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Super-biderivations and Linear Super-commuting Maps on a Class of Cartan Type Lie Superalgebras*

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Abstract. In this paper, super-biderivations on the Cartan type Lie superalgebra $W(n)$ are determined. As an application, linear super-commuting maps on $W(n)$ are obtained.

Keywords: Cartan type Lie superalgebra; Super-biderivation; W(n).

1. Introduction

In 1977, Kac gave the classification of finite-dimensional simple Lie superalgebras over a field of characteristic zero, which are divided into classical ones and Cartan type ones (see [6]). In 1997, Zhang constructed four classes of Cartan type simple Lie superalgebras W, S, H, and K (see [17]). Since then, further research has been conducted on the Witt type Lie superalgebra W (see [4, 7, 8]).

Throughout $\mathbb{C}$ is the complex field, $\mathbb{Z}$ the set of integers and $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ the additive group of two elements. Let $\Lambda(n)$ be the exterior algebra generated

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by $x_1, x_2, \ldots, x_n$ with the $\mathbb{Z}$-degree $\deg(x_i) = 1$ for $i = 1, 2, \ldots, n$. Then $\Lambda(n)$ is a $\mathbb{Z}$-graded algebra. Corresponding to this $\mathbb{Z}$-grading, $\Lambda(n)$ has the natural $\mathbb{Z}_2$-grading. Then $\Lambda(n) = \Lambda(n)_{\bar{0}} \bigoplus \Lambda(n)_{\bar{1}}$ is an associative superalgebra over $\mathbb{C}$. In this paper, $|a|$ is the parity ($\mathbb{Z}_2$-degree) of a in the superspace under consideration. Below δ_P means 1 if a proposition P is true, and 0 otherwise. For $i = 1, 2, \ldots, n$, define the distinguished partial derivations ∂_i with parity $|\partial_i| = |x_i|$ by letting

$$\partial_i(x_j) = \delta_{i=j} \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Write $W(n)$ for the superderivation algebra of $\Lambda(n)$, that is, $W(n) = \text{Der}\Lambda(\text{n})$ (see below for the definition), which is a Cartan type simple Lie superalgebra. Recall that

$$\begin{aligned} W(n) &= \text{span}_{\mathbb{C}}\{x_1^{i_1} x_2^{i_2} x_3^{i_3} \cdots x_n^{i_n} \partial_j \mid i_j = 0, 1, j = 1, 2, \ldots, n\}, \\ [x\partial_i, y\partial_j] &= x\partial_i(y)\partial_j - (-1)^{|x\partial_i||y\partial_j|} y\partial_j(x)\partial_i, \end{aligned}$$

where $x\partial_i, y\partial_j \in W(n)$ (see [7]). It is clear that $\dim W(n) = n \cdot 2^n$.

In recent years, biderivations on algebras or rings have been studied by many authors (see [1, 2, 9, 10, 11, 15]). In 2011, Wang, Yu and Chen introduced biderivations on Lie algebras (see [12]). In 2015, Xu and Wang discussed some properties of super-biderivations on Heisenberg superalgebras (see [14]). Later, many authors also study super-biderivations on Lie superalgebras (see [5, 13]). Now let us recall definitions of super-derivations or super-biderivations on Lie superalgebras in the following.

Definition 1.1. *Let L be a Lie superalgebra.*

(i) *Let $D : L \longrightarrow L$ be a linear map with parity $\beta \in \mathbb{Z}_2$. Then, D is called a super-derivation with parity β on L if it satisfies the following equation:*

$$D([x, y]) = [D(x), y] + (-1)^{\beta|x|}[x, D(y)], \quad x, y \in L.$$

Write $\text{Der}_{\bar{0}}L$ *(resp.* $\text{Der}_{\bar{1}}L$*) for the set of all super-derivations with parity $\bar{0} \in \mathbb{Z}_2$ (resp. $\bar{1} \in \mathbb{Z}_2$) on L. Write*

$$\text{Der}L = \text{Der}_{\bar{0}}L \oplus \text{Der}_{\bar{1}}L.$$

If $D \in \text{Der}L$, then D is called a super-derivation on L. For $x \in L$, let

$$\text{ad}x : L \longrightarrow L, \quad y \longmapsto [x, y].$$

Then $\text{ad}x$ *is a super-derivation on L, which is called an inner super-derivation. Write* $\text{Inn}L$ *for the set of all inner super-derivations on L.*

