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The Algebraic and Geometric Classification of Generalized Bott Projections of Matrix Algebras over Complex or Real Number

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Abstract. We review the Bott projections of the two by two matrix algebra over complex numbers, and then study projections of matrix algebras over complex or real numbers, to classify the generalized Bott projections (in our sense) and their spaces algebraically and geometrically, in the three by three, and four by four matrix cases, and in the general matrix case.

Keywords: Matrix algebra; Bott projection; K-theory; Projection; Homotopy theory; Noncommutative geometry.

1. Introduction

Without using Linear Algebra techniques such as for being diagonalizable of matrices (cf. [18], [33] as famous classical items), more directly, we would like to study projections of matrix algebras over complex or real numbers, to classify the generalized Bott projections (in our sense) and their spaces algebraically and geometrically, in the three by three, and four by four matrix cases, and in the general matrix case. In addition, as just a comparison, unitaries of the complex matrix algebras are only reviewed.

Our motivation for this study comes from deep and thorough understanding the so called Bott projection as well as its basic properties as in [34] (cf. [2], [26], [27], [28], and [29]). The Bott projection is viewed as a continuous, 2×2 matrix projection-valued function on the real two-dimensional sphere S^2, which also connects continuously the two standard rank one projections at north and south poles on S^2. The Bott projection does play an important role in the K-theory of C^*-algebras (cf. [31] and [34], as well as [2], [26], [27], [28], and [29]). Namely, it gives a non-trivial K-theory class of the C^*-algebra of all continuous, complex-valued functions on the 2-sphere S^2. As well, refer to [3] and [13] for the topological K-theory of spaces, and moreover, see [9] for other topics.

In this paper, we review the Bott projections in the 2×2 matrix algebra over complex numbers, and then obtain algebraic and geometric classification results on the generalized Bott projections (in our sense) of matrix algebras over complex or real numbers and on the spaces of the projections in the 3×3, and 4×4 matrix cases, and in the general matrix case. In the 2×2 matrix case, the Bott projection and its properties are considered in details, with some refinement or extension to the literature as in [34]. May as well refer to [31, 8.5], [32], [25], and moreover, [4], [12], [22], [24]. There may be more other items found in the literature. For some advanced or developed topics, may refer to [8], [10], [15]. [16], [17], and [21]. Furthermore, may refer to [1], [6], [7], [14], [19], [20], [23], and [30].

The explicit formulae obtained by determining those projections algebraically may be some useful as a convenient reference. As well, the geometric (or topological) structure for the spaces of the generalized Bott projections may have some applications such as to the theory of C^*-algebras. Certainly, the geometric (or topological) structure for the spaces may be considered as the first basic step towards yet a noncommutative geometry (or topology) theory for C^*-algebras (without stabilizing), such as the (stabilized) K-theory for C^*-algebras ([24], [34]). As well, the homotopical structure for those spaces is also deduced, but which is certainly well known (in Linear Algebra or Geometry) (cf. [18], [33]).

Unfortunately, this time we could not determine all the projections in those cases (except the 2×2 case), which may involve more further computation, but such computation in the 3×3 case and more may be known to some experts. This task may be considered in the future, but temporarily postponed.

Notation 1.1. We use the symbol $\equiv$ as meaning a definition. Let $i \in \mathbb{C}$ with $i^2 = -1$. We use the symbol $\approx$ as meaning a homemorphism. We denote by $M_n(\mathbb{C})$ the $n \times n$ matrix Banach or C^*-algebra over $\mathbb{C}$ of complex numbers. For convenience, we may consider the Euclidean norm on $M_n(\mathbb{C})$ as the complex n^2-dimensional Euclidean vector space $\mathbb{C}^{n^2}$, to equip its topolology. We denote by $M_n(\mathbb{R})$ the $n \times n$ matrix Banach algebra over $\mathbb{R}$ of real numbers.

Recall that an element $p \in M_n(\mathbb{C})$ is a projection if and only if $p = p^2 = p^*$ with p^* the adjoint of p, i.e, the complex conjugate transpose $\overline{p}^t$ of p. We denote by $P(M_n(\mathbb{C}))$ the space of all non-trivial projections of $M_n(\mathbb{C})$ with relative topology, and by

$$P(M_n(\mathbb{C}))^{\sim} = P(M_n(\mathbb{C})) \cup \{0_n, 1_n\} \quad \text{(as union)}$$

the space of all projections of $M_n(\mathbb{C})$, where 0_n is the zero matrix and 1_n is the identity matrix. Define $P(M_n(\mathbb{R}))$ and $P(M_n(\mathbb{R}))^{\sim}$ similarly.

For $a, b \in M_n(\mathbb{C})$ $(n \geq 1)$, we denote by $a \oplus b$ the diagonal sum in $M_{2n}(\mathbb{C})$.

2. The 2 × 2 Matrix Case

By solving the equation for the definition of 2×2 matrix projections, we obtain that, with several proper notations according to the calculation in the proof below,

Theorem 2.1. *If $p = (p_{ij})$ is a non-trivial projection of $M_2(\mathbb{C})$, then p is either*

(1) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \oplus 0 \equiv p_1, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 \oplus 1 \equiv p_2, \quad$ *or*

(2) $\begin{pmatrix} \frac{1 \pm \sqrt{1-4|z|^2}}{2} & \overline{z} \\ z & \frac{1 \mp \sqrt{1-4|z|^2}}{2} \end{pmatrix} \equiv p_{\pm}(z) \quad$ (*compound order*)

for any $z \in \mathbb{C} \setminus \{0\}$, with $1 > 1 - 4|z|^2 \geq 0$ if and only if $0 < |z| \leq \frac{1}{2}$.

We may as well define $p_{\pm}(0)$ as

$$p_+(0) = \lim_{z \to 0} p_+(z) = p_1 \quad \textit{and} \quad p_-(0) = \lim_{z \to 0} p_-(z) = p_2.$$

For $z = \frac{1}{2}e^{i\theta}$ with $\theta \in \mathbb{R}$ (mod 2π) and $|z| = \frac{1}{2}$,

$$p_{\pm}(\frac{1}{2}e^{i\theta}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}e^{-i\theta} \\ \frac{1}{2}e^{i\theta} & \frac{1}{2} \end{pmatrix} \equiv p(\frac{1}{2}e^{i\theta}).$$

In particular,

$$p_+(\pm\frac{1}{2}) = p_-(\pm\frac{1}{2}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \equiv p(\frac{1}{2}) \in M_2(\mathbb{R}).$$

Moreover, $p_+(z) = p_-(z)$ for $z \in \mathbb{C} \setminus \{0\}$ if and only if $|z| = \frac{1}{2}$. Furthermore, $p_{\pm}(z) \in M_2(\mathbb{R})$ if and only if $z \in \mathbb{R}$. Also, the extended $p_{\pm}(z)$ are viewed as injective, continuous, projection-valued functions from the closed ball $\{z \in \mathbb{C} \,|\, |z| \leq \frac{1}{2}\} \equiv B(0, \frac{1}{2})$ with center 0 and radius $\frac{1}{2}$ in $\mathbb{C}$ to $P(M_2(\mathbb{C}))$.

Proof. Suppose that $p = p^* = (\overline{p_{ji}})$. Then

$$p = \begin{pmatrix} a & \overline{z} \\ z & d \end{pmatrix} \quad a, d \in \mathbb{R}, z \in \mathbb{C}.$$

In addition, if $p^2 = p$, then $a^2 + |z|^2 = a$, $|z|^2 + d^2 = d$, and $(a + d)\overline{z} = \overline{z}$.

If $z = 0$, then $a = 0$ or 1, and $d = 0$ or 1.

If $z \neq 0$, then $a + d = 1$, so that if $|z| \leq \frac{1}{2}$, then

$$a = \frac{1 \pm \sqrt{1 - 4|z|^2}}{2} \quad \text{and} \quad d = \frac{1 \mp \sqrt{1 - 4|z|^2}}{2} \quad \text{in } \mathbb{R}$$

by solving the quadratic equations above with respect to a and d. And if $|z| \geq \frac{1}{2}$, then

$$a = \frac{1}{2} \pm i\sqrt{|z|^2 - \frac{1}{4}} \quad \text{and} \quad d = \frac{1}{2} \pm i\sqrt{|z|^2 - \frac{1}{4}} \quad \text{not in } \mathbb{R},$$

and hence, this case does not exist.

If $p_+(z) = p_-(z)$, then $|z| = \frac{1}{2}$ by computing the $(1, 1)$-entry in the equation.

The injectivity and continuity for $p_\pm(z)$ are clear, ■

We may say that the functions $p_\pm(z)$ are the **Bott projection** (function) on $B(0, \frac{1}{2})$, whose domain can be converted as given later below.

Corollary 2.2. *The space $P(M_2(\mathbb{R}))$ consists of p_1, p_2, and $p_\pm(t)$ for $t \in \mathbb{R}$ with $0 < |t| \leq \frac{1}{2}$, where $p_1 = \lim_{t\to 0} p_+(t) \equiv p_+(0)$, $p_2 = \lim_{t\to 0} p_-(t) = p_-(0)$, and $p_+(\pm\frac{1}{2}) = p(\frac{1}{2}) = p_-(\pm\frac{1}{2})$.*

Now let X and Y be topological spaces and K be a space viewed as a subspace in both X and Y. We denote by $X \sqcup_K Y$ the K**-jointed sum** of X and Y (we call so), which is defined to be the space obtained from attaching X and Y on the space K, or in other words, as that in the disjoint union $X \sqcup Y$ of X and Y, the space K viewed in X is identified with K viewed in Y.

We denote by S^2 the real 2-dimensional sphere in $\mathbb{R}^3$.

Theorem 2.3. *There is a homeomorphism between the space $P(M_2(\mathbb{C}))^\sim$ and the disjoint union $\{0_2\} \sqcup \{1_2\} \sqcup (B(0, \frac{1}{2}) \sqcup_{\frac{1}{2}S^1} B(0, \frac{1}{2}))$, with*

$$S^2 \approx B(0, \frac{1}{2}) \sqcup_{\frac{1}{2}S^1} B(0, \frac{1}{2}) \approx P(M_2(\mathbb{C}))$$

of all rank 1 projections of $M_2(\mathbb{C})$, where $B(0, 2^{-1}) \sqcup_{2^{-1}S^1} B(0, 2^{-1})$ is the space obtained from attaching two copies of $B(0, 2^{-1})$ along the set $\frac{1}{2}S^1 = \{z \in \mathbb{C} \,|\, |z| = \frac{1}{2}\}$, as a $\frac{1}{2}S^1$-jointed sum or a circle-jointed sum, where $\frac{1}{2}S^1$ is homeomorphic to the real 1-dimensional sphere $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$.

Proof. Define a homeomorphism from $B(0, \frac{1}{2}) \sqcup_{\frac{1}{2}S^1} B(0, \frac{1}{2})$ onto $P(M_2(\mathbb{C}))$ by sending $z \in B(0, 2^{-1})$ one copy with $|z| \leq \frac{1}{2}$ to $p_+(z)$, and $w \in B(0, 2^{-1})$ the other copy with $|w| \leq \frac{1}{2}$ to $p_-(w)$.

We may identify the space $\frac{1}{2}S^1 = \{z \in \mathbb{C} \,|\, |z| = \frac{1}{2}\}$ with S^1. It follows from elementary Topology that we can make the 2-sphere from attaching two distinct closed unit balls as in $\mathbb{R}^2$ along their boundary as S^1, in $\mathbb{R}^3$. ■

Corollary 2.4. *There is a homeomorphism between the space* $P(M_2(\mathbb{R}))^{\sim}$ *and the disjoint union* $\{0_2\} \sqcup \{1_2\} \sqcup ([-\frac{1}{2}, \frac{1}{2}] \sqcup_{\frac{1}{2}S^0} [-\frac{1}{2}, \frac{1}{2}])$*, with*

$$S^1 \approx ([-\frac{1}{2}, \frac{1}{2}] \sqcup_{\frac{1}{2}S^0} [-\frac{1}{2}, \frac{1}{2}]) \approx P(M_2(\mathbb{R}))$$

of all rank 1 *projections of* $M_2(\mathbb{R})$*, where* $[-2^{-1}, 2^{-1}] \sqcup_{\frac{1}{2}S^0} [-2^{-1}, 2^{-1}]$ *is the space obtained from attaching two copies of the closed interval* $[-2^{-1}, 2^{-1}]$ *at the set* $\frac{1}{2}S^0 = \{t \in \mathbb{R} \,|\, |t| = \frac{1}{2}\} = \{\pm\frac{1}{2}\}$*, as a two-points-jointed sum, where* $\frac{1}{2}S^0$ *is homeomorphic to the real* 0*-dimensional sphere* $S^0 = \{t \in \mathbb{R} \,|\, |t| = 1\} = \{\pm 1\}$*.*

We say that two projections p and q of $P(M_2(\mathbb{C}))^{\sim}$ are homotopic if there is a continuous path in $P(M_2(\mathbb{C}))^{\sim}$ connecting p and q. We denote by $[p]$ the homotopy class of p and by $P(M_2(\mathbb{C}))^{\sim}/\sim$ the set of all homotopy classes of elements of $P(M_2(\mathbb{C}))^{\sim}$.

