Southeast Asian Bulletin of Mathematics (2022) 46: 141–150

Southeast Asian Bulletin of Mathematics
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Ball Convergence of an Efficient High Order Iterative Method for Solving Banach Valued Equations

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Received 12 June 2019
Accepted 25 August 2021

Communicated by Pengyee Lee

AMS Mathematics Subject Classification(2020): 65G99, 65H10, 49M15, 65J15

Abstract. The aim of this article is to extend the applicability of a multi-point iterative method for solving equations involving Banach space valued operators. The method uses evaluations of : two vector values, two linear operators, one inverse of a linear operator and one frozen inverse of a linear operator per iteration. The ball convergence was established in earlier works on the m-dimensional space and requiring very high order derivatives. We only use hypotheses on the first derivative. Hence, we extend the applicability of this method. Moreover, estimations are given on a radius of convergence and the error distances based on generalized Lipschitz conditions not given before. Numerical examples are used to complete this article.

Keywords: Multi-step iterative method; Banach space; Ball convergence; frozen linear operator.

1. Introduction

Let B_1, B_2 be Banach spaces and $\Omega \subseteq B_1$ be a nonempty, open and convex set.

By $LB(B_1, B_2)$, we denote the space of bounded linear operators from B_1 into B_2. A plethora of problems initiated from diverse disciplines can be converted to an equation like

$$F(x) = 0, \tag{1}$$

where $F : \Omega \to B_2$ is a continuously differentiable operator in the sense of Fréchet. Iterative methods are mostly used to approximate a solution x_* of equation (1), since closed form solutions can be obtained only in special cases. The most popular iterative method of convergence order two is Newtons [1, 2, 3, 4, 5, 6, 7, 8, 16, 17, 18] defined for each $n = 0, 1, 2, \cdots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \tag{2}$$

where $x_0 \in \Omega$ is an initial point. The convergence order has been increased by introducing higher convergence order iterative methods. Recently, the following multi-step iterative method was introduced on $\mathbb{R}^m$ in [9] as

$$\begin{aligned}
\varphi_n^1 &= x_n - F'(x_n)^{-1}F(x_n)\\
\varphi_n^2 &= x_n - 2(F'(x_n) + F'(\varphi_n^1))^{-1}F(x_n)\\
\varphi_n^3 &= \varphi_n^2 - MF(\varphi_n^2)\\
\varphi_n^4 &= \varphi_n^3 - MF(\varphi_n^3)\\
&\vdots\\
\varphi_n^{i-1} &= \varphi_n^{i-2} - MF(\varphi_n^{i-2})\\
\varphi_n^{i} &= \varphi_n^{i-1} - MF(\varphi_n^{i-1}),\ i = 3, 4, \ldots, k\\
\varphi_n^{k+1} &= \varphi_n^{k} - MF(\varphi_n^{k})
\end{aligned} \tag{3}$$

where $M = (3F'(\varphi_n^1) - F'(x_n))^{-1}(F'(x_n) + F'(\varphi_n^1))F'(x_n)^{-1}$ and k is a fixed natural number. Method (3) extends the method by [15] given on the real line

$$\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n)\\
z_n &= x_n - 2(F'(x_n) + F'(y_n))^{-1}F(x_n)\\
x_{n+1} &= z_n - (3F'(y_n) - F'(x_n))^{-1}(F'(x_n) + F'(y_n))F'(x_n)^{-1}F(z_n).
\end{aligned} \tag{4}$$

We study the ball convergence of method (3), but in the more general setting of Banach space valued operators. Notice that the ball convergence of method (3) was given using hypothesis up to the seventh derivative to establish the $3(i-1)$ convergence order. The usage of these high convergence order derivatives that do not appear in method (3) significantly limit the usage of method (3) (or method (4)).