(ii) *Let $f : L \times L \longrightarrow L$ be a bilinear map with parity $\beta \in \mathbb{Z}_2$. f is called a super-biderivation with parity β on L if for any $x, y, z \in L$,*

$$\begin{aligned} f([y, z], x) &= [y, f(z, x)] + (-1)^{(\beta+|x|)|z|}[f(y, x), z], \\ f(x, [y, z]) &= [f(x, y), z] + (-1)^{(\beta+|x|)|y|}[y, f(x, z)]. \end{aligned}$$

Write $\mathrm{BDer}_\beta L$ *for the set of all super-biderivations with parity* β *on* L. *Denote by*

$$\mathrm{BDer}L = \mathrm{BDer}_{\bar{0}}L \oplus \mathrm{BDer}_{\bar{1}}L.$$

If $f \in \mathrm{BDer}L$, f *is called a super-biderivation on* L. *For* $\lambda \in \mathbb{C}$, *it is easy to verify that the map*

$$f : L \times L \longrightarrow L, \quad (x, y) \longmapsto \lambda[x, y]$$

is a super-biderivation on L, *which is said to be inner.*

In [5] and [13], the authors determine skew-symmetric super-biderivations on (centerless) super-Virasoro algebras. The super-biderivations without the skew-symmetric conditions on classical simple Lie superalgebras are also studied in [16]. In this paper, our aim is to determine super-biderivations on $W(n)$.

2. Super-biderivations on $W(n)$

We first recall and establish several auxiliary results.

Lemma 2.1. [7] $\mathrm{Der}W(n) = \mathrm{Inn}W(n)$.

Lemma 2.2. *Let* f *be a super-biderivation on* $W(n)$. *Then there exist two linear maps* φ_f *and* ψ_f *from* $W(n)$ *into itself such that*

$$f(x, y) = [\varphi_f(x), y] = [x, \psi_f(y)], \quad \forall x, y \in W(n).$$

Proof. For any fixed element $x \in W(n)$, $f(x, \cdot)$ is a super-derivation on $W(n)$. By Lemma 2.1, there exists a map $\phi_f : W(n) \longrightarrow W(n)$ such that $f(x, y) = [\phi_f(x), y]$, where $y \in W(n)$. Since f is bilinear, we have

$$f(\lambda x_1 + x_2, y) = \lambda f(x_1, y) + f(x_2, y).$$

for any $\lambda \in \mathbb{C}, x_1, x_2, y \in W(n)$. Consequently,

$$[\varphi_f(\lambda x_1 + x_2), y] = \lambda[\varphi_f(x_1), y] + [\varphi_f(x_2), y],$$

that is,

$$[\varphi_f(\lambda x_1 + x_2), y] = [\lambda\varphi_f(x_1) + \varphi_f(x_2), y].$$

Note that y is arbitrary, and then $\varphi_f(\lambda x_1 + x_2) - \lambda\varphi_f(x_1) - \varphi_f(x_2)$ lies in the center of $W(n)$, which is trivial. Then $\varphi_f(\lambda x_1 + x_2) = \lambda\varphi_f(x_1) + \varphi_f(x_2)$, that is, φ_f is linear. Similarly, there exists a linear map ψ_f from $W(n)$ into itself such that $f(\cdot, y) = [\cdot, \psi_f(y)]$ for any $y \in W(n)$. The proof is completed. ∎

Now we are in the position to give the main result of this section.

Theorem 2.3. *A map $f : W(n) \times W(n) \longrightarrow W(n)$ is a super-biderivation on $W(n)$ if and only if there exists $\lambda \in \mathbb{C}$ such that*

$$f(x, y) = \lambda[x, y], \quad \forall x, y \in W(n).$$

That is, every super-biderivation on $W(n)$ is inner.

Proof. The "if" direction is easy to verify. We now prove the "only if" direction.

Below we only prove the result for the case $n = 3$ since the general case is similar. Now, we assume that f is a super-biderivation on $W(3)$. In view of Lemma 2.2, there are two linear maps φ_f and ψ_f from $W(3)$ into itself such that $f(x, y) = [\varphi_f(x), y] = [x, \psi_f(y)]$ for all $x, y \in W(3)$. For $i_1, i_2, i_3 \in \{0, 1\}$ and $j \in \{1, 2, 3\}$, let