The well known consequences are deduced:

Corollary 2.5. *The homotopy classes within* $P(M_2(\mathbb{C}))^{\sim}$ *are given by* $[0_2]$*,* $[1_2]$*, and* $[1 \oplus 0] = [0 \oplus 1] \in P(M_2(\mathbb{C}))^{\sim}/\sim$*.*

Proof. Note that $\lim_{z\to 0} p_+(z) = 1 \oplus 0$ and $\lim_{z\to 0} p_-(z) = 0 \oplus 1$ and that $p_\pm(z)$ take the same value $p(z)$ at $z = \frac{1}{2}e^{i\theta}$. ∎

Corollary 2.6. *The homotopy classes within* $P(M_2(\mathbb{R}))^{\sim}$ *are given by* $[0_2]$*,* $[1_2]$*, and* $[1 \oplus 0] = [0 \oplus 1] \in P(M_2(\mathbb{R}))^{\sim}/\sim$*.*

For $x = (x_{ij}) \in M_2(\mathbb{C})$, we denote by $\mathrm{tr}(x)$ the canonical trace of x, which is viewed as a function on $M_2(\mathbb{C})$, where $\mathrm{tr}(x) = x_{11} + x_{22}$. For $x \in M_2(\mathbb{C})$, we denote by $\mathrm{rk}(x)$ the rank of x, as a function.

Corollary 2.7. *There are bijections among the homotopy set* $P(M_2(\mathbb{C}))^{\sim}/\sim$*, the image* $\mathrm{tr}(P(M_2(\mathbb{C}))^{\sim}) = \{0, 1, 2\}$*, and the image* $\mathrm{rk}(P(M_2(\mathbb{C}))^{\sim}) = \{0, 1, 2\}$*.*

The same also holds for $P(M_2(\mathbb{R})))^{\sim}$*.*

Proof. The trace for any $p \in P(M_2(\mathbb{C}))$ determined explicitly above is easily computed. Only to determine the rank of p, we may use a well known fact in Linear Algebra as that each p has eigenvalues 0 or 1, and that there is a unitary matrix u of $M_2(\mathbb{C})$ such that upu^* is equal to either 0, 1_2, or p_1. Without using this, we may do computation as in Lemma 2.11 below, but complicated in general. Note also that $\mathrm{tr}(upu^*) = \mathrm{tr}(p)$ and that the trace on $P(M_2(\mathbb{C}))$ takes only 1. ∎

Let $B(0, r) = \{z \in \mathbb{C} \,|\, |z| \le r\}$ be the closed unit ball in $\mathbb{C}$ with center 0 and radius r. By changing variables as in the statement below and computing the components of $p_\pm(z)$ in Theorem 2.1, it follows that

Proposition 2.8. [34, 5.I (e)] (Extended) *We obtain*

$$p_+(\frac{z}{1+|z|^2}) = \frac{1}{1+|z|^2}\begin{pmatrix}1 & \overline{z} \\ z & |z|^2\end{pmatrix} \equiv B_+(z), \quad z \in \mathbb{C} \setminus \{0\}$$

with $0 < |z| \leq 1$, *where* $\frac{|z|}{1+|z|^2} \leq \frac{1}{2}$ *if and only if* $|z| \in [0, \infty)$, *as well as*

$$p_+(\frac{z}{1+|z|^2}) = \frac{1}{1+|z|^2}\begin{pmatrix}|z|^2 & \overline{z} \\ z & 1\end{pmatrix} \equiv B_-(z), \quad z \in \mathbb{C} \setminus \{0\}$$

with $|z| \geq 1$.

On the other hand,

$$p_-(\frac{z}{1+|z|^2}) = \frac{1}{1+|z|^2}\begin{pmatrix}|z|^2 & \overline{z} \\ z & 1\end{pmatrix} \equiv B_-(z), \quad z \in \mathbb{C} \setminus \{0\}$$

with $0 < |z| \leq 1$, *as well as*

$$p_-(\frac{z}{1+|z|^2}) = \frac{1}{1+|z|^2}\begin{pmatrix}1 & \overline{z} \\ z & |z|^2\end{pmatrix} \equiv B_+(z), \quad z \in \mathbb{C} \setminus \{0\}$$

with $|z| \geq 1$.

Namely, the **Bott** *projection(-valued function) in* $M_2(\mathbb{C})$ *as in* [34] *is defined on* $\mathbb{C} \setminus \{0\}$ *as*

$$B_+(z) = \frac{1}{1+|z|^2}\begin{pmatrix}1 & \overline{z} \\ z & |z|^2\end{pmatrix} = \begin{cases} p_+(\frac{z}{1+|z|^2}) & 0 < |z| \leq 1, \\ p_-(\frac{z}{1+|z|^2}) & |z| \geq 1. \end{cases}$$

As well, we define

$$B_-(z) = \frac{1}{1+|z|^2}\begin{pmatrix}|z|^2 & \overline{z} \\ z & 1\end{pmatrix} = \begin{cases} p_-(\frac{z}{1+|z|^2}) & 0 < |z| \leq 1, \\ p_+(\frac{z}{1+|z|^2}) & |z| \geq 1, \end{cases}$$

which may be called the **dual** *Bott projection-valued function on* $\mathbb{C} \setminus \{0\}$.

In addition, we may set

$$B_+(0) = 1 \oplus 0 = p_1 \quad \text{and} \quad B_-(0) = 0 \oplus 1 = p_2,$$

and we have at infinity in any direction,

$$B_+(\infty) \equiv \lim_{|z| \to \infty} B_+(z) = 0 \oplus 1 = p_2 \quad \text{and} \quad B_-(\infty) \equiv \lim_{|z| \to \infty} B_-(z) = 1 \oplus 0 = p_1,$$

but within $P(M_2(\mathbb{C}))$.

Moreover, $B_\pm(z)$ *are injective as continuous functions from* $\mathbb{C}$ *into* $P(M_2(\mathbb{C}))$. *Also,* $B_+(z) = B_-(z)$ *if and only if* $|z| = 1$, *but* $B_+(z) = B_-(w)$ *such that for any* $z \in \mathbb{C} \setminus \{0\}$, *there is* $w \in \mathbb{C} \setminus \{0\}$ *such that* $z \neq w$ *with* $|z| < 1 < |w|$ *or* $|w| < 1 < |z|$, *as well as* $B_+(\mathbb{C} \setminus \{0\}) = B_-(\mathbb{C} \setminus \{0\})$ *as images* *with $B_+(B(0,1)\setminus\{0\}) = B_-(B^\circ(0,1)^c)$ and $B_+(B^\circ(0,1)^c) = B_-(B(0,1)\setminus\{0\})$, where $B^\circ(0,r) = \{z \in \mathbb{C} \,|\, |z| < r\}$ is the open ball as the interior of $B(0,r)$.*

Furthermore, $B_\pm(z)$ respectively can extend to injective continuous functions from the real 2-dimensional torus $S^2 \approx \mathbb{C} \sqcup \{\infty\}$ the one-point compactfication of $\mathbb{C}$ into $P(M_2(\mathbb{C}))$, with $B_+(S^2) = B_-(S^2)$. Namely, by the same symbol extended, $B_\pm$ are $P(M_2(\mathbb{C}))$-valued, continuous functions on S^2.

Proof. Let $f(x) = \frac{x}{1+x^2}$ for $x \in \mathbb{R}$. Then the derivative $f'(x) = 0$ if and only if $x = \pm 1$. Note that $f(\pm 1) = \pm\frac{1}{2}$ the maximum and the minimum of $f(x)$ respectively, $f(0) = 0$, and $\lim_{|x|\to\infty} f(x) = 0$, and that $f(x)$ is not injective on $[0,\infty)$, as that for any $0 < x < 1$, there is $y > 1$ such that $f(x) = f(y)$, and that $f((0,1]) = f([1,\infty))$ as images.

Compute only the $(1,1)$-entry of $p_+(\frac{z}{1+|z|^2})$ and $p_-(\frac{z}{1+|z|^2})$ case by case as

$$\frac{1+\sqrt{1-4\left(\frac{z}{1+|z|^2}\right)^2}}{2} = \frac{1+\left(\frac{\sqrt{(1-|z|^2)^2}}{1+|z|^2}\right)}{2} = \begin{cases} \frac{1}{1+|z|^2} & |z| \le 1, \\ \frac{|z|^2}{1+|z|^2} & |z| \ge 1, \end{cases}$$

$$\frac{1-\sqrt{1-4\left(\frac{z}{1+|z|^2}\right)^2}}{2} = \frac{1-\left(\frac{\sqrt{(1-|z|^2)^2}}{1+|z|^2}\right)}{2} = \begin{cases} \frac{|z|^2}{1+|z|^2} & |z| \le 1, \\ \frac{1}{1+|z|^2} & |z| \ge 1. \end{cases}$$

If $B_\pm(z) = B_\pm(w)$ respectively, then $z = w$.

If $B_+(z) = B_-(z)$, then $|z| = 1$. ■

Theorem 2.9. *There are identities and homeomorphisms as*

$$\begin{aligned} &B_+(S^2) \equiv \{B_+(z) \,|\, z \in \mathbb{C}\cup\{\infty\}\} = \{B_-(z) \,|\, z \in \mathbb{C}\cup\{\infty\}\} \equiv B_-(S^2) \\ =&P(M_2(\mathbb{C})) \approx B(0,\frac{1}{2}) \sqcup_{\frac{1}{2}S^1} B(0,\frac{1}{2}) \approx B(0,1) \sqcup_{S^1} B(0,1) \approx S^2. \end{aligned}$$

In other words, the Bott projection-valued function B_+ and the dual Bott projection-valued function B_- extended on S^2 are homeomorphisms to $P(M_2(\mathbb{C}))$.

Recall now that the complex projective plane $\mathbb{C}P(1)$ is defined to be the set of all equivalence classes $[(z_1,z_2)]$ (as directions) of points $(z_1,z_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$, where (z_1,z_2) is equivalent to (z_1',z_2') if and only if there is some non-zero $\lambda \in \mathbb{C}$ such that $\lambda(z_1,z_2) = (\lambda z_1, \lambda z_2) = (z_1', z_2')$.

Lemma 2.10. [34, 5.I (a)] *The complex projective space $\mathbb{C}P(1)$ is homeomorphic to the real 2-dimensional sphere S^2.*

Proof. Take $[(0,1)] \in \mathbb{C}P(1)$. Let $[(z_1,z_2)] \in \mathbb{C}P(1)\setminus\{[(0,1])\}$ with $z_1 \neq 0$. Then $(z_1,z_2) = z_1(1, z_1^{-1}z_2)$. Thus $[(z_1,z_2)] = [(1, z_1^{-1}z_2)]$. Since $[(1,z)] = [(1,w)]$ if and only if $z = w \in \mathbb{C}$, $\mathbb{C}P(1) \setminus \{[(0,1])\}$ is identified with $\mathbb{C}$, which is homeomorphic to $\mathbb{R}^2$. Note as well that for $(z_1, z_2) = z_1(1, z_1^{-1}z_2)$ with $z_1 \neq 0$, if z_1 converges to $0 \in \mathbb{C}$, then $\lim_{z_1 \to 0} z_1^{-1}z_2 = \infty$. Hence $[(z_1, z_2)] = [(1, z_1^{-1}z_2)]$ converges to $[(0,1)]$ in $\mathbb{C}P(1)$ as $z_1 \to 0$. It then follows that $\mathbb{C}P(1)$ is homeomorphic to S^2 as the one-point compactification of $\mathbb{R}^2$. ∎

Lemma 2.11. [34, 5.I (b)] (Extended) *The Bott projection $B_+(z)$ at $z \in \mathbb{C}$ is also the projection from $\mathbb{C}^2$ to the complex 1-dimensional subspace spanned by a point $(1, z) \in \mathbb{C}^2$ as a column vector.*

As well, the dual Bott projection $B_-(z)$ at $z \in \mathbb{C}$ is also the projection from $\mathbb{C}^2$ to the complex 1-dimensional subspace spanned by a point $(\overline{z}, 1) \in \mathbb{C}^2$ as a column vector.