Example 1.1. Define function F on $\Omega = [-\frac{1}{2}, \frac{3}{2})$ by

$$F(x) = \begin{cases} x^3 lnx^2 + x^5 - x^4 & \text{if } x \neq 0,\\ 0 & \text{if } x = 0. \end{cases} \tag{5}$$

We have $x^* = 1$. We also get that

$$\begin{aligned} F'(x) &= 3x^2 lnx^2 + 5x^4 - 4x^3 + 2x^2, \\ F''(x) &= 6xlnx^2 + 20x^3 + 12x^2 + 10x, \\ F'''(x) &= 6lnx^2 + 60x^2 - 24x + 22. \end{aligned}$$

Notice that $F'''(x)$ is unbounded on Ω.

Moreover, no radius of convergence, error estimations or uniqueness of the solution results are reported in [9, 15]. In the present article, we deal with these problems by using only hypothesis on the first Fréchet derivative. Furthermore, we provide computable radius of convergence, error estimate on $\|x_n - x_*\|$ and a uniqueness estimate utilizing generalized Lipschitz-type conditions. Concerning the convergence order, we used the computational order of convergence(COC) as well as the approximate computational order of convergence(ACOC) [10, 11, 12, 13, 14]. Notice that the computation of COC or ACOC do not require the usage of derivatives of order higher than one. Hence, we extend the applicability of method (3) (or (4)) and in the more general setting of a Banach space.

The layout of the rest of the article is set up as follows. The Ball convergence is presented in Section 2 and the numerical examples is given in Section 3.

2. Ball Convergence

Let us define some real valued functions and parameters to be used in the ball convergence analysis of method (3). Let $w_0 : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ be a continuous and increasing function such that $w_0(0) = 0$. Assume

$$w_0(t) = 1 \tag{6}$$

has a minimal positive solution ρ_1. Set $I_1 = [0, \rho_1)$. Let functions $w : I_1 \to \mathbb{R}$, $w_1 : I_1 \to \mathbb{R}$ be continuous and increasing such that $w(0) = 0$. Define functions g_1 and $\bar{g_1}$ on the interval I_1 by

$$\begin{aligned} g_1(t) &= \frac{\int_0^1 w((1-\tau)t)d\tau}{1 - w_0(t)}, \\ \bar{g_1}(t) &= g_1(t) - 1. \end{aligned}$$

We obtain $\bar{g_1}(0) = -1$ and $\bar{g_1}(t) \to \infty$ as $t \to \rho_1^-$. The intermediate value theorem assures the existence of at least one solution of equation $\bar{g_1}(t) = 0$ in $(0, \rho_1)$. Denote by R_1 the smallest such solution. Assume

$$p(t) = 1 \tag{7}$$

has a minimal positive solution ρ_p, where $p(t) = \frac{1}{2}(w_0(t) + w_0(g_1(t)))$. Set $I_2 = [0, \bar{\rho_1})$, where $\bar{\rho_1} = \min\{\rho_1, \rho_p\}$. Define functions g_2 and $\bar{g_2}$ on I_2 by

$$\begin{aligned} g_2(t) &= g_1(t) + \frac{(w_0(t) + w_0(g_1(t)t)) \int_0^1 w_1(\tau t)d\tau}{2(1 - w_0(t))(1 - p(t))}, \\ \bar{g_2}(t) &= g_2(t) - 1. \end{aligned}$$

We also get $\bar{g}_2(0) = -1$ and $\bar{g}_2(t) \to \infty$ as $\bar{\rho_1}^-$. Denote by R_2 the smallest solution of equation $\bar{g}_2(t) = 0$ in $(0, \bar{\rho_1})$. Assume

$$q(t) = 1 \tag{8}$$

has a minimal positive solution ρ_q, where $q(t) = \frac{1}{2}(w_0(t) + 3w_0(g_1(t)))$. Set $I_3 = [0, \bar{\rho_2})$, where $\bar{\rho_2} = \min \bar{\rho}_1, \rho_q$. Define functions g_3 and $\bar{g}_3$ on I_3 by

$$\begin{aligned} g_3(t) \;=\; & \Bigg[g_1(g_2(t)t) + \frac{1}{1 - w_0(g_2(t))} \\ & \times \left(w_1(t) + w_1(g_2(t)t)(1 + \frac{w_0(t) + w_0(g_1(t)t)}{1 - q(t)} \right) \frac{\int_0^1 w_1(\tau g_2(t)t)d\tau}{1 - w_0(t)} \Bigg] g_2(t), \\ \bar{g}_3(t) \;=\; & g_3(t) - 1. \end{aligned}$$