$$\begin{aligned}
\varphi_f(x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_j) &= \sum_{k=1}^{3} a_k^{(i_1i_2i_3j)}\partial_k + \sum_{k=1}^{3}\sum_{1\le l<t\le 3} c_{ltk}^{(i_1i_2i_3j)}x_lx_t\partial_k \\
&\quad + \sum_{k=1}^{3}\sum_{l=1}^{3} b_{lk}^{(i_1i_2i_3j)}x_l\partial_k + \sum_{k=1}^{3} d_k^{(i_1i_2i_3j)}x_1x_2x_3\partial_k, \\
\psi_f(x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_j) &= \sum_{k=1}^{3} a_k^{\langle i_1i_2i_3j\rangle}\partial_k + \sum_{k=1}^{3}\sum_{1\le l<t\le 3} c_{ltk}^{\langle i_1i_2i_3j\rangle}x_lx_t\partial_k \\
&\quad + \sum_{k=1}^{3}\sum_{l=1}^{3} b_{lk}^{\langle i_1i_2i_3j\rangle}x_l\partial_k + \sum_{k=1}^{3} d_k^{\langle i_1i_2i_3j\rangle}x_1x_2x_3\partial_k,
\end{aligned}$$

where the above coefficients are in $\mathbb{C}$. Then we get

$$\begin{aligned}
[\varphi_f(x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_j), \partial_p] &= -\sum_{k=1}^{3}\left(\delta_{p=2}c_{12k}^{(i_1i_2i_3j)} + \delta_{p=3}c_{13k}^{(i_1i_2i_3j)}\right)x_1\partial_k \\
&\quad - \sum_{k=1}^{3} d_k^{(i_1i_2i_3j)}\left(\delta_{p=3}x_1x_2\partial_k + \delta_{p=1}x_2x_3\partial_k\right) \\
&\quad + \sum_{k=1}^{3}\left(\delta_{p=1}c_{12k}^{(i_1i_2i_3j)} - \delta_{p=3}c_{23k}^{(i_1i_2i_3j)}\right)x_2\partial_k \\
&\quad + \sum_{k=1}^{3}\left(\delta_{p=1}c_{13k}^{(i_1i_2i_3j)} + \delta_{p=2}c_{23k}^{(i_1i_2i_3j)}\right)x_3\partial_k \\
&\quad + \sum_{k=1}^{3}\left(\delta_{p=2}d_k^{(i_1i_2i_3j)}x_1x_3\partial_k - b_{pk}^{(i_1i_2i_3j)}\partial_k\right), \qquad (1) \\
[\partial_j, \psi_f(x_1^{j_1}x_2^{j_2}x_3^{j_3}\partial_p)] &= \sum_{k=1}^{3} d_k^{\langle j_1j_2j_3p\rangle}\left(\delta_{j=1}x_2x_3\partial_k - \delta_{j=2}x_1x_3\partial_k + \delta_{j=3}x_1x_2\partial_k\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{3}\left(\delta_{j=1}b_{1k}^{\langle j_1j_2j_3p\rangle}+\delta_{j=2}b_{2k}^{\langle j_1j_2j_3p\rangle}+\delta_{j=3}b_{3k}^{\langle j_1j_2j_3p\rangle}\right)\partial_k\\
&+\sum_{k=1}^{3}\left(-\delta_{j=2}c_{12k}^{\langle j_1j_2j_3p\rangle}-\delta_{j=3}c_{13k}^{\langle j_1j_2j_3p\rangle}\right)x_1\partial_k\\
&+\sum_{k=1}^{3}\left(\delta_{j=1}c_{12k}^{\langle j_1j_2j_3p\rangle}-\delta_{j=3}c_{23k}^{\langle j_1j_2j_3p\rangle}\right)x_2\partial_k\\
&+\sum_{k=1}^{3}\left(\delta_{j=1}c_{13k}^{\langle j_1j_2j_3p\rangle}+\delta_{j=2}c_{23k}^{\langle j_1j_2j_3p\rangle}\right)x_3\partial_k. \qquad (2)
\end{aligned}
$$

Next, we shall complete the proof by verifying the following seven claims.

Claim 1. Let $j \in \{1,2,3\}$. Then

$$
\begin{aligned}
\varphi_f(\partial_j) &= \sum_{k=1}^{3}\left(a_k^{(000j)}\partial_k+\sum_{l=1}^{3}b_{lk}^{(000j)}x_l\partial_k\right),\\
\psi_f(\partial_j) &= \sum_{k=1}^{3}\left(a_k^{\langle 000j\rangle}\partial_k+\sum_{l=1}^{3}b_{lk}^{\langle 000j\rangle}x_l\partial_k\right),
\end{aligned}
$$

where $-b_{st}^{(000r)}=b_{rt}^{\langle 000s\rangle}$ for $r,s,t\in\{1,2,3\}$.