Moreover, for $z \in \mathbb{C}$ we obtain

$$B_+(z)\begin{pmatrix} -\overline{z} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \textit{and} \quad B_-(z)\begin{pmatrix} 1 \\ -z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Proof. Indeed, compute that for $(z_1, z_2) \in \mathbb{C}^2$,

$$B_+(z)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{1+|z|^2}\begin{pmatrix} 1 & \overline{z} \\ z & |z|^2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{z_1 + \overline{z}z_2}{1+|z|^2}\begin{pmatrix} 1 \\ z \end{pmatrix} = \frac{\langle (z_1, z_2), (1, z)\rangle}{\|(1, z)\|^2}\begin{pmatrix} 1 \\ z \end{pmatrix},$$

where $\langle (z_1, z_2), (w_1, w_2)\rangle = z_1\overline{w_1} + z_2\overline{w_2}$ is the complex inner product for $\mathbb{C}^2$ and $\|(z_1, z_2)\| = \sqrt{\langle (z_1, z_2), (z_1, z_2)\rangle}$ the norm. As well,

$$B_-(z)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{1+|z|^2}\begin{pmatrix} |z|^2 & \overline{z} \\ z & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{z_1 z + z_2}{1+|z|^2}\begin{pmatrix} \overline{z} \\ 1 \end{pmatrix} = \frac{\langle (z_1, z_2), (\overline{z}, 1)\rangle}{\|(\overline{z}, 1)\|^2}\begin{pmatrix} \overline{z} \\ 1 \end{pmatrix} \blacksquare$$

Lemma 2.12. [34, 5.I (c)] (Extended) *There is a homeomorphism from $\mathbb{C}P(1) \setminus \{[(0,1)]\}$ to the subspace $\{B_+(z) \mid z \in \mathbb{C}\}$ of $P(M_2(\mathbb{C}))$, defined by sending $[(1, z)]$ to $B_+(z)$ for $z \in \mathbb{C}$.*

There is also a homeomorphism from $\mathbb{C}P(1) \setminus \{[(0,1)]\}$ to the subspace $\{B_-(z) \mid z \in \mathbb{C}\}$ of $P(M_2(\mathbb{C}))$, defined by sending $[(1, z)]$ to $B_-(z)$ for $z \in \mathbb{C}$.

Corollary 2.13. [34, 5.I (d)] (Extended) *There are homeomorphisms as follows*:

$$\begin{aligned} \mathbb{C}P(1) \approx S^2 &\approx \{B_+(z) \mid z \in \mathbb{C}\} \cup \{p_2\} = B_+(S^2) \\ &= \{B_-(z) \mid z \in \mathbb{C}\} \cup \{p_1\} = B_-(S^2) = P(M_2(\mathbb{C})). \end{aligned}$$

Unitaries 2.14. We now denote by $U_2(\mathbb{C})$ the group of 2×2 unitary matrices of $M_2(\mathbb{C})$. Namely, $u \in U_2(\mathbb{C})$ if and only if $u^*u = 1_2 = uu^*$.

For $z \in \mathbb{C}$, we define

$$u_+(z) = \frac{1}{\sqrt{1+|z|^2}}\begin{pmatrix} 1 & -\overline{z} \\ z & 1 \end{pmatrix} \quad \text{and} \quad u_-(z) = \frac{1}{\sqrt{1+|z|^2}}\begin{pmatrix} \overline{z} & 1 \\ 1 & -z \end{pmatrix}.$$

Lemma 2.15. [34, 5.I (f)] (Extended) *For any $z \in \mathbb{C}$, it follows that $u(z)_\pm \in U_2(\mathbb{C})$. As well, $u_+(z) = u_+(z)^*$ if and only if $z = 0$ and $u_-(z) = u_-(z)^*$ if and only if $z \in \mathbb{R}$, and that*

$$u_+(z)p_1u_+(z)^* \equiv \mathrm{Ad}(u_+(z))p_1 = B_+(z) \quad \textit{and} \quad \mathrm{Ad}(u_-(z))p_1 = B_-(z).$$

Now let

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \equiv \begin{pmatrix} u_1 & u_2 \end{pmatrix} \in U_2(\mathbb{C})$$

with u_1 and u_2 as column vectors in $\mathbb{C}^2$. As a well known fact, there is an equivalence between $u \in U_2(\mathbb{C})$ and $u^*u = 1_2$. Equivalently, $\|u_1\|^2 = |u_{11}|^2 + |u_{21}|^2 = 1 = \|u_2\|^2 = |u_{12}|^2 + |u_{22}|^2$ and $\langle u_1, u_2 \rangle = 0$.

We denote by $SU_2(\mathbb{C})$ the normal subgroup of $U_2(\mathbb{C})$ with determinant 1.

Lemma 2.16. [5, pp. 7] *There is a homeomorphism between $SU_2(\mathbb{C})$ and S^3 the 3-dimensional sphere.*

Proof. It follows from the norm 1, the inner product 0, and determinant 1 that if $u \in SU_2(\mathbb{C})$, then

$$u = \begin{pmatrix} u_{11} & -\overline{u_{21}} \\ u_{21} & \overline{u_{11}} \end{pmatrix}, \quad \|u_1\|^2 = 1.$$

Indeed, check that

$$\begin{aligned}
&\langle u_1, u_2 \rangle u_{12} + (\det u)\overline{u_{22}} = 0\, u_{12} + 1\, \overline{u_{22}} = \overline{u_{22}} \\
=&(u_{11}\overline{u_{12}} + u_{21}\overline{u_{22}})u_{12} + (u_{11}u_{22} - u_{12}u_{21})\overline{u_{22}} \\
=&u_{11}(|u_{12}|^2 + |u_{22}|^2) = u_{11}, \\
&\langle u_1, u_2 \rangle u_{22} - (\det u)\overline{u_{12}} = 0\, u_{22} - 1\, \overline{u_{12}} = -\overline{u_{12}} \\
=&(u_{11}\overline{u_{12}} + u_{21}\overline{u_{22}})u_{22} - (u_{11}u_{22} - u_{12}u_{21})\overline{u_{12}} \\
=&u_{21}(|u_{22}|^2 + |u_{12}|^2) = u_{21}.
\end{aligned}$$

If $u_{i1} = s_{i1} + it_{i1}$ with $s_{i1}, t_{i1} \in \mathbb{R}$, then

$$|u_{11}|^2 + |u_{21}|^2 = s_{11}^2 + t_{11}^2 + s_{21}^2 + t_{21}^2 = 1,$$

so that the vector u_1 is identified with $(s_{11}, t_{11}, s_{21}, t_{21}) \in S^3$. ■

As just a comparison with the above case of projections,

Lemma 2.17. [5, 11] *There is a short exact sequence of groups or spaces as*

$$1 \to SU_2(\mathbb{C}) \approx S^3 \to U_2(\mathbb{C}) \xrightarrow{\det} U_2(\mathbb{C})/SU_2(\mathbb{C}) \cong S^1 \to 1,$$

induced by determinant det.

Proof. If $u \in U_2(\mathbb{C})$, then $u^*u = 1_2$, which implies $\overline{\det u} \det u = 1$. ■

3. The 3 × 3 Matrix Case

Solving the equation for the definition of 3×3 matrix projections implies that

Lemma 3.1. *If $p = (p_{ij})$ is a projection of $M_3(\mathbb{C})$, then*

$$p = \begin{pmatrix} a & \overline{z_1} & \overline{z_2} \\ z_1 & b & \overline{z_3} \\ z_2 & z_3 & c \end{pmatrix} \quad a, b, c \in \mathbb{R}, z_1, z_2, z_3 \in \mathbb{C},$$

where

$$a^2 + |z_1|^2 + |z_2|^2 = a, \quad |z_1|^2 + b^2 + |z_3|^2 = b, \quad |z_2|^2 + |z_3|^2 + c^2 = c,$$

and

$$(a+b)\overline{z_1} + \overline{z_2}z_3 = \overline{z_1}, \quad (a+c)\overline{z_2} + \overline{z_1 z_3} = \overline{z_2}, \quad z_1\overline{z_2} + (b+c)\overline{z_3} = \overline{z_3},$$

so that

$$\begin{aligned} a = a(z_1, z_2) &\equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2 - |z_2|^2} \quad \text{if } 0 \le |z_1|^2 + |z_2|^2 \le \frac{1}{4}, \\ b = b(z_1, z_3) &\equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2 - |z_3|^2} \quad \text{if } 0 \le |z_1|^2 + |z_3|^2 \le \frac{1}{4}, \\ c = c(z_2, z_3) &\equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_2|^2 - |z_3|^2} \quad \text{if } 0 \le |z_2|^2 + |z_3|^2 \le \frac{1}{4}. \end{aligned}$$

Corollary 3.2. *It follows from Lemma* 3.1 *the following*:

(1) *If $z_1 = z_2 = z_3 = 0$, then $a = 0$ or 1, and $b = 0$ or 1, and $c = 0$ or 1.*

(2) *If $z_1 \neq 0$ and $z_2 = z_3 = 0$, or if $z_1 = z_2 = 0$ and $z_3 \neq 0$, then these correspond to the 2×2 case. If $z_1 = z_3 = 0$ and $z_2 \neq 0$, then this also correspond to the 2×2 case similarly.*

(1)′, (2)′ *If $z_1 = 0$ is zero, then z_2 or z_3 is zero. If $z_2 = 0$ is zero, then z_1 or z_3 is zero. If $z_3 = 0$, then z_1 or z_2 is zero. These cases correspond to the cases done above.*

(3) *The rest is the case where z_1, z_2, z_3 are all non-zero.*

Remark 3.3. Further computation and determination in the case (3) above are postponed, and would be given in the future. It is certainly noticed by some experts that the case (3) does happen.

We may say that a projection $p \in M_3(\mathbb{C})$ is **degenerate** (on the off diagonal part) if it is in the case (1), and a non-trivial projection of $M_3(\mathbb{C})$ is a **generalized Bott** projection if in the cases (1) or (2), and is **generically non-degenerate** (on the off diagonal part) if in the case (3). We denote by $P_B(M_3(\mathbb{C}))$ the space of all generalized Bott projections of $M_3(\mathbb{C})$. Set

$$P_B(M_3(\mathbb{C}))^\sim = P_B(M_3(\mathbb{C})) \cup \{0_3, 1_3\}.$$

Define $P_B(M_3(\mathbb{R}))$ and $P_B(M_3(\mathbb{R}))^\sim$ similarly. In what follows, we consider only the cases (1) or (2).

With suitable notations according to the computation above, we formulate the following theorem:

Theorem 3.4. *If $p = (p_{ij})$ is a generalized Bott projection of $M_3(\mathbb{C})$, then p is either* (1) $p_1 = 1 \oplus 0_2$, $p_2 = 0 \oplus 1 \oplus 0$, $p_3 = 0_2 \oplus 1$, *or* $p_1 + p_2$, $p_2 + p_3$, $p_1 + p_3$, *or* (2)

$$\begin{aligned}
&p_\pm(z_1) \oplus \{0,1\}, \text{ for } z_1 \in \mathbb{C} \setminus \{0\} \text{ with } 0 < |z_1| \le \frac{1}{2},\\
&\{0,1\} \oplus p_\pm(z_3), \text{ for } 0 < |z_3| \le \frac{1}{2}, \text{ or}
\end{aligned}$$

$$p_\pm(z_2)^\sim \oplus^\sim \{0,1\} \equiv \begin{pmatrix} \frac{1\pm\sqrt{1-4|z_2|^2}}{2} & 0 & \overline{z_2} \\ 0 & \{0,1\} & 0 \\ z_2 & 0 & \frac{1\mp\sqrt{1-4|z_2|^2}}{2} \end{pmatrix} \quad \text{(split)}$$

for any $z_2 \in \mathbb{C} \setminus \{0\}$, with $0 < |z_2| \le \frac{1}{2}$, where each $p_\pm(z_j)$ for $1 \le j \le 3$ are defined as in Theorem 2.1.

