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# Rokhlin Property for Group Actions on Hilbert $C^*$-modules*

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**Abstract.** We introduce Rokhlin properties for certain discrete group actions on $C^*$-correspondences as well as on Hilbert bimodules and analyze them. It turns out that the group actions on any $C^*$- correspondence $E$ with Rokhlin property induces group actions on the associated $C^*$-algebra $\mathcal{O}_E$ with Rokhlin property and the group actions on any Hilbert bimodule with Rokhlin property induces group actions on the linking algebra with Rokhlin property. Permanence properties of several notions such as the nuclear dimension and the $\mathcal{D}$-absorbing property with respect to the crossed product

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of Hilbert $C^*$-modules with groups, where group actions have Rokhlin property, are studied.

**Keywords:** Crossed products; Cuntz-Pimsner algebra; Group action; Hilbert $C^*$- modules; Linking algebras; Nuclear dimension; Rokhlin property; Self-absorbing $C^*$- algebras.

## 1. Introduction

Kishimoto in [20] showed that the reduced crossed product of a simple $C^*$-algebra, with respect to an outer action of a discrete group, is simple. Herman and Ocneanu in [10] defined the Rokhlin property of finite group actions on $C^*$-algebras in terms of projections and this notion is stronger than the outerness. The modern definition of the Rokhlin property of finite group actions on $C^*$-algebras is due to Izumi (see [12]). Several classes including AF-algebras, AI-algebras, AT-algebras are closed under finite group actions with the Rokhlin property (cf. [24]). We begin Section 2 with Santiago's definition for finite group actions on not necessarily unital $C^*$-algebras from [32] and list there some classes that are preserved under crossed products by finite group actions with this Rokhlin property.

In [29], from $C^*$-correspondences Pimsner constructed $C^*$-algebras which are known as Cuntz-Pimsner algebras. The class of Cuntz-Pimsner algebras includes Cuntz algebras, Cuntz-Krieger algebras and the crossed products by $\mathbb{Z}$. For every $C^*$- correspondence $(E, \mathcal{A}, \phi)$, Katsura in [16] defined a $C^*$-algebra $\mathcal{O}_E$. The algebra $\mathcal{O}_E$ is the same as the Cuntz-Pimsner algebra when $\phi$ is injective. The algebras $\mathcal{O}_E$ also generalize the crossed product, as defined in [1], by Hilbert bimodules and graph algebras (cf. [8]). The graph $C^*$-algebras of topological graphs can also be realized as $\mathcal{O}_E$ for some $C^*$-correspondence $(E, \mathcal{A}, \phi)$ (see [17] for details).

In Subsection 3.1 we recall the definition compatible action, $(\eta, \alpha)$, of a locally compact group on a $C^*$-correspondence. Hao and Ng in [9] proved that each compatible action of a locally compact group $G$ on a $C^*$-correspondence $(E, \mathcal{A}, \phi)$ induces an action of $G$ on the associated $C^*$-algebra $\mathcal{O}_E$. It is of interest to determine at least the sufficient conditions under which for a compatible action $(\eta, \alpha)$ of $G$, the permanence property with respect to the crossed product is exhibited by several notions related to this $C^*$-algebra, associated to the Hilbert module. We define the Rokhlin property for finite group actions on $C^*$-correspondences in Subsection 3.2 and provide an answer of the above question regarding sufficient conditions in Subsection 3.4 when group $G$ is finite.

For any class of unital and separable $C^*$-algebras $\mathcal{C}$, Osaka and Phillips in [24] introduced the notion of local $\mathcal{C}$-algebra. Santiago in [32] extended this notion by considering non-unital $C^*$-algebras. The notion of closed under local approximation (cf. [32, Definition 3]) is defined in terms of local $\mathcal{C}$-algebra. If $\mathcal{C}$ denotes certain class of $C^*$-algebras such as purely infinite $C^*$-algebras, separable

$\mathcal{D}$-absorbing $C^*$-algebras etc. listed in Theorem 2.3, then $\mathcal{C}$ is closed under local approximation and under crossed product with a finite group action with the Rokhlin property. As an application of the observation made in Subsection 3.3 we show that if an action $(\eta, \alpha)$ of a finite group $G$ on $(E, \mathcal{A}, \phi)$ has the Rokhlin property and $\mathcal{O}_E$ belongs to one of the classes mentioned above, then $\mathcal{O}_{E\times_\eta G}$ belongs to the same class (see Corollary 3.9). At the end of Section 3, we point out that the gauge action on the graph $C^*$-algebra is saturated by [14, Theorem 6.3], but the corresponding action on the $C^*$-correspondence does not have the Rokhlin property.

We introduce, in Section 4, the Rokhlin property for compatible finite group actions on Hilbert bimodules and prove the following: If we realize a Hilbert $\mathcal{A}$-module $E$ as a Hilbert $\mathcal{K}(E)$-$\mathcal{A}$ bimodule, and if $\mathcal{A}$ belongs to a class $\mathcal{C}$ in the previous paragraph and if a group action $\eta$ of a finite group $G$ on the Hilbert $\mathcal{A}$-module $E$ has the Rokhlin property as a certain compatible action on the bimodule, then the linking algebra of the crossed product Hilbert $\mathcal{A} \times_\alpha G$-module $E \times_\eta G$ belongs to the class $\mathcal{C}$. To obtain this result we first prove that any compatible action of a finite group $G$ with the Rokhlin property on a Hilbert bimodule induces an action of $G$ with the Rokhlin property on the linking algebra. We introduce the Rokhlin property for compatible $\mathbb{Z}$-actions on Hilbert bimodules and take a similar approach as in the previous section to analyse such actions of $\mathbb{Z}$.