In fact, it is true that $[\varphi_f(\partial_j),\partial_p]=[\partial_j,\psi_f(\partial_p)]$ by Lemma 2.2. Then by Equations (1) and (2), we get the following equations:

$$
\begin{aligned}
-\delta_{p=1}d_k^{(000j)} &= \delta_{j=1}d_k^{\langle 000p\rangle},\\
\delta_{p=2}d_k^{(000j)} &= -\delta_{j=2}d_k^{\langle 000p\rangle},\\
-\delta_{p=3}d_k^{(000j)} &= \delta_{j=3}d_k^{\langle 000p\rangle},\\
-b_{pk}^{(000j)} &= \delta_{j=1}b_{1k}^{\langle 000p\rangle}+\delta_{j=2}b_{2k}^{\langle 000p\rangle}+\delta_{j=3}b_{3k}^{\langle 000p\rangle}
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_{p=2}c_{12k}^{(000j)}+\delta_{p=3}c_{13k}^{(000j)} &= \delta_{j=2}c_{12k}^{\langle 000p\rangle}+\delta_{j=3}c_{13k}^{\langle 000p\rangle},\\
\delta_{p=1}c_{12k}^{(000j)}-\delta_{p=3}c_{23k}^{(000j)} &= \delta_{j=1}c_{12k}^{\langle 000p\rangle}-\delta_{j=3}c_{23k}^{\langle 000p\rangle},\\
\delta_{p=1}c_{13k}^{(000j)}+\delta_{p=2}c_{23k}^{(000j)} &= \delta_{j=1}c_{13k}^{\langle 000p\rangle}+\delta_{j=2}c_{23k}^{\langle 000p\rangle}.
\end{aligned}
$$

Consequently, we complete the proof of Claim 1.

For convenience, for $k\in\{1,2,3\}$ we set the symbols $\Phi_k\in\{1,2,3\}\backslash\{k\}$ and $e_k\in\mathbb{C}^3$ with 1 at the k-th position and 0 at others.

Claim 2. Let $j,p,q \in \{1,2,3\}$. Then

$$\begin{aligned}
\varphi_f(\partial_j) &= \sum_{k=1}^{3} \left(a_k^{(000j)} \partial_k + b_{11}^{(000j)} x_k \partial_k \right), \\
\psi_f(\partial_j) &= \sum_{k=1}^{3} \left(a_k^{\langle 000j \rangle} \partial_k - b_{11}^{(000k)} x_k \partial_j \right), \\
\psi_f(x_p \partial_q) &= \sum_{k=1}^{3} \left(a_k^{\langle e_p q \rangle} \partial_k + a_p^{(000k)} x_k \partial_q \right).
\end{aligned}$$

In the following, we fix $j,p,q \in \{1,2,3\}$. It is true that

$$[\varphi_f(\partial_j), x_p\partial_q] = [\partial_j, \psi_f(x_p\partial_q)]$$

from Lemma 2.2. By Claim 1, we get

$$\begin{aligned}
[\varphi_f(\partial_j), x_p\partial_q] &= a_p^{(000j)} \partial_q + \left(b_{pp}^{(000j)} - b_{qq}^{(000j)} \right) x_p \partial_q \\
&\quad + \sum_{l \neq p} b_{lp}^{(000j)} x_l \partial_q - \sum_{k \neq q} b_{qk}^{(000j)} x_p \partial_k. \qquad (3)
\end{aligned}$$

Then by Equations (2) and (3), we get the following equations:

$$d_1^{\langle e_p q \rangle} = d_2^{\langle e_p q \rangle} = d_3^{\langle e_p q \rangle} = b_{j\Phi_q}^{\langle e_p q \rangle} = 0, \quad a_p^{(000j)} = b_{jq}^{\langle e_p q \rangle}.$$

In addition, we also obtain some other equations by the following two steps.

Step 1. Let $p = q$. Then

$$\begin{aligned}
\delta_{q\neq 2} b_{2q}^{(000j)} &= \delta_{j=1} c_{12q}^{\langle e_q q \rangle} - \delta_{j=3} c_{23q}^{\langle e_q q \rangle}, \\
b_{1\Phi_1}^{(000j)} &= \delta_{j=2} c_{12\Phi_1}^{\langle 1001 \rangle} + \delta_{j=3} c_{13\Phi_1}^{\langle 1001 \rangle}, \\
\delta_{q\neq 1} b_{1q}^{(000j)} &= -\delta_{j=2} c_{12q}^{\langle e_q q \rangle} - \delta_{j=3} c_{13q}^{\langle e_q q \rangle}, \\
b_{2\Phi_2}^{(000j)} &= -\delta_{j=1} c_{12\Phi_2}^{\langle 0102 \rangle} + \delta_{j=3} c_{23\Phi_2}^{\langle 0102 \rangle}, \\
\delta_{q\neq 3} b_{3q}^{(000j)} &= \delta_{j=1} c_{13q}^{\langle e_q q \rangle} - \delta_{j=2} c_{23q}^{\langle e_q q \rangle}, \\
b_{3\Phi_3}^{(000j)} &= \delta_{j=2} c_{23\Phi_3}^{\langle 0013 \rangle} - \delta_{j=1} c_{13\Phi_3}^{\langle 0013 \rangle}, \\
\delta_{(rst)\neq(231)} c_{rst}^{\langle 1001 \rangle} &= \delta_{(rst)\neq(132)} c_{rst}^{\langle 0102 \rangle} = \delta_{(rst)\neq(123)} c_{rst}^{\langle 0013 \rangle} = 0.
\end{aligned}$$