We may define as well

$$\begin{aligned}
&p_+(0) \oplus 0 \equiv \lim_{z_1 \to 0} p_+(z_1) \oplus 0 = p_1 = p_+(0)^\sim \oplus^\sim 0 \equiv \lim_{z_2 \to 0} p_+(z_2)^\sim \oplus^\sim 0,\\
&p_-(0) \oplus 0 \equiv \lim_{z_1 \to 0} p_-(z_1) \oplus 0 = p_2 = 0 \oplus p_+(0) \equiv 0 \oplus \lim_{z_3 \to 0} p_+(z_3),\\
&0 \oplus p_-(0) \equiv 0 \oplus \lim_{z_3 \to 0} p_-(z_3) = p_3 = p_-(0)^\sim \oplus^\sim 0 \equiv \lim_{z_2 \to 0} p_-(z_2)^\sim \oplus^\sim 0,
\end{aligned}$$

and

$$\begin{aligned}
&p_+(0) \oplus 1 \equiv \lim_{z_1 \to 0} p_+(z_1) \oplus 1 = p_1 + p_3 = 1 \oplus p_-(0) \equiv 1 \oplus \lim_{z_3 \to 0} p_-(z_3),\\
&p_+(0)^\sim \oplus^\sim 1 \equiv \lim_{z_2 \to 0} p_+(z_2)^\sim \oplus^\sim 1 = p_1 + p_2 = 1 \oplus p_+(0) \equiv 1 \oplus \lim_{z_3 \to 0} p_+(z_3),\\
&p_-(0) \oplus 1 \equiv \lim_{z_1 \to 0} p_-(z_1) \oplus 1 = p_2 + p_3 = p_-(0)^\sim \oplus^\sim 1 \equiv \lim_{z_2 \to 0} p_-(z_2)^\sim \oplus^\sim 1.
\end{aligned}$$

Corollary 3.5. *The space $P_B(M_3(\mathbb{R}))$ consists of p_1, p_2, p_3, $p_1 + p_2$, $p_2 + p_3$, $p_1 + p_3$, and $p_\pm(t_1) \oplus \{0,1\}$ for $t_1 \in \mathbb{R}$ with $0 < |t_1| \le \frac{1}{2}$, $\{0,1\} \oplus p_\pm(t_3)$ for*

$t_3 \in \mathbb{R}$ *with* $0 < |t_3| \leq \frac{1}{2}$, *and* $p_\pm(t_2)^\sim \oplus^\sim \{0,1\}$ *for* $t_2 \in \mathbb{R}$ *with* $0 < |t_2| \leq \frac{1}{2}$, *where*

$$\begin{aligned}
p_1 &= \lim_{t_1\to 0} p_+(t_1) \oplus 0 \equiv p_+(0) \oplus 0 = \lim_{t_2\to 0} p_+(t_2)^\sim \oplus^\sim 0 \equiv p_+(0)^\sim \oplus^\sim 0, \\
p_2 &= \lim_{t_1\to 0} p_-(t_1) \oplus 0 \equiv p_-(0) \oplus 0 = 0 \oplus \lim_{t_3\to 0} p_+(t_3) \equiv 0 \oplus p_+(0), \\
p_3 &= \lim_{t_2\to 0} p_-(t_2)^\sim \oplus^\sim 0 \equiv p_-(0)^\sim \oplus^\sim 0 = 0 \oplus \lim_{t_3\to 0} p_-(t_3) \equiv 0 \oplus p_-(0),
\end{aligned}$$

and

$$\begin{aligned}
p_1 + p_3 &= \lim_{t_1\to 0} p_+(t_1) \oplus 1 \equiv p_+(0) \oplus 1 = 1 \oplus \lim_{t_3\to 0} p_-(t_3) \equiv 1 \oplus p_-(0), \\
p_1 + p_2 &= \lim_{t_2\to 0} p_+(t_2)^\sim \oplus^\sim 1 \equiv p_+(0)^\sim \oplus^\sim 1 = 1 \oplus \lim_{t_3\to 0} p_+(t_3) \equiv 1 \oplus p_+(0), \\
p_2 + p_3 &= \lim_{t_1\to 0} p_-(t_1) \oplus 1 \equiv p_-(0) \oplus 1 = \lim_{t_2\to 0} p_-(t_2)^\sim \oplus^\sim 1 \equiv p_-(0)^\sim \oplus^\sim 1.
\end{aligned}$$

Now let X, Y, and Z be topological spaces, K a space viewed as a subspace of X and Y, L a space viewed as a subspace of Y and Z, and M a space viewed as a subspace of Z and X. Then we denote by $X \sqcup_K Y \sqcup_L Z \sqcup_M \circlearrowleft$ a **cyclic** K, L, M**-jointed sum** of X, Y, and Z (we call so), which is defined to be the space obtained from attaching X and Y on K and attaching Y and Z on L and attaching Z and X on M.

Theorem 3.6. *There is a homeomorphism between the subspace* $P_B(M_3(\mathbb{C}))^\sim$ *of* $M_3(\mathbb{C})$ *and the following disjoint union*:

$$\begin{aligned}
&(\textit{rank } 0,3) \quad \{0_3\} \sqcup \{1_3\} \sqcup \\
&(\textit{rank } 1) \quad [(B(0,\tfrac{1}{2}) \sqcup_{\frac{1}{2}S^1} B(0,\tfrac{1}{2})) \sqcup_{p_2} ((B(0,\tfrac{1}{2}) \sqcup_{\frac{1}{2}S^1} B(0,\tfrac{1}{2})) \\
&\qquad\qquad \sqcup_{p_3} ((B(0,\tfrac{1}{2}) \sqcup_{\frac{1}{2}S^1} B(0,\tfrac{1}{2})) \sqcup_{p_1} \circlearrowleft] \sqcup \\
&(\textit{rank } 2) \quad [((B(0,\tfrac{1}{2}) \sqcup_{\frac{1}{2}S^1} B(0,\tfrac{1}{2})) \sqcup_{p_2+p_3} (B(0,\tfrac{1}{2}) \sqcup_{\frac{1}{2}S^1} B(0,\tfrac{1}{2})) \\
&\qquad\qquad \sqcup_{p_1+p_2} (B(0,\tfrac{1}{2}) \sqcup_{\frac{1}{2}S^1} B(0,\tfrac{1}{2})) \sqcup_{p_1+p_3} \circlearrowleft],
\end{aligned}$$

with complex variables as

$$\begin{aligned}
&[(\{z_1\} \sqcup_{\frac{1}{2}S^1} \{w_1\}) \sqcup_{w_1=0=z_3} (\{z_3\} \sqcup_{\frac{1}{2}S^1} \{w_3\}) \\
&\quad \sqcup_{w_3=0=w_2} (\{z_2\} \sqcup_{\frac{1}{2}S^1} \{w_2\}) \sqcup_{z_2=0=z_1} \circlearrowleft],
\end{aligned}$$

and

$$\begin{aligned}
&[(\{z_1\} \sqcup_{\frac{1}{2}S^1} \{w_1\}) \sqcup_{w_1=0=w_2} (\{z_2\} \sqcup_{\frac{1}{2}S^1} \{w_2\}) \\
&\quad \sqcup_{z_2=0=z_3} (\{z_3\} \sqcup_{\frac{1}{2}S^1} \{w_3\}) \sqcup_{w_3=0=z_1} \circlearrowleft],
\end{aligned}$$

respectively, where the first cyclic points-jointed sum $[\cdots \circlearrowleft]$ *is obtained from attaching cyclically* 3 *copies of* $B(0,2^{-1}) \sqcup_{2^{-1}S^1} B(0,2^{-1})$ *at the zero point of* $B(0,2^{-1})$ *as its right component with the zero point of* $B(0,2^{-1})$ *as its left component, and the space corresponds to the space of all rank* 1 *generalized Bott projections of* $M_3(\mathbb{C})$*, and the common zero points correspond to the projections* p_1, p_2, p_3 *respectively, and the variable* z_j *for* $j = 1, 3$ *corresponds to* $p_+(z_j)$*, and* w_j *for* $j = 1, 3$ *corresponds to* $p_-(w_j)$*, and* z_2 *corresponds to* $p_+(z_2)^\sim$*, and* w_2 *corresponds to* $p_-(w_2)^\sim$*, and the second cyclic points-jointed sum* $[\cdots \circlearrowleft]$ *is obtained similarly, with slightly different identifications of variables as above, and it corresponds to the space of all rank* 2 *generalized Bott projections of* $M_3(\mathbb{C})$*, and the common zero points correspond to the projections* p_2+p_3*,* p_1+p_2*,* p_1+p_3 *respectively.*

Corollary 3.7. *There is a homeomorphism between the subspace* $P_B(M_3(\mathbb{R}))^\sim$ *of* $M_3(\mathbb{R})$ *and the following disjoint union*:

$$
\begin{aligned}
&(\text{rank } 0,3) \quad \{0_3\} \sqcup \{1_3\} \sqcup \\
&(\text{rank } 1) \quad \{([-\tfrac{1}{2}, \tfrac{1}{2}] \sqcup_{\frac{1}{2}S^0} [-\tfrac{1}{2}, \tfrac{1}{2}]) \sqcup_{p_2} ([-\tfrac{1}{2}, \tfrac{1}{2}] \sqcup_{\frac{1}{2}S^0} [-\tfrac{1}{2}, \tfrac{1}{2}]) \sqcup_{p_3} \\
&\qquad\qquad ([-\tfrac{1}{2}, \tfrac{1}{2}] \quad \sqcup_{\frac{1}{2}S^0} [-\tfrac{1}{2}, \tfrac{1}{2}]) \sqcup_{p_1} \circlearrowleft\} \sqcup \\
&(\text{rank } 2) \quad \{([-\tfrac{1}{2}, \tfrac{1}{2}] \sqcup_{\frac{1}{2}S^0} [-\tfrac{1}{2}, \tfrac{1}{2}]) \sqcup_{p_2+p_3} ([-\tfrac{1}{2}, \tfrac{1}{2}] \sqcup_{\frac{1}{2}S^0} [-\tfrac{1}{2}, \tfrac{1}{2}]) \\
&\qquad\qquad \sqcup_{p_1+p_2} ([-\tfrac{1}{2}, \tfrac{1}{2}] \sqcup_{\frac{1}{2}S^0} [-\tfrac{1}{2}, \tfrac{1}{2}]) \sqcup_{p_1+p_3} \circlearrowleft\},
\end{aligned}
$$

with real variables as the same as given above.

Theorem 3.8. *There is a homeomorphism between* $P_B(M_3(\mathbb{C}))$ *and the disjoint union*

$$[S^2 \sqcup_{p_2} S^2 \sqcup_{p_3} S^2 \sqcup_{p_1} \circlearrowleft] \sqcup [S^2 \sqcup_{p_2+p_3} S^2 \sqcup_{p_1+p_2} S^2 \sqcup_{p_1+p_3} \circlearrowleft],$$

which is homeomorphic to the disjoint union of the cyclic points-jointed sums of 3 *copies of* S^2 *at the north and south poles* $n \in S^2$ *(one) and* $s \in S^2$ *(the next) identified respectively as*

$$[S^2 \sqcup_{n=s} S^2 \sqcup_{n=s} S^2 \sqcup_{n=s} \circlearrowleft] \sqcup [S^2 \sqcup_{n=s} S^2 \sqcup_{n=s} S^2 \sqcup_{n=s} \circlearrowleft],$$

both path-connected components of which are homeomorphic.

Corollary 3.9. *There is a homeomorphism between* $P_B(M_3(\mathbb{R}))$ *and the disjoint union*

$$[S^1 \sqcup_{p_2} S^1 \sqcup_{p_3} S^1 \sqcup_{p_1} \circlearrowleft] \sqcup [S^1 \sqcup_{p_2+p_3} S^1 \sqcup_{p_1+p_2} S^1 \sqcup_{p_1+p_3} \circlearrowleft],$$

which is homeomorphic to the disjoint union of two cyclic points-jointed sums of 3 *copies of* S^1 *at* $+1 \in S^1$ *(one) and* $-1 \in S^1$ *(the next) identified respectively as*

$$[S^1 \sqcup_{+1=-1} S^1 \sqcup_{+1=-1} S^1 \sqcup_{+1=-1} \circlearrowleft] \sqcup [S^1 \sqcup_{+1=-1} S^1 \sqcup_{+1=-1} S^2 \sqcup_{+1=-1} \circlearrowleft],$$

both path-connected components of which are homeomorphic.

Corollary 3.10. *The homotopy classes of* $P_B(M_3(\mathbb{C}))^{\sim}$ *are given by*

$$P_B(M_3(\mathbb{C}))^{\sim}/\sim = \{[0_3], [1_3], [p_1] = [p_2] = [p_3], [p_1 + p_2] = [p_2 + p_3] = [p_1 + p_3]\}.$$

The same also holds for $P_B(M_3(\mathbb{R}))^{\sim}$.

Proof. The homotopies among p_1, p_2, and p_3 and among p_1+p_2, p_2+p_3, and p_1+p_3 within $P_B(M_3(\mathbb{C}))$ and $P_B(M_3(\mathbb{R}))$ are constructed explicitly and respectively as in Theorems 2.1 and 3.3 above. ■

Corollary 3.11. *There are bijections among the homotopy set* $P_B(M_3(\mathbb{C}))^{\sim}/\sim$, *the trace image* $\mathrm{tr}(P_B(M_3(\mathbb{C}))^{\sim}) = \{0, 1, 2, 3\}$, *and the rank image* $\mathrm{rk}(P_B(M_3(\mathbb{C}))^{\sim})$.

The same also holds for $P_B(M_3(\mathbb{R}))^{\sim}$.

It is then deduced that

Corollary 3.12. *There is a continuous path from a generically non-degenerate projection of* $M_3(\mathbb{C})$ *to some generalized Bott projection of* $M_3(\mathbb{C})$ *within* $P(M_3(\mathbb{C}))$.

Namely, there is a continuous deformation from $P(M_3(\mathbb{C}))$ *to* $P_B(M_3(\mathbb{C}))$.

The same also holds for $P(M_3(\mathbb{R}))$.