We get again $\bar{g}_3(0) = -1$ and $\bar{g}_3(t) \to \infty$ as $\bar{\rho_2}^-$. Denote by R_3 the smallest solution of equation $\bar{g}_3(t) = 0$ in $(0, \rho_2)$. Define functions g_i, $\bar{g}_i$, $i = 4, 5, \cdots, k+1$ on the interval I_3 by

$$g_i(t) \;=\; g_3^{i-1}(t) g_2(t), \quad \bar{g}_i(t) = g_i(t) - 1.$$

We get $\bar{g}_i(0) = -1$ and $\bar{g}_i(t) \to \infty$ as $t \to \bar{\rho_2}^-$. Denote by R_i the smallest positive solution of equation $\bar{g}_i(t) = 0$. Define a radius of convergence R by

$$R = \min\{R_j\}, \qquad j = 1, 2, \cdots, k+1. \tag{9}$$

By these definitions, we get for each $t \in [0, R)$

$$0 \le g_j(t) < 1, \tag{10}$$

$$0 \le p(t) < 1, \tag{11}$$

$$0 \le q(t) < 1. \tag{12}$$

Let $S(x, d) = \{y \in B_1 : \|y - x\| < d\}$ and let also $\bar{S}(x, d)$ stand for the closure of $S(x, d)$.

The hypotheses $(\mathcal{H})$ shall be used in the ball convergence of method (3):

$(\mathcal{H}_1)$ $F : \Omega \to B_2$ is a continuously differentiable operator in the sense of Fréchet, and there exists $x_* \in \Omega$ with $F(x_*) = 0$ and $F'(x_*)^{-1} \in LB(B_2, B_1)$.

$(\mathcal{H}_2)$ There exists function $w_0 : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ continuous and increasing with $w_0(0) = 0$ such that for each $x \in \Omega$,

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \le w_0(\|x - x_*\|).$$

Set $\Omega_0 = \Omega \cap S(x_*, \rho_1)$, where ρ_1 is given in (6).

$(\mathcal{H}_3)$ There exist functions $w : I_1 \to \mathbb{R}$, $w_1 : I_1 \to \mathbb{R}$ with $w_0(0) = 0$ such that for each $x, y \in \Omega_0$

$$\begin{aligned} \|F'(x_*)^{-1}(F'(y) - F'(x))\| \;&\le\; w(\|y - x\|), \\ \|F'(x_*)^{-1}F'(x)\| \;&\le\; w_1(\|x - x_*\|). \end{aligned}$$

($\mathcal{H}_4$) $\bar{S}(x_*, R) \subseteq \Omega, \rho_1, \rho_p, \rho_q$ given in (6)-(8), respectively exist and R is defined as in (9).

($\mathcal{H}_5$) There exists $R_* \geq R$ such that $\int_0^1 w_0(\tau R_*)d\tau < 1$. Set $\Omega_1 = \Omega \cap \bar{S}(x_*, R_*)$.

Next, the ball convergence of method (3) is provided under the conditions ($\mathcal{H}$) and the aforementioned notation.

Theorem 2.1. *Under the conditions* ($\mathcal{H}$) *for* $x_0 \in S(x_*, R) - \{x_*\}$, *the following items hold :* $\{x_n\} \subseteq S(x_*, R)$ *and* $\lim_{n\to\infty} x_n = x_*$. *Moreover,* x_* *is the only solution of equation* $F(x) = 0$ *in the set* Ω_1.