Step 2. Let $p \neq q$. Then

$$\begin{aligned}
b_{\Phi_p p}^{(000j)} = \; & \delta_{p=1}\delta_{\Phi_p=2} \left(\delta_{j=1} c_{12q}^{\langle 100q \rangle} - \delta_{j=3} c_{23q}^{\langle 100q \rangle} \right) \\
& + \delta_{p=1}\delta_{\Phi_p=3} \left(\delta_{j=1} c_{13q}^{\langle 100q \rangle} - \delta_{j=2} c_{23q}^{\langle 100q \rangle} \right) \\
& + \delta_{p=2}\delta_{\Phi_p=1} \left(-\delta_{j=2} c_{12q}^{\langle 010q \rangle} - \delta_{j=3} c_{13q}^{\langle 010q \rangle} \right)
\end{aligned}$$

$$
\begin{aligned}
&+\delta_{p=2}\delta_{\Phi_p=3}\left(\delta_{j=1}c_{13q}^{\langle 010q\rangle}-\delta_{j=2}c_{23q}^{\langle 010q\rangle}\right)\\
&+\delta_{p=3}\delta_{\Phi_p=1}\left(-\delta_{j=2}c_{12q}^{\langle 001q\rangle}-\delta_{j=3}c_{13q}^{\langle 001q\rangle}\right)\\
&+\delta_{p=3}\delta_{\Phi_p=2}\left(\delta_{j=1}c_{12q}^{\langle 001q\rangle}-\delta_{j=3}c_{23q}^{\langle 001q\rangle}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
b_{qk}^{(000j)}-\delta_{k=q}b_{pp}^{(000j)} = \;& \delta_{p=1}\left(\delta_{j=2}c_{12k}^{\langle 100q\rangle}+\delta_{j=3}c_{13k}^{\langle 100q\rangle}\right)\\
&+\delta_{p=2}\left(\delta_{j=3}c_{23k}^{\langle 010q\rangle}-\delta_{j=1}c_{12k}^{\langle 010q\rangle}\right)\\
&+\delta_{p=3}\left(\delta_{j=2}c_{23k}^{\langle 001q\rangle}-\delta_{j=1}c_{13k}^{\langle 001q\rangle}\right).
\end{aligned}
$$

Besides, all the coefficients of $x_{\Phi_p}\partial_{\Phi_q}$ on the right side of Equation (2) are 0.

After the above two steps, we get

$$
\begin{aligned}
\varphi_f(\partial_j) &= \sum_{k=1}^{3}\left(a_k^{(000j)}\partial_k+b_{11}^{(000j)}x_k\partial_k\right),\\
\psi_f(x_p\partial_q) &= \sum_{k=1}^{3}\left(a_k^{\langle e_pq\rangle}\partial_k+a_p^{(000k)}x_k\partial_q\right).
\end{aligned}
$$

It is true that $-b_{st}^{(000r)}=b_{rt}^{\langle 000s\rangle}$ for $r,s,t=1,2,3$ from Claim 1. Then

$$
\psi_f(\partial_j)=\sum_{k=1}^{3}\left(a_k^{\langle 000j\rangle}\partial_k-b_{11}^{(000k)}x_k\partial_j\right).
$$

Consequently, we complete the proof of Claim 2.

Claim 3. Let $j,q\in\{1,2,3\}$ and $i_1,i_2,i_3\in\{0,1\}$. Then

$$
\begin{aligned}
\varphi_f(\partial_j) &= a_1^{(0001)}\partial_j,\\
\psi_f(x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_q) &= \delta_{(i_1i_2i_3)\neq(0,0,0)}a_1^{(0001)}x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_q+\sum_{k=1}^{3}a_k^{\langle i_1i_2i_3q\rangle}\partial_k.
\end{aligned}
$$