Proof. Corollaries 3.9 and 3.10 hold for $P(M_3(\mathbb{C}))^{\sim}$ and $P(M_3(\mathbb{R}))^{\sim}$ as well, by using Linear Algebra. Indeed, such a deformation may be obtained by letting the variables z_j $(1 \le j \le 3)$ going to 0 separately, within $P(M_3(\mathbb{C}))^{\sim}$. ■

4. The 4 × 4 Matrix Case

Solving the equation for 4×4 matrix projections we obtain

Lemma 4.1. *If* $p = (p_{ij})$ *is a projection of* $M_4(\mathbb{C})$, *then*

$$p = \begin{pmatrix} a_{11} & \overline{z_{21}} & \overline{z_{31}} & \overline{z_{41}} \\ z_{21} & a_{22} & \overline{z_{32}} & \overline{z_{42}} \\ z_{31} & z_{32} & a_{33} & \overline{z_{43}} \\ z_{41} & z_{42} & z_{43} & a_{44} \end{pmatrix}$$

with $a_{jj} \in \mathbb{R}$ *for* $1 \leq j \leq 4$, $z_{ji} \in \mathbb{C}$ *for* $1 \leq i < j \leq 4$, *where*

$$\begin{aligned}
a_{11}^2 + |z_{21}|^2 + |z_{31}|^2 + |z_{41}|^2 &= a_{11}, \\
|z_{21}|^2 + a_{22}^2 + |z_{32}|^2 + |z_{42}|^2 &= a_{22}, \\
|z_{31}|^2 + |z_{32}|^2 + a_{33}^2 + |z_{43}|^2 &= a_{33}, \\
|z_{41}|^2 + |z_{42}|^2 + |z_{43}|^2 + a_{44} &= a_{44},
\end{aligned}$$

and

$$\begin{aligned}
&(a_{11} + a_{22})\overline{z_{21}} + \overline{z_{31}}z_{32} + \overline{z_{41}}z_{42} = \overline{z_{21}}, (a_{11} + a_{33})\overline{z_{31}} + \overline{z_{21}z_{32}} + \overline{z_{41}}z_{43} = \overline{z_{31}}, \\
&(a_{11} + a_{44})\overline{z_{41}} + \overline{z_{21}z_{42}} + \overline{z_{31}z_{43}} = \overline{z_{41}}, (a_{22} + a_{33})\overline{z_{32}} + z_{21}\overline{z_{31}} + \overline{z_{42}}z_{43} = \overline{z_{32}}, \\
&(a_{22} + a_{44})\overline{z_{42}} + z_{21}\overline{z_{41}} + \overline{z_{32}z_{43}} = \overline{z_{42}}, (a_{33} + a_{44})\overline{z_{43}} + z_{31}\overline{z_{41}} + z_{32}\overline{z_{42}} = \overline{z_{43}},
\end{aligned}$$

so that

$$a_{11} = a_{11}(z_{21}, z_{31}, z_{41}) \equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_{21}|^2 - |z_{31}|^2 - |z_{41}|^2}$$

if $0 \leq |z_{21}|^2 + |z_{31}|^2 + |z_{41}|^2 \leq \frac{1}{4}$,

$$a_{22} = a_{22}(z_{21}, z_{32}, z_{42}) \equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_{21}|^2 - |z_{32}|^2 - |z_{42}|^2}$$

if $0 \leq |z_{21}|^2 + |z_{32}|^2 + |z_{42}|^2 \leq \frac{1}{4}$,

$$a_{33} = a_{33}(z_{31}, z_{32}, z_{43}) \equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_{31}|^2 - |z_{32}|^2 - |z_{43}|^2}$$

if $0 \leq |z_{31}|^2 + |z_{32}|^2 + |z_{43}|^2 \leq \frac{1}{4}$,

$$a_{44} = a_{44}(z_{41}, z_{42}, z_{43}) \equiv \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_{41}|^2 - |z_{42}|^2 - |z_{43}|^2}$$

if $0 \leq |z_{41}|^2 + |z_{42}|^2 + |z_{43}|^2 \leq \frac{1}{4}$.

Corollary 4.2. *It follows from Lemma* 4.1 *the following*:

(1) *If all* $z_{ji} = 0$ *for* $j > i$, *then* $a_{jj} = 0$ *or* 1 *for* $1 \leq j \leq 4$.
(2) *If only one* $z_{ji} \neq 0$ *with* $j > i$ *and the other all* z_{kl} *zero for* $k > l$, *then these correspond to the* 2×2 *case or the split* 2×2 *case.*
(3) *There are also the cases by choosing two split or not, blocks of the split or not,* 2×2 *cases.*
(4) *There is the rest of the other cases.*

Remark 4.3. Further computation and determination in the case (4) above are postponed, and would be given in the future. Note that the case (4) does happen, as in the 3×3 case.

We may say that a projection $p \in M_4(\mathbb{C})$ is **degenerate** (on the off diagonal part) if it is in the case (1), and a non-trivial projection of $M_4(\mathbb{C})$ is a (combinatorically) **generalized Bott** projection if in the cases (1), (2) or (3), and otherwise, it is **generically non-degenerate** (on the off diagonal part) if in the case (4). We denote by $P_B(M_4(\mathbb{C}))$ the space of all generalized Bott projections of $M_4(\mathbb{C})$. Set

$$P_B(M_4(\mathbb{C}))^{\sim} = P_B(M_4(\mathbb{C})) \cup \{0_4, 1_4\}.$$

Define $P_B(M_4(\mathbb{R}))$ and $P_B(M_4(\mathbb{R}))^{\sim}$ similarly.

With suitable notations,

Theorem 4.4. *If $p = (p_{ij})$ is a generalized Bott projection of $M_4(\mathbb{C})$, then p is either* (1) $p_1 = 1 \oplus 0_3$, $p_2 = 0 \oplus 1 \oplus 0_2$, $p_3 = 0_2 \oplus 1 \oplus 0$, $p_4 = 0_3 \oplus 1$, *or* $p_i + p_j$ *for* $1 \le i < j \le 4$ ($_4C_2 = 6$ *many*), *or* $p_i + p_j + p_k$ *for* $1 \le i < j < k \le 4$ ($_4C_3 = 4$ *many*), *or* (2)

$$\begin{aligned}
&p_\pm(z_{21}) \oplus \{0,1\} \oplus \{0,1\}, \text{ for } z_{21} \in \mathbb{C} \setminus \{0\} \text{ with } 0 < |z_{21}| \le \frac{1}{2}, \text{ or} \\
&\{0,1\} \oplus p_\pm(z_{32}) \oplus \{0,1\}, \text{ for } 0 < |z_{32}| \le \frac{1}{2}, \text{ or} \\
&\{0,1\} \oplus \{0,1\} \oplus p_\pm(z_{43}), \text{ for } 0 < |z_{43}| \le \frac{1}{2}, \text{ or}
\end{aligned}$$

$$p_\pm(z_{31})^{\sim} \oplus^{\sim} (\oplus^2\{0,1\}) \equiv \begin{pmatrix} \frac{1\pm\sqrt{1-4|z_{31}|^2}}{2} & 0 & \overline{z_{31}} \\ 0 & \{0,1\} & 0 \\ z_{31} & 0 & \frac{1\mp\sqrt{1-4|z_{31}|^2}}{2} \end{pmatrix} \oplus \{0,1\} \quad \text{(split)}$$

for any $z_{31} \in \mathbb{C} \setminus \{0\}$, *with* $0 < |z_{31}| \le \frac{1}{2}$, *or*

$$p_\pm(z_{42})^{\sim} \oplus^{\sim} (\oplus^2\{0,1\}) \equiv \{0,1\} \oplus \begin{pmatrix} \frac{1\pm\sqrt{1-4|z_{42}|^2}}{2} & 0 & \overline{z_{42}} \\ 0 & \{0,1\} & 0 \\ z_{42} & 0 & \frac{1\mp\sqrt{1-4|z_{42}|^2}}{2} \end{pmatrix} \quad \text{(split)}$$

for any $z_{42} \in \mathbb{C} \setminus \{0\}$, *with* $0 < |z_{42}| \le \frac{1}{2}$, *or*

$$p_\pm(z_{41})^{\sim} \oplus^{\sim} (\oplus^2\{0,1\}) \equiv \begin{pmatrix} \frac{1\pm\sqrt{1-4|z_{41}|^2}}{2} & (0,0) & \overline{z_{41}} \\ (0,0)^t & \{0,1\} \oplus \{0,1\} & (0,0)^t \\ z_{41} & (0,0) & \frac{1\mp\sqrt{1-4|z_{41}|^2}}{2} \end{pmatrix} \quad \text{(split)}$$

for any $z_{41} \in \mathbb{C} \setminus \{0\}$, *with* $0 < |z_{41}| \leq \frac{1}{2}$, *or* (3)

$$\begin{aligned} &p_{\pm}(z_{21}) \oplus p_{\pm}(z_{43}) \quad \textit{for } 0 < |z_{21}| \leq \frac{1}{2} \textit{ and } 0 < |z_{43}| \leq \frac{1}{2}, \textit{ or} \\ &p_{\pm}(z_{41})^{\sim} \oplus^{\sim} p_{\pm}(z_{32}) \quad \textit{for } 0 < |z_{41}| \leq \frac{1}{2} \textit{ and } 0 < |z_{32}| \leq \frac{1}{2}, \textit{ or} \\ &p_{\pm}(z_{31})^{\sim} \oplus^{\sim} p_{\pm}(z_{42})^{\sim} \quad \textit{for } 0 < |z_{31}| \leq \frac{1}{2} \textit{ and } 0 < |z_{42}| \leq \frac{1}{2}, \end{aligned}$$

and in total there are

$$(\sum_{k=0}^{4} {}_4C_k) + 2^2 {}_4C_2 + \frac{1}{2} {}_4C_2 = 2^4 + 24 + 3 = 43 \quad \textit{cases},$$

where each $p_{\pm}(z_{j,i})$ *for* $1 \leq i < j \leq 4$ *are defined as in Theorem* 2.1.

We may define as well the respective values at respective zeros with respect to non-zero parameters such as $z_{j,i}$, *as the respective limits as the standard diagonal projections such as* p_j, $p_i + p_j$, *and* $p_i + p_j + p_k$ *with* $i < j < k$, *as in Theorem* 3.3, *but omitted.*

Theorem 4.5. *There is a homeomorphism between the subspace* $P_B(M_4(\mathbb{C}))^{\sim}$ *of* $M_4(\mathbb{C})$ *and the following disjoint union*:

$$\begin{aligned} &(\textit{rank } 0,4) \quad \{0_4\} \sqcup \{1_4\} \sqcup \\ &(\textit{rank } 1) \quad [\sqcup_{p_1,\cdots,p_4}(\sqcup_{i<j}^{6}(S^2, p_i, p_j))] \sqcup \\ &(\textit{rank } 2) \quad \sqcup_{p_i+p_j, i<j} [(\sqcup_{i<j}^{6}(S^2, p_i, p_j) + \{p_k, p_l\}) \sqcup \\ &\qquad \{(S^2, p_1, p_2) \oplus (S^2, p_3, p_4)\} \sqcup \{(S^2, p_1, p_4) \oplus (S^2, p_2, p_3)\} \\ &\qquad \sqcup \{(S^2, p_1, p_3) \oplus (S^2, p_2, p_4)\}] \sqcup \\ &(\textit{rank } 3) \quad \sqcup_{p_i+p_j+p_k, i<j<k} [(\sqcup_{i<j}^{6}(S^2, p_i, p_j) + \{p_k + p_l\}], \end{aligned}$$

where $\sqcup_{p_1,\cdots,p_4}(\sqcup_{i<j}^{6}(S^2, p_i, p_j))$ *means the* $\{p_1, \cdots, p_4\}$*-jointed sum of* 6 *copies of* S^2, *each of which is pointed with* p_i *and* p_j *for some and any* $1 \leq i < j \leq 4$ *as that* p_i *and* p_j *identified with the north and south poles of* S^2 *respectively, such that*

$$\begin{bmatrix} p_1 & & & \\ (S^2, p_1, p_2) & p_2 & & \\ (S^2, p_1, p_3) & (S^2, p_2, p_3) & p_3 & \\ (S^2, p_1, p_4) & (S^2, p_2, p_4) & (S^2, p_3, p_4) & p_4 \end{bmatrix}$$

(*a matrix as a picture*), *where each* p_j *in the picture is identified with the same other* p_j, *and the* $\{p_1, \cdots, p_4\}$*-jointed sum corresponds to the space of all rank* 1 *generalized Bott projections.*

And $\sqcup_{p_i+p_j, i<j}(\sqcup_{i<j}^{6}(S^2, p_i, p_j) + \{p_k, p_l\})$ (*shorten*) *means the* $\{p_i + p_j \mid 1 \leq i < j \leq 4\}$*-jointed sum of* 2×6 *copies of* S^2, *each of which is pointed with* p_i

and p_j for some and any $1 \leq i < j \leq 4$ as well as $p_i + p_k$ and $p_j + p_k$ with some p_k (or p_l) for $k \neq i$ and $k \neq j$ such that

$$\begin{bmatrix} & p_1+p_2 & p_1+p_3 & p_1+p_4 \\ (S^2,p_1,p_2)+\{p_3,p_4\} & & p_2+p_3 & p_2+p_4 \\ (S^2,p_1,p_3)+\{p_2,p_4\} & (S^2,p_2,p_3)+\{p_1,p_4\} & & p_3+p_4 \\ (S^2,p_1,p_4)+\{p_2,p_3\} & (S^2,p_2,p_4)+\{p_1,p_3\} & (S^2,p_3,p_4)+\{p_1,p_2\} & \end{bmatrix}$$

where we define

$$(S^2,p_i,p_j)+\{p_k,p_l\} \equiv \{(S^2,p_i+p_k,p_j+p_k),(S^2,p_i+p_l,p_j+p_l)\},$$

and each $p_i + p_j$ in the picture is identified with the same other $p_i + p_j$

In addition to the jointed sum above, each $(S^2, p_i, p_j) \oplus (S^2, p_k, p_l)$ means the disjoint union of two copies of S^2, each of which is pointed with distinct $\{p_i, p_j\}$ or $\{p_k, p_l\}$ as well as $p_i + p_k$, $p_i + p_l$, $p_j + p_k$, and $p_j + p_l$, and in that and this cases, each $p_i + p_j$ is identified with the same other $p_i + p_j$, and the $\{p_i + p_j \,|\, i < j\}$-jointed sum in total of this and that cases corresponds to the space of all rank 2 generalized Bott projections.