Proof. Let $x \in S(x_*, R) - \{x_*\}$. Then, by (6), ($\mathcal{H}_1$) and ($\mathcal{H}_2$), we have that

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq w_0(\|x - x_*\|) < w_0(R) < 1,$$

so $F'(x)^{-1} \in LB(B_2, B_1)$ and

$$\|F'(x)^{-1}F'(x_*)\| \leq \frac{1}{1 - w_0(\|x - x_*\|)} \tag{13}$$

(by Banach perturbation Lemma in [2, 17]). The point φ_0^1 is also well defined by the first substep of method (3) for $n = 0$, and

$$\begin{aligned}
\|\varphi_0^1 - x_*\| &= \|x_0 - x_* - F'(x_0)^{-1}F(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(x_*)\| \\
&\quad \|\int_0^1 F'(x_*)^{-1}(F'(x_* + \tau(x_0 - x_*)) - F'(x_0))d\tau(x_0 - x_*)\| \\
&\leq \frac{\int_0^1 w((1-\tau)\|x_0 - x_*\|)d\tau\|x_0 - x_*\|}{1 - w_0(\|x_0 - x_*\|)} \\
&= g_1(\|x_0 - x_*\|)\|x_0 - x_*\| \\
&\leq \|x_0 - x_*\| < R, \qquad (14)
\end{aligned}$$

by (9),(10) (for $j = 1$), ($\mathcal{H}$) and (13). We must show $(F'(x_0) + F'(\varphi_0^1))^{-1} \in LB(B_2, B_1)$, then φ_0^2 will be well defined by the second substep of mehod (3) for $n = 0$. In view of (9),(11) and ($\mathcal{H}_2$),

$$\begin{aligned}
&\|(2F'(x_*))^{-1}[(F'(x_0) - F'(x_*)) + (F'(\varphi_0^1) - F'(x_*))]\| \\
\leq\ & \frac{1}{2}[\|F'(x_*)^{-1}(F'(x_0) - F'(x_*))\| \\
&+\|F'(x_*)^{-1}(F'(\varphi_0^1) - F'(x_*))\|] \\
\leq\ & \frac{1}{2}[w_0(\|x_0 - x_*\|) + w_0(\|\varphi_0^1 - x_*\|)] \\
\leq\ & \frac{1}{2}[w_0(\|x_0 - x_*\|) + w_0(g_1(\|x_0 - x_*\|)\|x_0 - x_*\|)] \\
=\ & p(\|x_0 - x_*) < p(R) < 1,
\end{aligned}$$

so

$$\|(F'(x_0)+F'(\varphi_0^1))^{-1}F'(x_*)\| \le \frac{1}{2(1-p(\|x-x_*\|))}. \tag{15}$$

We can have by $(\mathcal{H}_1)$, and $(\mathcal{H}_3)$

$$F(x)=F(x)-F(x_*)=\int_0^1 F'(x_*+\tau(x-x_*))d\tau(x-x_*).$$

so

$$\|F'(x_*)^{-1}F(x)\| \le \int_0^1 w_1(\tau\|x-x_*\|)d\tau\|x-x_*\|. \tag{16}$$

By the second substep method (3), we can write

$$\begin{aligned}\varphi_0^2-x_* &= x_0-x_*-F'(x_0)^{-1}F(x_0)+(F'(x_0)^{-1}-2(F'(x_0)+F'(\varphi_0^1))^{-1}F(x_0)\\ &= x_0-x_*-F'(x_0)^{-1}F(x_0)+F'(x_0)^{-1}(F'(\varphi_0^1)-F'(x_k))\\ &\quad +(F'(x_k)-F'(x_0))(F'(x_0)+F'(\varphi_0^1))^{-1}F(x_0).\end{aligned} \tag{17}$$

Then, by (9),(10) (for $j=2$), (13) (for $x=x_0$) and (14)-(17), we get in turn that

$$\begin{aligned}&\|\varphi_0^2-x_*\|\\ \le&\Bigg[g_1(\|x_0-x_*\|)\\ &+\frac{(w_0(\|x_0-x_*\|)+w_0(\|\varphi_0^1-x_*\|)\int_0^1 w_1(\tau\|x_0-x_*\|)d\tau}{2(1-w_0(\|x_0-x_*\|))(1-p(\|x_0-x_*\|)))}\Bigg]\|x_0-x_*\|\\ \le& g_2(\|x_0-x_*\|)\|x_0-x_*\| \le \|x_0-x_*\| < R.\end{aligned} \tag{18}$$