In the following, we fix $j,q\in\{1,2,3\}$ and $i_1,i_2,i_3\in\{0,1\}$. It is true that

$$
[\varphi_f(\partial_j),x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_q]=[\partial_j,\psi_f(x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_q)]
$$

from Lemma 2.2. By Claim 2, we get

$$
\begin{aligned}
[\varphi_f(\partial_j),x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_q] = \;& \delta_{i_1=1}a_1^{(000j)}x_2^{i_2}x_3^{i_3}\partial_q+(-1)^{i_1}\delta_{i_2=1}a_2^{(000j)}x_1^{i_1}x_3^{i_3}\partial_q\\
&+(i_1+i_2+i_3-1)b_{11}^{(000j)}x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_q\\
&+(-1)^{i_1+i_2}\delta_{i_3=1}a_3^{(000j)}x_1^{i_1}x_2^{i_2}\partial_q.
\end{aligned}
$$

Then by the above equation and (2), we get the following equations:

$$\begin{aligned}
\delta_{j=1}d_k^{\langle i_1i_2i_3q\rangle} &= \delta_{k=q}\delta_{(i_1i_2i_3)=(111)}a_1^{(000j)},\\
\delta_{j=2}d_k^{\langle i_1i_2i_3q\rangle} &= \delta_{k=q}\delta_{(i_1i_2i_3)=(111)}a_2^{(000j)},\\
\delta_{j=3}d_k^{\langle i_1i_2i_3q\rangle} &= \delta_{k=q}\delta_{(i_1i_2i_3)=(111)}a_3^{(000j)}
\end{aligned}$$

and

$$\begin{aligned}
&\delta_{j=2}c_{12k}^{\langle i_1i_2i_3q\rangle} + \delta_{j=3}c_{13k}^{\langle i_1i_2i_3q\rangle}\\
=\ &\delta_{k=q}\delta_{(i_1i_2i_3)=(110)}a_2^{(000j)} + \delta_{k=q}\delta_{(i_1i_2i_3)=(101)}a_3^{(000j)},\\
&\delta_{j=1}c_{12k}^{\langle i_1i_2i_3q\rangle} - \delta_{j=3}c_{23k}^{\langle i_1i_2i_3q\rangle}\\
=\ &\delta_{k=q}\delta_{(i_1i_2i_3)=(110)}a_1^{(000j)} - \delta_{k=q}\delta_{(i_1i_2i_3)=(011)}a_3^{(000j)},\\
&\delta_{j=1}c_{13k}^{\langle i_1i_2i_3q\rangle} + \delta_{j=2}c_{23k}^{\langle i_1i_2i_3q\rangle}\\
=\ &\delta_{k=q}\delta_{(i_1i_2i_3)=(101)}a_1^{(000j)} + \delta_{k=q}\delta_{(i_1i_2i_3)=(011)}a_2^{(000j)},\\
&\delta_{j=1}b_{1k}^{\langle i_1i_2i_3q\rangle} + \delta_{j=2}b_{2k}^{\langle i_1i_2i_3q\rangle} + \delta_{j=3}b_{3k}^{\langle i_1i_2i_3q\rangle}\\
=\ &\delta_{k=q}\delta_{(i_1i_2i_3)=(010)}a_2^{(000j)} + \delta_{k=q}\delta_{(i_1i_2i_3)=(001)}a_3^{(000j)}\\
&+\delta_{k=q}\delta_{(i_1i_2i_3)=(100)}a_1^{(000j)},
\end{aligned}$$

where $k \in \{1,2,3\}$. Consequently, we complete the proof of Claim 3.

Claim 4. Let $j \in \{1,2,3\}$ and $i_1, i_2, i_3 \in \{0,1\}$. Then

$$\begin{aligned}
\psi_f(\partial_j) &= a_1^{\langle 0001\rangle}\partial_j,\\
\varphi_f(x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_j) &= \delta_{(i_1i_2i_3)\neq(000)}\left(a_1^{\langle 0001\rangle}x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_j + \sum_{k=1}^{3}a_k^{(i_1i_2i_3j)}\partial_k\right).
\end{aligned}$$

In the following, we fix $j, p \in \{1,2,3\}$ and $i_1, i_2, i_3 \in \{0,1\}$. It is true that

$$[\varphi_f(x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_j), \partial_p] = [x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_j, \psi_f(\partial_p)]$$

from Lemma 2.2. By Claim 3, we get

$$\begin{aligned}
[x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_j, \psi_f(\partial_p)] = \ &(-1)^{i_1+i_2+i_3}\delta_{i_1=1}a_1^{\langle 000p\rangle}x_2^{i_2}x_3^{i_3}\partial_j\\
&+(-1)^{i_1+i_2+i_3}\delta_{i_3=1}a_3^{\langle 000p\rangle}x_1^{i_1}x_2^{i_2}\partial_j\\
&+(-1)^{i_2+i_3}\delta_{i_2=1}a_2^{\langle 000p\rangle}x_1^{i_1}x_3^{i_3}\partial_j.
\end{aligned}$$