And $\sqcup_{p_i+p_j+p_k, i<j<k}[(\sqcup_{i<j}^6 (S^2, p_i, p_j) + \{p_k + p_l\}]$ means the $\{p_i + p_j + p_k \,|\, 1 \leq i < j < k \leq 4\}$-jointed sum of 6 copies of S^2, each of which is pointed with p_i and p_j for some and any $1 \leq i < j \leq 4$ as well as $p_i + p_k + p_l$ and $p_j + p_k + p_l$ with $\{i, j\}$ and $\{k, l\}$ distinct in $\{1, \cdots, 4\}$ such that

$$\begin{bmatrix} p_2+p_3+p_4 & & & \\ (S^2,p_1,p_2)+p_3+p_4 & p_1+p_3+p_4 & & \\ (S^2,p_1,p_3)+p_2+p_4 & (S^2,p_2,p_3)+p_1+p_4 & p_1+p_2+p_4 & \\ (S^2,p_1,p_4)+p_2+p_3 & (S^2,p_2,p_4)+p_1+p_3 & (S^2,p_3,p_4)+p_1+p_2 & \sum_{s=1}^3 p_s \end{bmatrix}$$

where we define

$$(S^2,p_i,p_j)+p_k+p_l \equiv (S^2,p_i+p_k+p_l,p_j+p_k+p_l),$$

and each $p_i + p_j + p_k$ in the picture is identified with the same other $p_i + p_j + p_k$, and the $\{p_i + p_j + p_k \,|\, i < j < k\}$-jointed sum corresponds to the space of all rank 3 generalized Bott projections.

Corollary 4.6. *There is a homeomorphism between the subspace $P_B(M_4(\mathbb{R}))^\sim$ of $M_4(\mathbb{R})$ and the following disjoint union:*

$$\begin{aligned} &(\textit{rank } 0,4) && \{0_4\} \sqcup \{1_4\} \sqcup \\ &(\textit{rank } 1) && [\sqcup_{p_1,\cdots,p_4}(\sqcup_{i<j}^6 (S^1,p_i,p_j))] \sqcup \\ &(\textit{rank } 2) && \sqcup_{p_i+p_j,i<j}\, [(\sqcup_{i<j}^6 (S^1,p_i,p_j)+\{p_k,p_l\}) \sqcup \\ & && \quad \{(S^1,p_1,p_2)\oplus(S^1,p_3,p_4)\} \sqcup \{(S^1,p_1,p_4)\oplus(S^1,p_2,p_3)\} \\ & && \quad \sqcup \{(S^1,p_1,p_3)\oplus(S^1,p_2,p_4)\}] \sqcup \\ &(\textit{rank } 3) && \sqcup_{p_i+p_j+p_k,i<j<k}\, [(\sqcup_{i<j}^6 (S^1,p_i,p_j)+\{p_k+p_l\}], \end{aligned}$$

where each (S^1, p_i, p_j) *means the* S^1 *pointed with* p_i *and* p_j *at the points* $\pm 1 \in S^1$ *respectively, and the other pointed* $(S^1, p_i, p_j)+\{p_k, p_l\}$, $(S^1, p_i, p_j)\oplus(S^1, p_k, p_l)$, *and* $(S^1, p_i, p_j)+\{p_k+p_l)$ *as well as their points-jointed sums are defined similarly as in the theorem above.*

Corollary 4.7. *The homotopy classes of* $P_B(M_4(\mathbb{C}))^{\sim}$ *are given by*

$$\begin{aligned} P_B(M_4(\mathbb{C}))^{\sim}/\sim = \{&[0_4], [1_4], [p_1] = [p_2] = [p_3] = [p_4], \\ &[p_1 + p_2] = [p_i + p_j] \quad (1 \le i < j \le 4), \\ &[p_1 + p_2 + p_3] = [p_i + p_j + p_k] \quad (1 \le i < j < k \le 4)\}. \end{aligned}$$

The same also holds for $P_B(M_4(\mathbb{R}))^{\sim}$.

Proof. The homotopies among p_1, p_2, p_3, and p_4 and among $p_i + p_j$ for $1 \le i < j \le 4$ and among $p_i + p_j + p_k$ for $1 \le i < j < k \le 4$ within $P_B(M_4(\mathbb{C}))$ and $P_B(M_4(\mathbb{R}))$ are constructed explicitly and respectively as in Theorems 2.1, 3.3, and 4.3 above. ■

Corollary 4.8. *There are bijections among the homotopy set* $P_B(M_4(\mathbb{C}))^{\sim}/\sim$, *the trace image* $\mathrm{tr}(P_B(M_4(\mathbb{C}))^{\sim}) = \{0, 1, 2, 3, 4\}$, *and the rank image* $\mathrm{rk}(P_B(M_4(\mathbb{C}))^{\sim})$.

The same also holds for $P_B(M_4(\mathbb{R}))^{\sim}$.

It is then deduced that

Corollary 4.9. *There is a continuous path from a generically non-degenerate projection of* $M_4(\mathbb{C})$ *to some generalized Bott projection of* $M_4(\mathbb{C})$ *within* $P(M_4(\mathbb{C}))$.

Namely, there is a continuous deformation from $P(M_4(\mathbb{C}))$ *to* $P_B(M_4(\mathbb{C}))$.

The same also holds for $P(M_4(\mathbb{R}))$.

Proof. Corollaries 4.6 and 4.7 hold for $P(M_4(\mathbb{C}))^{\sim}$ and $P(M_4(\mathbb{R}))^{\sim}$ as well, by using Linear Algebra. ■

5. The General Matrix Case

Solving the equation for $n \times n$ matrix projections implies that

Lemma 5.1. *If* $p = (p_{ij})$ *is a projection of* $M_n(\mathbb{C})$, *then*

$$p = \begin{pmatrix} a_{11} & \overline{z_{21}} & \cdots & \overline{z_{n1}} \\ z_{21} & a_{22} & \cdots & \overline{z_{n2}} \\ \vdots & \ddots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & a_{nn} \end{pmatrix}$$

with $a_{jj} \in \mathbb{R}(1 \le j \le n)$, $z_{ji} \in \mathbb{C}(1 \le i < j \le n)$, *where*

$$a_{11}^2 + \sum_{k=2}^{n} |z_{k1}|^2 = a_{11}, \quad |z_{21}|^2 + a_{22}^2 + \sum_{k=3}^{n} |z_{k2}|^2 = a_{22}, \quad \cdots,$$
$$\sum_{k=1}^{l-1} |z_{lk}|^2 + a_{ll}^2 + \sum_{k=l+1}^{n} |z_{kl}|^2 = a_{ll}, \quad \cdots, \quad \sum_{k=1}^{n-1} |z_{nk}|^2 + a_{nn}^2 = a_{nn},$$

and

$$(a_{11} + a_{22})\overline{z_{21}} + \sum_{k=3}^{n} \overline{z_{k1}} z_{k2} = \overline{z_{21}}, \quad \cdots, \quad (a_{11} + a_{nn})\overline{z_{n1}} + \sum_{k=2}^{n-1} \overline{z_{k1} z_{nk}} = \overline{z_{n1}},$$
$$z_{21}\overline{z_{31}} + (a_{22} + a_{33})\overline{z_{32}} + \sum_{k=4}^{n} \overline{z_{k2}} z_{k3} = \overline{z_{32}}, \quad \cdots,$$
$$\cdots, \quad \sum_{k=1}^{n-2} z_{nk}\overline{z_{n-1,k}} + (a_{n-1,n-1} + a_{nn}) z_{n,n-1} = z_{n,n-1},$$

so that each of the diagonal components a_{jj} *is solved as* (*for instance*),

$$a_{11} = a_{11}(z_{21}, \cdots, z_{n1}) \equiv \begin{cases} \frac{1}{2} \pm \sqrt{\frac{1}{4} - \sum_{k=2}^{n} |z_{k1}|^2} \in \mathbb{R} & \text{if } 0 \le \sum_{k=2}^{n} |z_{k1}|^2 \le \frac{1}{4}, \\ \frac{1}{2} \pm i\sqrt{\sum_{k=2}^{n} |z_{k1}|^2 - \frac{1}{4}} \notin \mathbb{R} & \text{if } \sum_{k=2}^{n} |z_{k1}|^2 > \frac{1}{4}, \end{cases}, \quad \cdots,$$
$$a_{nn} = a_{nn}(z_{n1}, \cdots, z_{n,n-1}) \equiv \begin{cases} \frac{1}{2} \pm \sqrt{\frac{1}{4} - \sum_{k=1}^{n-1} |z_{nk}|^2} \in \mathbb{R} & \text{if } 0 \le \sum_{k=1}^{n-1} |z_{nk}|^2 \le \frac{1}{4}, \\ \frac{1}{2} \pm i\sqrt{\sum_{k=1}^{n-1} |z_{nk}|^2 - \frac{1}{4}} \notin \mathbb{R} & \text{if } \sum_{k=1}^{n-1} |z_{nk}|^2 > \frac{1}{4}. \end{cases}$$

Corollary 5.2. *It follows from Lemma* 5.1 *the following statements hold:*

(1) *If all* $z_{ij} = 0$ *with* $i > j$, *then* $a_{jj} = 0$ *or* 1 *for* $1 \le j \le n$.

(2) *If only one* $z_{ij} \ne 0$ *with* $i > j$ *and the other all* $z_{kl} = 0$ *for* $k > l$, *then these correspond to the* 2×2 *case or the split* 2×2 *case.*

(3) *Moreover, there are certainly the cases where there are distinct block-wise many of the* 2×2 *cases or the split* 2×2 *cases, as shown in the* 3×3 *or* 4×4 *cases.*

(4) *There is the rest of the other cases.*

Remark 5.3. Further computation and determination in the case (4) above are postponed, unfortunately. Note that the case (4) does happen.

We may say that a projection $p \in M_n(\mathbb{C})$ is **degenerate** (on the off diagonal part) if it is in the case (1), and a non-trivial projection of $M_n(\mathbb{C})$ is a (combinatorically) **generalized Bott** projection if in the cases (1), (2) or (3), and otherwise, is **generically non-degenerate** (on the off diagonal part) if in the case (4). We denote by $P_B(M_n(\mathbb{C}))$ the space of all generalized Bott projections of $M_n(\mathbb{C})$. Set

$$P_B(M_n(\mathbb{C}))^\sim = P_B(M_n(\mathbb{C})) \cup \{0_n, 1_n\}.$$

Define $P_B(M_n(\mathbb{R}))$ and $P_B(M_n(\mathbb{R}))^\sim$ similarly.