Next, we must show $(3F'(\varphi_0^1)-F'(x_0))^{-1}\in LB(B_2,B_1)$. Using (9), (12), $(\mathcal{H}_2)$ and (14), obtain in turn that

$$\begin{aligned}&\|(2F'(x_*))^{-1}[3(F'(\varphi_0^1)-F'(x_*))-(F'(x_0)-F'x_*))]\|\\ \le&\frac{1}{2}[3\|F'(x_*0^{-1}(F'(\varphi_0^1)-F'(x_*))\|\\ &+\|F'(x_*)^{-1}(F'(x_0)-F'x_*))]\|\\ \le&\frac{1}{2}[3w_0(\|\varphi_0^1-x_*\|)+w_0(\|x_0-x_*\|)]\\ \le& g(\|x_0-x_*\|<q(R)<1,\end{aligned} \tag{19}$$

so

$$\|(3F'(\varphi_0^1)-F'(x_0))^{-1}F'(x_*)\| \le \frac{1}{2(1-q(\|x_0-x_*\|))}. \tag{20}$$

We can write by the third substep of method (3) for $n=0$

$$\begin{aligned}\varphi_0^3-x_* &= \varphi_0^2-x_*-F'(\varphi_0^2)^{-1}F(\varphi_0^2)+[F'(\varphi_0^2)^{-1}-(3F'(\varphi_0^2)-F'(x_0))^{-1}\\ &\quad (F'(x_0)+F'(\varphi_0^1))F'(x_0)^{-1}]F(\varphi_0^2)\\ &= \varphi_0^2-x_*-F'(\varphi_0^2)^{-1}F(\varphi_0^2)+F'(\varphi_0^2)^{-1}\\ &\quad [F'(x_0)-F'(\varphi_0^2)(I+2(3F'(\varphi_0^1)-F'(x_0))^{-1}(F'(x_0)\\ &\quad -F'(\varphi_0^1)F'(x_0)^{-1}F(\varphi_0^2)].\end{aligned} \tag{21}$$

Then, by (9), (10) (for $j = 3$), (13), (14), (18), (20) and (21), we get in turn that

$$\begin{aligned}
\|\varphi_0^3 - x_*\| \leq & \Bigg[g_1(\|\varphi_0^2 - x_*\|) \\
& + \frac{1}{1 - w_0(\|\varphi_0^2 - x_*\|)} \big[w_1(\|x_0 - x_*\|) + w_1(\|\varphi_0^2 - x_*\|) \\
& \left(1 + \frac{w_0(\|x_0 - x_*\|) + w_0(\|\varphi_0^1 - x_*\|)}{1 - q(\|x_0 - x_*\|)}\right) \\
& \times \frac{\int_0^1 w_1(\tau\|\varphi_0^2 - x_*\|)d\tau}{1 - w_0(\|x_0 - x_*\|)})\big]\Bigg] \|\varphi_0^2 - x_*\| \\
\leq & \; g_3(\|x_0 - x_*\|)\|\varphi_0^2 - x_*\| \\
\leq & \; g_3(\|x_0 - x_*)g_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < R. \quad (22)
\end{aligned}$$

Similarly to (22) by the fourth substep of method (3), we obtain

$$\begin{aligned}
\|\varphi_0^i - x_*\| \leq & \; g_3(\|x_0 - x_*\|)^{i-3}\|\varphi_0^3 - x_*\|, \\
\|\varphi_0^{k+1} - x_*\| \leq & \; g_3(\|x_0 - x_*\|)^{k-2}\|\varphi_0^3 - x_*\| \\
\leq & \; g_3^{k-2}(\|x_0 - x_*\|)g_3(\|x_0 - x_*\|)g_2(\|x_0 - x_*\|)\|x_0 - x_*\|, \\
\|x_1 - x_*\| \leq & \; g_k(\|x_0 - x_*\|)\|x_0 - x_*\|, \\
& \vdots \\
\|x_{n+1} - x_*\| \leq & \; g_k^{n+1}(\|x_0 - x_*\|)\|x_0 - x_*\| = c\|x_0 - x_*\| < R
\end{aligned}$$