Then by the above equation and (1), we get the following equations:

$$\begin{aligned}
\delta_{p=1}d_k^{(i_1i_2i_3j)} &= \delta_{k=j}\delta_{(i_1i_2i_3)=(111)}a_1^{\langle 000p\rangle},\\
\delta_{p=2}d_k^{(i_1i_2i_3j)} &= \delta_{k=j}\delta_{(i_1i_2i_3)=(111)}a_2^{\langle 000p\rangle},\\
\delta_{p=3}d_k^{(i_1i_2i_3j)} &= \delta_{k=j}\delta_{(i_1i_2i_3)=(111)}a_3^{\langle 000p\rangle},\\
b_{pk}^{(i_1i_2i_3j)} &= \delta_{k=j}\delta_{(i_1i_2i_3)=(001)}a_3^{\langle 000p\rangle} + \delta_{k=j}\delta_{(i_1i_2i_3)=(010)}a_2^{\langle 000p\rangle}\\
&\quad +\delta_{k=j}\delta_{(i_1i_2i_3)=(100)}a_1^{\langle 000p\rangle}
\end{aligned}$$

and

$$\begin{aligned}
&\delta_{p=2}c_{12k}^{(i_1i_2i_3j)} + \delta_{p=3}c_{13k}^{(i_1i_2i_3j)}\\
=\ &\delta_{k=j}\delta_{(i_1i_2i_3)=(110)}a_2^{\langle 000p\rangle} - \delta_{k=j}\delta_{(i_1i_2i_3)=(101)}a_3^{\langle 000p\rangle},\\
&\delta_{p=1}c_{12k}^{(i_1i_2i_3j)} - \delta_{p=3}c_{23k}^{(i_1i_2i_3j)}\\
=\ &\delta_{k=j}\delta_{(i_1i_2i_3)=(110)}a_1^{\langle 000p\rangle} + \delta_{k=j}\delta_{(i_1i_2i_3)=(011)}a_3^{\langle 000p\rangle},\\
&\delta_{p=1}c_{13k}^{(i_1i_2i_3j)} + \delta_{p=2}c_{23k}^{(i_1i_2i_3j)}\\
=\ &\delta_{k=j}\delta_{(i_1i_2i_3)=(101)}a_1^{\langle 000p\rangle} - \delta_{k=j}\delta_{(i_1i_2i_3)=(011)}a_2^{\langle 000p\rangle},
\end{aligned}$$

where $k \in \{1,2,3\}$. Consequently, we complete the proof of Claim 4.

Claim 5. Let $i, j \in \{1,2,3\}$. Then

$$\psi_f(x_i\partial_j) = \varphi_f(x_i\partial_j) = a_1^{(0001)}x_i\partial_j, \quad \psi_f(\partial_j) = a_1^{(0001)}\partial_j.$$

In the following, we fix $i, j, p, q \in \{1,2,3\}$. It is true that

$$[\varphi_f(x_p\partial_q), x_i\partial_j] = [x_p\partial_q, \psi_f(x_i\partial_j)]$$

from Lemma 2.2. By Claim 3, we get

$$\begin{aligned}
[x_p\partial_q, \psi_f(x_i\partial_j)] &= \left[x_p\partial_q, \quad a_1^{(0001)}x_i\partial_j + \sum_{k=1}^{3} a_k^{\langle e_ij\rangle}\partial_k\right]\\
&= a_1^{(0001)}\left(\delta_{q=i}x_p\partial_j - \delta_{p=j}x_i\partial_q\right) - a_p^{\langle e_ij\rangle}\partial_q.
\end{aligned}$$

By Claim 4, we get

$$\begin{aligned}
[\varphi_f(x_p\partial_q), x_i\partial_j] &= \left[a_1^{\langle 0001\rangle}x_p\partial_q + \sum_{k=1}^{3} a_k^{(e_pq)}\partial_k, \quad x_i\partial_j\right]\\
&= a_1^{\langle 0001\rangle}\left(\delta_{q=i}x_p\partial_j - \delta_{p=j}x_i\partial_q\right) + a_i^{(e_pq)}\partial_j.
\end{aligned}$$

As a result, it is true that

$$a_1^{\langle 0001\rangle} = a_1^{(0001)}, \quad a_i^{(e_pq)} = -\delta_{q=j}a_p^{\langle e_ij\rangle}.$$

Then the last equation implies that $a_i^{(e_p q)}$ and $a_p^{\langle e_i j\rangle}$ are 0, which is because $i, j, p, q \in \{1, 2, 3\}$ are arbitrary. Consequently, we complete the proof of Claim 5.

