, S)$ *lie in the same component if, and only if, they are conjugate in* G*.*

Proof. Let S be a set of generators of G and let u and v be any two vertices of the digraph$\mathcal{C}(G, S)$, which are conjugate in G. Then $v = g^{-1}ug$, for some g in G. Since S is a set of generators of G and $g \in G$, we have $g = s_1s_2...s_n$, for some $s_{1,}s_{2,}...s_n \in S$. Thus,

$$v = (s_1s_2...s_n)^{-1}u(s_1s_2...s_n) = s_n^{-1}s_{(n-1)}^{-1}...s_2^{-1}s_1^{-1}us_1s_2...s_n.$$

Setting $u = u_0, u_1 = s_1^{-1}u_0s_1, u_2 = s_2^{-1}u_1s_2, ..., v = u_n = s_n^{-1}u_{n-1}s_n$, one can see that $(u_0, u_1), (u_1, u_2), ..., (u_{n-1}, u_n)$ are edges in $\mathcal{C}(G, S)$ and the vertices u and v are connected by a directed path $u_0s_1u_1s_2u_2...s_nu_n$, where $u = u_0$ and $u_n = v$, showing that the vertices u and v lie in the same component of $\mathcal{C}(G, S)$.

Conversely, let us assume that the vertices u and v lie in the same component of the digraph $\mathcal{C}(G, S)$. Since a component of $\mathcal{C}(G, S)$ is strongly connected, the vertices u and v are connected by a directed path, say $u_0r_1u_1r_2u_2...r_nu_n$, where $u = u_0, v = u_n, r_1, r_2, ..., r_n \in S$ and $u_0, u_1, ..., u_n \in G$. Since the vertices $u_{i-1}$and u_i are joined by the edge corresponding to the label r_i, we have $u_i = r_i^{-1}u_{i-1}r_i, 1 \leq i \leq n$. Therefore

$$\begin{aligned} v &= u_n = r_n^{-1}r_{n-1}^{-1}r_{n-2}^{-1}...r_2^{-1}r_1^{-1}u_0r_1r_2...r_{n-1}r_n \\ &= (r_1r_2...r_{n-1}r_n)^{-1}u(r_1r_2...r_{n-1}r_n) = g^{-1}ug, \end{aligned}$$

where $g = r_1r_2...r_{n-1}r_n \in G$, so that u and v are conjugate in G. ■

Remark 3.4. The above theorem need not be true if S is not a set of generators of G. In the digraph given in the Example 2.4, S is not a generating set of G. The vertices a_8 and a_9 are $3-$cycles, so that they are conjugate to each other in S_4, but they lie in different components.

Remark 3.5. If S generates G, from Theorem 3.3, all vertices in a conjugacy class of G lie in the same component of $\mathcal{C}(G, S)$ and vice versa. So, the number of components of the digraph $\mathcal{C}(G, S)$ is equal to the number of conjugacy classes of G. Since the components of a graph induce a partition of the vertex set of the graph, the conjugacy classes of G partition the group G, which is nothing but the class equation for the group G. We use this fact in the following theorem.

Theorem 3.6. *If S generates G, a component of $\mathcal{C}(G,S)$ has at most $(|G|)/2$ vertices.*

Proof. Let G be a group and let S be a subset of G such that $< S >= G$. Let v be any vertex of a component W of $\mathcal{C}(G,S)$. By Theorem 3.3, all vertices of W are conjugate to v in G. So $|W| = C(v)$, where $C(v) = \{g^{-1}vg/g \in G\}$ is the conjugacy class of v in G. But $|C(v)| = |G|/|N(v)|$, where $N(v)$ is the normalizer of v in G. Since $N(v)$ contains at least two elements, namely, v and e, the identity element of G, $|N(v)| \geq 2$. Thus $|W| = |C(v)| \leq |G|/2$. ■

Remark 3.7. Let v be any vertex of $\mathcal{C}(G,S)$. Clearly each element s in S induces an out-edge at the vertex v in $\mathcal{C}(G,S)$. If $S = G$, then v has $|G|$ out-edges in $\mathcal{C}(G,S)$. Since any component of $\mathcal{C}(G,S)$ has at most $|G|/2$ vertices, the component containing the vertex v also has at most $|G|/2$ vertices. Thus, at least two out-edges at v induced by different colors, say s_1 and s_2 have the same terminating vertex, say w. This gives the possibility of parallel edges in the digraph $\mathcal{C}(G,S)$. The following theorem gives an important criterion regarding

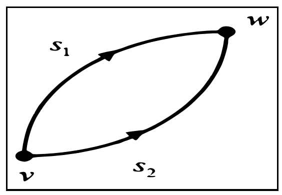

Figure 4: Parallel edges

parallel edges in $\mathcal{C}(G,S)$.