With suitable notations,

Theorem 5.4. *If $p = (p_{ij})$ is a generalized Bott projection of $M_n(\mathbb{C})$, then p is either (1) the standard, rank k, diagonal projections $p_{i_1} + \cdots + p_{i_k}$ for $1 \leq i_1 < \cdots < i_k \leq n$ (${}_nC_k = \frac{n!}{k!(n-k)!}$ cases) for $1 \leq k \leq n$, or (2)*

$$p_\pm(z_{i_2,i_1}) \oplus^\sim (\oplus^{n-2}\{0,1\}) \text{ for } z_{i_2,i_1} \in \mathbb{C} \setminus \{0\} \text{ with } 0 < |z_{i_2,i_1}| \leq \frac{1}{2},$$

for $1 \leq i_1 < i_2 \leq n$ (${}_nC_2 \times 2^{n-2}$ cases), or (3) for $z_{i_2,i_1}, z_{i_4,i_3} \in \mathbb{C} \setminus \{0\}$ with $0 < |z_{i_2,i_1}| \leq \frac{1}{2}$ and $0 < |z_{i_4,i_3}| \leq \frac{1}{2}$,

$$p_\pm(z_{i_2,i_1}) \oplus^\sim p_\pm(z_{i_4,i_3}) \oplus^\sim (\oplus^{n-4}\{0,1\}),$$

for $1 \leq i_1 < i_2 \leq n$ chosen first and for $1 \leq i_3 < i_4 \leq n$ chosen next with the sets $\{i_1, i_2\}$ and $\{i_3, i_4\}$ distinct (${}_nC_2 \times {}_{n-2}C_2 \times 2^{n-4}$ cases), or in general, for $z_{i_2,i_1}, z_{i_4,i_3}, \cdots, z_{i_{2k},i_{2k-1}} \in \mathbb{C} \setminus \{0\}$ with $0 < |z_{i_2,i_1}| \leq \frac{1}{2}$, $\cdots$, and $0 < |z_{i_{2k},i_{2k-1}}| \leq \frac{1}{2}$,

$$p_\pm(z_{i_2,i_1}) \oplus^\sim p_\pm(z_{i_4,i_3}) \oplus^\sim \cdots \oplus^\sim p_\pm(z_{i_{2k},i_{2k-1}}) \oplus^\sim (\oplus^{n-2k}\{0,1\}),$$

for $1 \leq i_1 < i_2 \leq n$ chosen first and for $1 \leq i_3 < i_4 \leq n$ chosen next with the sets $\{i_1, i_2\}$ and $\{i_3, i_4\}$ distinct, and $\cdots$ for $1 < i_{2k-1} < i_{2k} \leq n$ chosen similarly, inductively, and distinctly (${}_nC_2 \times {}_{n-2}C_2 \times \cdots \times {}_{n-2k+2}C_2 \times 2^{n-2k}$ cases), for $2 \leq 2k \leq n-1$ when n is odd and for $2 \leq 2k \leq n$ when n is odd, where each $p_\pm(z_{j,i})$ for $1 \leq i < j \leq n$ are defined as in Theorem 2.1, and $\oplus^\sim$ here by the same symbol as before means both the usual diagonal sum $\oplus$ as well as the split diagonal sum $\oplus^\sim$ as in Theorem 3.3 and its naturally extended split diagonal sum such that for instance,

$$p_\pm(z_{j,i})^\sim \oplus^\sim (\oplus^{n-2}\{0,1\}) \equiv$$

$$(\oplus^{l_1}\{0,1\}) \oplus \begin{pmatrix} \frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_{j,i}|^2} & 0 \cdots 0 & \overline{z_{j,i}} \\ (0 \cdots 0)^t & \oplus^{l_2}\{0,1\} & (0 \cdots 0)^t \\ z_{j,i} & 0 \cdots 0 & \frac{1}{2} \mp \sqrt{\frac{1}{4} - |z_{j,i}|^2} \end{pmatrix} \oplus (\oplus^{l_3}\{0,1\}),$$

with $l_1 + l_2 + l_3 = n - 2$ for some $l_1 \geq 0$, $l_2 \geq 0$, $l_3 \geq 0$, for any $z_{j,i} \in \mathbb{C} \setminus \{0\}$ with $0 < |z_{j,i}| \leq \frac{1}{2}$. We may define as well the respective values at respective zeros with respect to non-zero parameters such as $z_{j,i}$, as the respective limits as the standard diagonal projections such as $p_{i_1} + \cdots + p_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$, as in Theorem 3.3, but omitted.

Theorem 5.5. *There is a homeomorphism between the subspace $P_B(M_n(\mathbb{C}))^\sim$ of $M_n(\mathbb{C})$ and the following disjoint union*:

$$
\begin{aligned}
&(\textit{rank } 0, n) && \{0_n\} \sqcup \{1_n\} \sqcup \\
&(\textit{rank } 1) && [\sqcup_{p_1, \cdots, p_n} (\sqcup_{i<j}^{{}_nC_2} (S^2, p_i, p_j))] \sqcup \\
&(\textit{rank } 2) && [\sqcup_{p_i + p_j, i<j} [(\sqcup_{i<j}^{{}_nC_2 \cdot (n-2)} (S^2, p_i, p_j) + \{p_{k_1}, \cdots, p_{k_{n-2}}\}) \sqcup \\
& && \quad (\sqcup_{(i<j) \sqcup (k<l)} \{(S^2, p_i, p_j) \oplus (S^2, p_k, p_l)\})]] \sqcup \\
&(\textit{rank } 3) && [\sqcup_{p_i + p_j + p_k, i<j<k} \\
& && [(\sqcup_{i<j}^{{}_nC_2 \cdot {}_{n-2}C_2} (S^2, p_i, p_j) + \{p_{k_1} + p_{k_2} \,|\, (i<j) \sqcup (k_1 < k_2)\}) \sqcup \\
& && \quad (\sqcup_{(i<j) \sqcup (k<l)} \{(S^2, p_i, p_j) \oplus (S^2, p_k, p_l) + \{p_{k_1}, \cdots, p_{k_{n-4}}\}\}) \sqcup \\
& && (\sqcup_{(i<j) \sqcup (k_1<k_2) \sqcup (k_3<k_4)} \{(S^2, p_i, p_j) \oplus (S^2, p_{k_1}, p_{k_2}) \\
& && \oplus (S^2, p_{k_3}, p_{k_4})\})]] \sqcup \cdots \sqcup
\end{aligned}
$$

$$
\begin{aligned}
&(\textit{rank } k \le \tfrac{n}{2}) && [\sqcup_{p_{i_1} + \cdots + p_{i_k}, i_1 < \cdots < i_k} \\
& && [(\sqcup_{i<j}^{{}_nC_2 \cdot {}_{n-2}C_{k-1}} (S^2, p_i, p_j) \\
& && \quad + \{p_{l_1} + \cdots + p_{l_{k-1}} \,|\, (i<j) \sqcup (l_1 < \cdots < l_{k-1})\}) \sqcup \\
& && \quad (\sqcup_{(i_1<i_2) \sqcup (i_3<i_4)} \{(S^2, p_{i_1}, p_{i_2}) \oplus (S^2, p_{i_3}, p_{i_4}) \\
& && \quad + \{p_{j_1}, \cdots, p_{j_{k-2}} \,|\, \{i_1, i_2, i_3, i_4\} \sqcup (j_1 < \cdots < j_{k-2})\}\}) \\
& && \quad \sqcup \cdots \sqcup \\
& && (\sqcup_{(i_1<i_2) \sqcup \cdots \sqcup (i_{2k-1} < i_{2k})} \{(S^2, p_{i_1}, p_{i_2}) \oplus \cdots \oplus (S^2, p_{i_{2k-1}}, p_{i_{2k}})\})]] \\
&(\textit{rank } k > \tfrac{n}{2}) && \quad \sqcup \cdots (\textit{omitted, see below}) \cdots \sqcup \\
&(\textit{rank } n-1) && [\sqcup_{p_{i_1} + \cdots + p_{i_{n-1}}, i_1 < \cdots < i_{n-1}} (\sqcup_{i<j}^{{}_nC_2} (S^2, p_i, p_j) + p_{j_1} + \cdots + p_{j_{n-2}})]
\end{aligned}
$$

where $\sqcup_{p_1, \cdots, p_n} (\sqcup_{i<j}^{{}_nC_2} (S^2, p_i, p_j))$ *means the* $\{p_1, \cdots, p_n\}$*-jointed sum of* ${}_nC_2$ *many copies of* S^2*, each of which is pointed with* p_i *and* p_j *for some and any* $1 \le i < j \le n$ *as that* p_i *and* p_j *identified with the north and south poles of* S^2 *respectively, such that*

$$
\begin{bmatrix}
p_1 & & & \\
(S^2, p_1, p_2) & p_2 & & \\
\vdots & \ddots & \ddots & \\
(S^2, p_1, p_n) & \cdots & (S^2, p_{n-1}, p_n) & p_n
\end{bmatrix}
$$

(*a matrix as a picture*)*, where each* p_j *in the picture is identified with the same other* p_j*, and the* $\{p_1, \cdots, p_n\}$*-jointed sum corresponds to the space of all rank* 1 *generalized Bott projections.*

And $\sqcup_{p_i + p_j, i<j} (\sqcup_{i<j}^{{}_nC_2 \cdot (n-2)} (S^2, p_i, p_j) + \{p_{k_1}, \ldots, p_{k_{n-2}}\})$ (*shorten*) *means the* $\{p_i + p_j \,|\, 1 \le i < j \le n\}$*-jointed sum of* ${}_nC_2 \cdot (n-2)$ *copies of* S^2*, each of which* *is pointed with* p_i *and* p_j *for some and any* $1 \leq i < j \leq n$ *as well as* $p_i + p_{k_s}$ *and* $p_j + p_{k_s}$ *with some* p_{k_s} *for* $k_s \neq i$ *and* $k_s \neq j$ *such that*

$$\begin{bmatrix} & p_1 + p_2 & \cdots & p_1 + p_n \\ (S^2, p_1, p_2) + \{p_{k_1}, \cdots, p_{k_{n-2}}\} & & \ddots & p_2 + p_n \\ \vdots & \ddots & & p_{n-1} + p_n \\ (S^2, p_1, p_n) + \{p_{k_1}, \cdots, p_{k_{n-2}}\} & \cdots & (S^2, p_{n-1}, p_n) + \{p_{k_s}\}_{s=1}^{n-2} & \end{bmatrix}$$

where we define

$$(S^2, p_i, p_j) + \{p_{k_1}, \cdots, p_{k_{n-2}}\} \equiv \{(S^2, p_i + p_{k_s}, p_j + p_{k_s}) \,|\, 1 \leq s \leq n-2\}.$$

and each $p_i + p_j$ *in the picture is identified with the same other* $p_i + p_j$.

In addition to the jointed sum above, each $(S^2, p_i, p_j) \oplus (S^2, p_k, p_l)$ *means the disjoint union of two copies of* S^2*, each of which is pointed with distinct* $\{p_i, p_j\}$ *or* $\{p_k, p_l\}$ *as well as* $p_i + p_k$*,* $p_i + p_l$*,* $p_j + p_k$*, and* $p_j + p_l$*, so that*

$$\begin{aligned} &(S^2, p_i, p_j) \oplus (S^2, p_k, p_l) \\ \equiv &(S^2, p_i + p_k, p_i + p_l, p_j + p_k, p_j + p_l) \oplus (S^2, p_k + p_i, p_k + p_j, p_l + p_i, p_l + p_j). \end{aligned}$$

In that and this cases, each $p_i + p_j$ *in the picture is identified with the same other* $p_i + p_j$*, and the* $\{p_i + p_j \,|\, i < j\}$*-jointed sum in total of this and that cases corresponds to the space of all rank* 2 *generalized Bott projections.*

And

$$\sqcup_{p_i+p_j+p_k, i<j<k}[\sqcup_{i<j}^{{}_nC_2 \cdot {}_{n-2}C_2}((S^2, p_i, p_j) + \{p_{k_1} + p_{k_2} \,|\, (i < j) \sqcup (k_1 < k_2)\})]$$

(shorten) means the $\{p_i + p_j + p_k \,|\, 1 \leq i < j < k \leq n\}$*-jointed sum of* ${}_nC_2 \cdot {}_{n-2}C_2$ *copies of* S^2*, each of which is pointed with* p_i *and* p_j *for some and any* $1 \leq i < j \leq n$ *as well as* $p_i + p_{k_1} + p_{k_2}$ *and* $p_j + p_{k_1} + p_{k_2}$ *with* $\{i, j\}$ *and* $\{k_1, k_2\}$ *distinct in* $\{1, \cdots, n\}$*, without such a picture, where*

$$\begin{aligned} &(S^2, p_i, p_j) + \{p_{k_1} + p_{k_2} \,|\, (i < j) \sqcup (k_1 < k_2)\} \\ \equiv &\{(S^2, p_i + p_{k_1} + p_{k_2}, p_j + p_{k_1} + p_{k_2}) \,|\, (i < j) \sqcup (k_1 < k_2)\} \end{aligned}$$

the set of ${}_{n-2}C_2$ *many elements, and each* $p_i + p_j + p_k$ *in the picture is identified with the same other* $p_i + p_j + p_k$.