Claim 6. Let $j \in \{1, 2, 3\}$ and $i_1, i_2, i_3 \in \{0, 1\}$. Then

$$\varphi_f(x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j) = \psi_f(x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j) = a_1^{(0001)} x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j.$$

It is sufficient to prove the case that there is at most one among i_1, i_2, i_3 being 0 by Claims 3 and 5. In the following, we fix $j, p, q \in \{1, 2, 3\}$ and make the convention that there is at most one among $i_1, i_2, i_3 \in \{0, 1\}$ being 0. It is true that

$$\begin{aligned} [\varphi_f(x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j), x_p \partial_q] &= [x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j, \psi_f(x_p \partial_q)], \\ [x_p \partial_q, \psi_f(x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j)] &= [\varphi_f(x_p \partial_q), x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j] \end{aligned}$$

from Lemma 2.2. By Claim 6, we get

$$\begin{aligned} [\varphi_f(x_p \partial_q), x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j] &= [a_1^{(0001)} x_p \partial_q, x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j], \\ [x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j, \psi_f(x_p \partial_q)] &= [x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j, a_1^{(0001)} x_p \partial_q)]. \end{aligned}$$

By Claims 3 and 4, we have the following two equations respectively,

$$\begin{aligned} [x_p \partial_q, \psi_f(x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j)] &= \left[x_p \partial_q, a_1^{(0001)} x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j + \sum_{k=1}^{3} a_k^{\langle i_1 i_2 i_3\rangle} \partial_k \right], \\ [\varphi_f(x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j), x_p \partial_q] &= \left[a_1^{(0001)} x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_j + \sum_{k=1}^{3} a_k^{(i_1 i_2 i_3)} \partial_k, x_p \partial_q \right]. \end{aligned}$$

As a result, we get

$$\left[\sum_{k=1}^{3} a_k^{(i_1 i_2 i_3)} \partial_k, x_p \partial_q \right] = \left[x_p \partial_q, \sum_{k=1}^{3} a_k^{\langle i_1 i_2 i_3\rangle} \partial_k \right] = 0.$$

Then it is true that

$$a_p^{(i_1 i_2 i_3)} = a_p^{\langle i_1 i_2 i_3\rangle} = 0.$$

Consequently, we complete the proof of Claim 6.

Claim 7. $f(x, y) = a_1^{(0001)}[x, y]$ for all $x, y \in W(3)$. That is, f is inner.

In fact, by Claim 6 and the linear property of φ_f or ψ_f, we get

$$\varphi_f(x) = a_1^{(0001)} x = \psi_f(x), \quad \forall x \in W(3).$$

This, together with Lemma 2.2, implies

$$f(x, y) = [\varphi_f(x), y] = [x, \psi_f(y)] = a_1^{(0001)}[x, y], \quad \forall x, y \in W(3).$$

Extend this method to $n \geq 3$, the proof is completed. ■

3. Linear Super-commuting Maps

An important result on linear (or additive) commuting maps is Posner's Theorem in 1957. Since then many scholars have studied commuting maps on various algebras. The commuting maps also are generalized to other similar forms (see [3]). In [13], the authors introduce the definition of linear super-commuting maps on Lie superalgebras. Let L be a Lie superalgebra. A linear map $\varphi : L \longrightarrow L$ is called a linear super-commuting map on L if

$$[\varphi(x), x] = 0, \quad \forall x \in L.$$

The following Theorem determines linear super-commuting maps on $W(n)$.

Theorem 3.1. *Every linear super-commuting map on $W(n)$ is a scalar map.*

Proof. Let φ be a linear super-commuting map on $W(n)$. Define a bilinear map

$$f : W(n) \times W(n) \longrightarrow W(n), \quad (x, y) \longmapsto [\varphi(x), y].$$

Then f is a super-biderivation on $W(n)$ (see [5]). Fix $x \in W(n)$. For any $y \in W(n)$, we get $f(x, y) = \lambda[x, y]$ from Theorem 2.3, where $\lambda \in \mathbb{C}$. This implies that $[\varphi(x), y] = \lambda[x, y]$, i.e., $[\varphi(x) - \lambda x, y] = 0$. In other words, $\varphi(x) - \lambda x$ lies in the center of $W(n)$ since y is arbitrary, which is 0. Hence, $\varphi(x) = \lambda x$ for all $x \in W(n)$. Conversely, if $\varphi(x) = \lambda x$ for all $x \in W(n)$, φ is a linear super-commuting map on $W(n)$. ■

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