Theorem 3.8. *The vertex v in the digraph $\mathcal{C}(G,S)$ has parallel out-edges if, and only if, for some $s \in S$, the coset $N(v)s$ contains an element of S other than s.*

Proof. A vertex v in $\mathcal{C}(G,S)$ has parallel out-edges

$\iff$ for some $s \in S$ there exists $t \in S$,$t \neq s$ such that $(v, s^{-1}vs)$ and $(v, t^{-1}vt)$ are parallel edges

$\iff$ for some $s \in S$ there exists $t \in S$, $t \neq s$ such that $s^{-1}vs = t^{-1}vt$

$\iff$ for some $s \in S$ there exists $t \in S$, $t \neq s$ such that $ts^{-1}v = vts^{-1}$

$\iff$ for some $s \in S$ there exists $t \in S$, $t \neq s$ such that $ts^{-1} \in N(v)$

$\iff$ for some $s \in S$ there exists $t \in S$,$t \neq s$ such that $t \in N(v)s$. ■

Conclusion. The structure of component preserving automorphisms and component preserving inner automorphisms of $\mathcal{C}(G,S)$ are studied separately.

Acknowledgement. The authors express their thanks to Prof. L. Nagamuni Reddy for his valuable suggestions during the preparation of this paper. The authors also thank Prof. K.P. Shum for his valuable suggestions and the referees for their critical comments, which enhanced the clarity of the paper.

References

[1] M. Afkhami, K.K. Hasshyarmanesh, On the cozero-divizor graphs of commutative rings, *Southeast Asian Bull. Math.* **35** (3) (2011) 753–762.

[2] M. Alaeiyan, On normal edge-transitive Cayley graphs of some abelian groups, *Southeast Asian Bull. Math.* **33** (1) (2009) 13–19.

[3] N.L. Biggs, *Algebraic Graph Theory*, Cambridge University Press, 1974.

[4] J.A. Bondy, U.S.R. Murty, *Graph Theory and Related Topics*, Acad. Press, New York, 1979.

[5] M. Budden, N. Calkins, W.N. Hack, J. Lambert, K. Thompson, Enumeration of triangles in quartic residue graphs, *Integers* **11** #A48 (2011), 16 pages.

[6] N. Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall of India Pvt. Ltd., New Delhi, 1979.

[7] R. Frucht, Graphs of degree three with a given abstract group, *Canad. J. Math.* **1** (1949) 365–378.

[8] F. Harary, *Graph Theory*, Addison Wesley Reading, 1969.

[9] I.N. Herstein, *Topics in Algebra*, **2nd** Ed., Vikas Publishing House Private Limited, New Delhi, 1975.

[10] W. Imrich, M.E. Watkins, On automorphism groups of cayley graphs, *Period Math. Hungar.* **7** (1976) 243–258.

[11] G.M. Jian and S.H. Fan, On undirected Cayley graphs of some completely simple semigroups, *Southeast Asian Bull. Math.* **33** (2009) 741–749.

[12] L. John, A.N.P. Kumari, Semigroup theoretical study of Cayley graphs of rectangular bands, *Southeast Asian Bull. Math.* **35** (4) (2011) 943–950.

[13] C.H. Li, On isomorphisms of finite Cayley graphs: A survey, *Discrete Mathematics* **256** (2002) 301–334.

[14] A.V. Kelerav, C.E. Praeger, On transitive Cayley graphs of groups and semigroups, *European Journal of Combinatorics* **24** (2003) 59–72.

[15] E. Knill, Notes on connectivity of Cayley coset digraphs, arXiv:math/9411221v1.

[16] D. Konig, *Theorie der Endlichen und Unendlichen Graphen*, Leipzig, 1936; Reprinted Chelsea, New York, 1950.

[17] M. Levaku, *Studies on Domination Parameters and Enumeration of Cycles in Some Arithmetic Graphs*, Ph.D. Thesis, Sri Venkateswara University, Tirupati, India, 2002.

[18] B. Maheswari, L. Madhavi, Enumeration of triangles and Hamilton cycles in quadratic residue Cayley graphs, *Chamchuri Journal of Mathematics* **1** (1) (2009) 95-103.

[19] B. Maheswari, L. Madhavi, Enumeration of Hamiltion cycles and triangles in Euler Totient Cayley graphs, *Graph Theory Notes of New York* **LIX** (2010) 28–31.

[20] J. Morris, *Connectvity of Cayley graphs: A special family, (A survey article)*, NSERC Undergraduate Research Award Article, Dept. of Mathematics and Statistics, Trent University, Peterborough, 2004.

[21] Y. Nam, Isomorphism classes of Cayley permutation graphs, *J. Korean Math. Soc.* **34** (2) (1997) 337–344.

[22] K.R. Parthasarathy, *Basic Graph Theory*, Tata Mc.Graw-Hill Publishing Company Limited, 1994.

[23] P. Wang and J.S. Li, On the normality of Cayley digraphs of valancy 2 in nonabelian group of odd order, *Southeast Asian Bull. Math.* **24** (2) (2000) 323–329.