This $\{p_i + p_j + p_k \,|\, i < j < k\}$*-jointed sum together with the other two disjoint unions*

$$\begin{aligned} &(\sqcup_{(i<j)\sqcup(k<l)}\{(S^2, p_i, p_j) \oplus (S^2, p_k, p_l) + \{p_{k_1}, \cdots, p_{k_{n-4}}\}\})\sqcup \\ &(\sqcup_{(i<j)\sqcup(k_1<k_2)\sqcup(k_3<k_4)}\{(S^2, p_i, p_j) \oplus (S^2, p_{k_1}, p_{k_2}) \oplus (S^2, p_{k_3}, p_{k_4})\}) \end{aligned}$$

with such identifications of $p_i + p_j + p_k$ *corresponds to the space of all rank* 3 *generalized Bott projections, where each* $(S^2, p_i, p_j) \oplus (S^2, p_k, p_l) + \{p_{k_1}, \cdots, p_{k_{n-4}}\}$ *is pointed with*

$$\begin{cases} p_i \\ p_j \end{cases} + \begin{cases} p_k \\ p_l \end{cases} + \{p_{k_1}, \cdots, p_{k_{n-4}}\}$$

as respective additions, and each $(S^2, p_i, p_j) \oplus (S^2, p_{k_1}, p_{k_2}) \oplus (S^2, p_{k_3}, p_{k_4})$ *is pointed with*

$$\begin{cases} p_i \\ p_j \end{cases} + \begin{cases} p_{k_1} \\ p_{k_2} \end{cases} + \begin{cases} p_{k_3} \\ p_{k_4}. \end{cases}$$

If $2k \leq n$*, then the space of all rank* k *degenerate projections is written as the* $\{p_{i_1} + \cdots + p_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq n\}$*-pointed disjoint union of the disjoint unions of* k *types such as*

$$\begin{aligned}
&(S^2, p_i, p_j) + \{p_{l_1} + \cdots + p_{l_{k-1}}, \quad (i < j) \sqcup (l_1 < \cdots < l_{k-1}),\\
&(S^2, p_{i_1}, p_{i_2}) \oplus (S^2, p_{i_3}, p_{i_4}) + \{p_{j_1}, \cdots, p_{j_{k-2}}\}, \{i_1, i_2, i_3, i_4\} \sqcup (j_1 < \cdots < j_{k-2}),\\
&\quad \cdots\cdots,\\
&(S^2, p_{i_1}, p_{i_2}) \oplus \cdots \oplus (S^2, p_{i_{2k-1}}, p_{i_{2k}}), \quad (i_1 < i_2) \sqcup \cdots \sqcup (i_{2k-1} < i_{2k}).
\end{aligned}$$

If $2k > n$*, then the space of all rank* k *generalized Bott projections is written as the* $\{p_{i_1} + \cdots + p_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq n\}$*-pointed disjoint union of the disjoint unions of less by less* $k - s_k$ *types one by one diminishing from the bottom type among the maximum types* $k = \frac{n}{2}$ *if* n *is even and* $k = \frac{n-1}{2}$ *if* n *is odd as above, with* $1 \leq k - s_k \leq k - 1$ *for some* $s_k \geq 1$*, where* s_k *increases with respect to* k*.*

And $\sqcup_{p_{i_1}+\cdots+p_{i_{n-1}}, i_1<\cdots<i_{n-1}} (\sqcup_{i<j}^{{}_nC_2} (S^2, p_i, p_j) + p_{j_1} + \cdots + p_{j_{n-2}})$ *means the* $\{p_{i_1} + \cdots + p_{i_{n-1}} \,|\, 1 \leq i_1 < \cdots < i_{n-1} < k \leq n\}$*-jointed sum of* ${}_nC_2$ *copies of* S^2*, each of which is pointed with* p_i *and* p_j *for some and any* $1 \leq i < j \leq n$ *as well as* $p_i + p_{j_1} + \cdots + p_{j_{n-2}}$ *and* $p_j + p_{j_1} + \cdots + p_{j_{n-2}}$ *such that*

$$\begin{bmatrix}
p_2 + \cdots + p_n & & \\
(S^2, p_1, p_2) + p_3 + \cdots + p_n & \ddots & \\
\vdots & \ddots & \ddots \\
(S^2, p_1, p_n) + p_2 + \cdots + p_{n-1} & \cdots & (S^2, p_{n-1}, p_n) + \sum_{s=1}^{n-2} p_s \;\; \sum_{s=1}^{n-1} p_s
\end{bmatrix}$$

where each $p_{i_1} + \cdots + p_{i_{n-1}}$ *in the picture is identified with the same other* $p_{i_1} + \cdots + p_{i_{n-1}}$*. The* $\{p_{i_1} + \cdots + p_{i_{n-1}} \,|\, 1 \leq i_1 < \cdots < i_{n-1} < k \leq n\}$*-jointed sum corresponds to the space of all rank* $n-1$ *generalized Bott projections.*

Corollary 5.6. *There is a homeomorphism between the subspace* $P_B(M_n(\mathbb{R}))^\sim$ *of* $M_n(\mathbb{R})$ *and the same disjoint union of the points-disjoint unions of those types as in the theorem above, where every* S^2 *at many places in the statement of the theorem such as* (S^2, p_i, p_j) *is replaced with* S^1 *respectively, such as* (S^1, p_i, p_j)*.*

Corollary 5.7. *The homotopy classes of $P_B(M_n(\mathbb{C}))^\sim$ are given by*

$$\begin{aligned}
&P_B(M_n(\mathbb{C}))^\sim/\sim = \{[0_4],[1_4],[p_1]=[p_2]=\cdots=[p_n],\\
&[p_1+p_2]=[p_i+p_j] \quad (1\le i<j\le n),\\
&[p_1+p_2+p_3]=[p_i+p_j+p_k] \quad (1\le i<j<k\le n),\\
&\quad \cdots\cdots,\\
&[\sum_{s=1}^{n-1} p_s]=[p_{i_1}+p_{i_2}+\cdots+p_{i_{n-1}}] \quad (1\le i_1<i_2<\cdots<i_{n-1}\le n)\}.
\end{aligned}$$

The same also holds for $P_B(M_n(\mathbb{R}))^\sim$.

Proof. The homotopies among p_1, p_2, $\cdots$, p_n and among p_i+p_j for $1\le i<j\le n$ and among $p_i+p_j+p_k$ for $1\le i<j<k\le n$, and $\cdots$, and among $p_{i_1}+p_{i_2}+\cdots+p_{i_{n-1}}$ for $1\le i_1<i_2<\cdots<i_{n-1}\le n$ within $P_B(M_n(\mathbb{C}))$ and $P_B(M_n(\mathbb{R}))$ are constructed explicitly and respectively as in Theorems 2.1, 3.3, 4.3, and 5.3 above. ■

Corollary 5.8. *There are bijections among the homotopy set $P_B(M_n(\mathbb{C}))^\sim/\sim$, the trace image $\mathrm{tr}(P_B(M_n(\mathbb{C}))^\sim) = \{0,1,2,\cdots,n\}$, and the rank image $\mathrm{rk}(P_B(M_n(\mathbb{C}))^\sim)$.*

The same also holds for $P_B(M_n(\mathbb{R}))^\sim$.

It is then deduced that

Corollary 5.9. *There is a continuous path from a generically non-degenerate projection of $M_n(\mathbb{C})$ to some generalized Bott projection of $M_n(\mathbb{C})$ within $P(M_n(\mathbb{C}))$.*

Namely, there is a continuous deformation from $P(M_n(\mathbb{C}))$ to $P_B(M_n(\mathbb{C}))$.

The same also holds for $P(M_n(\mathbb{R}))$.

Proof. Corollaries 5.6 and 5.7 hold for $P(M_n(\mathbb{C}))^\sim$ and $P(M_n(\mathbb{R}))^\sim$ as well, by using Linear Algebra. ■

Unitaries 5.10. Now let $U_n(\mathbb{C})$ denote the group of unitary matrices in $M_n(\mathbb{C})$. Let $SU_n(\mathbb{C})$ be the normal subgroup of $U_n(\mathbb{C})$ with determinant 1.

It is known as [5, I. 4. (4.8), Page 36] (cf. [11]) that

Lemma 5.11. *There are homeomorphisms as in the following*:

$$U_n(\mathbb{C})/U_{n-1}(\mathbb{C}) \approx S^{2n-1} \quad \textit{and} \quad SU_n(\mathbb{C})/SU_{n-1}(\mathbb{C}) \approx S^{2n-1}$$

for $n\ge 2$.

Proof. Define the complex $(n-1)$-dimensional sphere in $\mathbb{C}^n$ as

$$S^{n-1}(\mathbb{C}) = \{(z_j) \in \mathbb{C}^n \mid \sum_{j=1}^{n} |z_j|^2 = 1\}.$$

Then there is a homeomorphism between $S^{n-1}(\mathbb{C})$ and S^{2n-1} the real $(2n-1)$-dimensional sphere.

The groups $U_n(\mathbb{C})$ and $SU_n(\mathbb{C})$ act transitively on $S^{n-1}(\mathbb{C})$ by matrix multiplication from the left. Indeed, for any $z = (z_j) \in S^{n-1}(\mathbb{C})$, there is an orthonormal basis $\{w_1, \cdots, w_n\}$ for $\mathbb{C}^n$ extending z, with $w_n = z$. Then $W = (w_1, \ldots, w_n) \in U_n(\mathbb{C})$ with each w_j as a column vector, so that $We_n = z$, where e_n is the standard n-th basis vector for $\mathbb{C}^n$. Note also that $W' = (\overline{\det W} w_1, w_2, \cdots, w_n) \in SU_n(\mathbb{C})$ with $\det W \in \mathbb{T}$ the 1-torus, so that $W'e_n = z$.

Then the isotropy subgroups $U_n(\mathbb{C})_{e_n}$ and $SU_n(\mathbb{C})_{e_n}$ of $U_n(\mathbb{C})$ and $SU_n(\mathbb{C})$ at the standard n-th basis vector e_n in $\mathbb{C}^n$ are isomorphic to $U_{n-1}(\mathbb{C})$ and $SU_{n-1}(\mathbb{C})$ respectively. Indeed, if $U = (u_{ij}) \in U_n(\mathbb{C})$ and $Ue_n = e_n$, then $u_{in} = 0$ for $1 \leq i \leq n-1$. It follows from the (n,n) component for $UU^* = 1_n$ that $u_{nj} = 0$ for $1 \leq j \leq n-1$. Hence $U = U' \oplus 1$ with $U' \in U_{n-1}(\mathbb{C})$. The same is applied for $SU_n(\mathbb{C})$.

There are surjective maps from $U_n(\mathbb{C})$ and $SU_n(\mathbb{C})$ to $S^{n-1}(\mathbb{C})$, defined as $U \mapsto Ue_n$, both of which factors through to the quotient groups as

$$\begin{array}{ccc} U_n(\mathbb{C}) \text{ and } SU_n(\mathbb{C}) & \longrightarrow & S^{n-1}(\mathbb{C}) \\ \big\downarrow & & \big\| \\ U_n(\mathbb{C})/U_n(\mathbb{C})_{e_n} \text{ and } SU_n(\mathbb{C})/SU_n(\mathbb{C})_{e_n} & \longrightarrow & S^{n-1}(\mathbb{C}), \end{array}$$

respectively, the arrow in the bottom line is a bijection at this moment. Note as well that $S^{n-1}(\mathbb{C})$ is a Hausdorff space and that $U_n(\mathbb{C})$ and $SU_n(\mathbb{C})$ are compact groups, and so are the quotient groups. It then follows that the arrow in the bottom line is a homeomorphism, and hence the statement in this lemma holds. ■

As just a comparison to the above case of projections,

Corollary 5.12. (see [5, I. 4. (4.8)], [11]) *There are decomposition series of* $U_n(\mathbb{C})$ *and* $SU_n(\mathbb{C})$ *by subquotient spaces as*

$$U_k(\mathbb{C})/U_{k-1}(\mathbb{C}) \approx S^{2k-1} \quad \textit{and} \quad SU_k(\mathbb{C})/SU_{k-1}(\mathbb{C}) \approx S^{2k-1}$$

for $2 \leq k \leq n$, *with* $U_1(\mathbb{C}) \approx S^1$ *and* $SU_1(\mathbb{C}) = \{1\}$ *trivial. Namely, inductively,*

$$\begin{aligned} U_n(\mathbb{C}) &\approx S^{2n-1} \times^{\sim} U_{n-1}(\mathbb{C}) \\ &\approx S^{2n-1} \times^{\sim} (S^{2n-3} \times^{\sim} U_{n-2}(\mathbb{C})) \end{aligned}$$

$$
\begin{aligned}
&\approx \cdots\cdots \\
&\approx S^{2n-1} \times^{\sim} (S^{2n-3} \times^{\sim} (\cdots \times^{\sim} [S^3 \times^{\sim} S^1] \cdots)), \quad \textit{and} \\
SU_n(\mathbb{C}) &\approx S^{2n-1} \times^{\sim} (S^{2n-3} \times^{\sim} (\cdots \times^{\sim} [S^3 \times^{\sim} \{1\}] \cdots)),
\end{aligned}
$$

where each space of the form $B \times^{\sim} F$ at the inductive steps means the fibered space over B the base space with F the fiber space.

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