by replacing $\varphi_0^1, \varphi_0^2, \cdots, \varphi_0^k$ by $\varphi_m^1, \varphi_m^2, \cdots, \varphi_m^k$ in the preceding estimations, so $\lim_{n\to\infty} x_n = x_*$, and $x_{n+1} \in S(x_*, R)$ since $c = g_k^{n+1}(\|x_0 - x_*\|) \in [0, 1]$. Let $Q = \int_0^1 F'(x_* + \tau(y_* - x_*))d\tau$ for $y_* \in \Omega_1$ with $F(y_*) = 0$. Then, by $(\mathcal{H}_2)$ and $(\mathcal{H}_5)$, we get that

$$\|F'(x_*)^{-1}(Q - F'(x_*))\| \leq \int_0^1 w_0(\tau\|y_* - x_*\|)d\tau \leq \int_0^1 w_0(\tau R_*)d\tau < 1, \quad (23)$$

so $Q^{-1} \in LB(B_2, B_1)$, and by the identity $0 = F(y_*) - F(x_*) = Q(y_* - x_*)$, we deduce that $x_* = y_*$. ∎

Remark 2.2.

(a) Let $w_0(t) = L_0 t$ and $w(t) = Lt$. Then the radius $r_1 = \frac{2}{2L_0 + L}$ was obtained by Argyros in [2] as a convergence radius for Newton's method under condition (17)–(19). Notice that the convergence radius for Newton's method given independently by Rheinboldt [17] and Traub [18] is given by

$$\rho = \frac{2}{3L_1} < r_1,$$

where L_1 is the Lipschitz constant on Ω. We have $L_0 < L_1$ and $L < L_1$ As an example, let us consider the function $f(x) = e^x - 1$. Then $x^* = 0$. Set $D = U(0,1)$. Then, we have that $L_0 = e - 1 < L = e^{\frac{1}{e-1}} < L_1 = e$, so $\rho = 0.24252961 < r_1 = 0.3827$.
Moreover, the new error bounds [2] are:

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones [17, 18]

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L_1$ or $L < L_1$. Clearly, the radius of convergence of method (2) given by R cannot be to be larger than r_1.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [4, 5, 6, 7].

(c) The results can be also be used to solve equations where the operator F' satisfies the autonomous differential equation [2, 3, 4, 5, 6, 7, 8]:

$$F'(x) = p(F(x)),$$

where p is a known continuous operator. Since $F'(x^*) = p(F(x^*)) = p(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $p(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (2) are not changing if we use the new instead of the old conditions [9, 15]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{ln\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \quad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{ln\frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \quad \text{for each } \ n = 0, 1, 2, \ldots$$

instead of the error bounds obtained in Theorem 2.1.

(e) In view of (13) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\|\end{aligned}$$

condition (15) can be dropped and w_1 can be replaced by

$$w_1(t) = 1 + w_0(t).$$

3. Numerical Examples

Example 3.1. Let $B_1 = B_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $T = \overline{U}(0,1)$. Define function F on T by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{24}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in T.$$

Then, we get that $x^* = 0$, $w_0(t) = 7.5t$, $w(t) = 15t$, $w_1(t) = 15$. This way, we have that

$$R_1 = 0.0666667, R_2 = 0.00954591, R_3 = 0.0000342244 = R.$$

Example 3.2. Returning back to the motivation example at the introduction on this paper, we have $w_0(t) = w(t) = 96.662907t$, $w_1(t) = 1.0631$. Then, the parameters for method (2) are

$$R_1 = 0.00689682, R_2 = 0.00039486, R_3 = 0.0000890418 = R.$$

Example 3.3. Let $B_1 = B_2 = \mathbb{R}^3$, $\Omega = S(0,1), x^* = (0,0,0)^T$ and define F on T by

$$F(x) = F(x_1, x_2, x_3) = (e^{x_1} - 1, \frac{e-1}{2}{x_2}^2 + x_2, x_3)^T. \tag{25}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$F'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, $F'(x^*) = diag(1,1,1)$, we have $w_0(t) = (e-1)t, w(t) = e^{\frac{1}{e-1}}t, w_1(t) = e^{\frac{1}{e-1}}$.

Then, we obtain that

$$R_1 = 0.382692, R_2 = 0.180065, R_3 = 0.0401757 = R.$$

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