## 2. Rokhlin Property for Finite Group Actions on $C^*$-Algebras

In this section, first we recall the definition of the Rokhlin property for finite group actions on a $C^*$-algebra, from [32], which involves positive contractions. The Rokhlin property for finite group actions on $C^*$-algebras was studied by several authors in [12, 23, 24, 28] and [32], and it was proved that many classes of $C^*$-algebras are preserved under thecrossed product when the action of the group has the Rokhlin property.

**Definition 2.1.** [32] *Let $\alpha : G \to Aut(\mathcal{A})$ be an action of a finite group $G$ on a $C^*$-algebra $\mathcal{A}$. We say that $\alpha$ has the* Rokhlin property *if for any $\epsilon > 0$ and every finite subset $S$ of $\mathcal{A}$ there exist orthogonal positive contractions $(f_g)_{g\in G} \subset \mathcal{A}$ satisfying*

(i) $\|(\sum_{g\in G} f_g)a - a\| < \epsilon$ *for all* $a \in S$,
(ii) $\|\alpha_h(f_g) - f_{hg}\| < \epsilon$ *for* $h, g \in G$,
(iii) $\|[f_g, a]\| < \epsilon$ *for* $g \in G$ *and* $a \in S$.

*The elements $(f_g)_{g\in G}$ are called* Rokhlin elements *for $\alpha$.*

*Remark 2.2.* When $\mathcal{A}$ is unital, $\alpha$ has the Rokhlin property in the sense of Izumi (see [12]). That is, we can take a partition of unity $(e_g)_{g\in G}$ consisting of projections in place of $(f_g)_{g\in G}$ (see [32, Corollary 1]).

The following notions are borrowed from [24] and [32]: Let $\mathcal{C}$ be a class of $C^*$-algebras. A *local* $\mathcal{C}$*-algebra* is a $C^*$-algebra $\mathcal{A}$ such that for every finite set $S \subset \mathcal{A}$ and every $\epsilon > 0$, there exists a $C^*$-algebra $\mathcal{B}$ in $\mathcal{C}$ and a $*$-homomorphism $\pi\colon \mathcal{B} \to \mathcal{A}$ such that $dist(a, \pi(\mathcal{B})) < \epsilon$ for all $a \in S$. We say that $\mathcal{C}$ is closed under *local approximation* if every local $\mathcal{C}$-algebra belongs to $\mathcal{C}$.

We recall the following result from [32] (cf. [25, 26, 24, 28, 11, 19] and [13]).

**Theorem 2.3.** [32] *The following classes are closed under local approximation and under crossed product with finite group actions with the Rokhlin property, respectively:*

(i) *purely infinite $C^*$-algebras,*
(ii) *$C^*$-algebras having stable rank one,*
(iii) *$C^*$-algebras with real rank zero,*
(iv) *$C^*$-algebras of nuclear dimension at most $n$, where $n \in \mathbb{Z}_+$,*
(v) *separable $\mathcal{D}$-absorbing $C^*$-algebras where $\mathcal{D}$ is a strongly self-absorbing $C^*$-algebra,*
(vi) *simple $C^*$-algebras,*
(vii) *simple $C^*$-algebras that are stably isomorphic to direct limits of sequences of $C^*$-algebras, in a class $\mathfrak{S}$, where $\mathfrak{S}$ is a class of finitely generated semiprojective $C^*$-algebras that is closed under taking tensor products by matrix algebras over $\mathbb{C}$,*
(viii) *separable AF-algebras,*
(ix) *separable simple $C^*$-algebras that are stably isomorphic to AI-algebras,*
(x) *separable simple $C^*$-algebras that are stably isomorphic to AT-algebras,*
(xi) *$C^*$-algebras that are stably isomorphic to sequential direct limits of one-dimensional noncommutative CW-complexes,*
(xii) *separable $C^*$-algebras whose quotients are stably projectionless,*
(xiii) *simple stably projectionless $C^*$-algebras,*
(xiv) *separable $C^*$-algebras with almost unperforated Cuntz semigroup,*
(xv) *simple $C^*$-algebras with strict comparison of positive elements,*
(xvi) *separable $C^*$-algebras whose closed two-sided ideals are nuclear and satisfy the Universal Coefficient Theorem.*

## 3. Rokhlin Property for Finite Group Actions on $C^*$-Correspondences

In this section we define and explore the Rokhlin property for a compatible group action on a $C^*$-correspondence when the group is finite.

### 3.1. $C^*$-Correspondence

Let $E$ be a vector space which is a right module over a $C^*$-algebra $\mathcal{A}$ and satisfying $\alpha(xa) = (\alpha x)a = x(\alpha a)$ for $x \in E, a \in \mathcal{A}, \alpha \in \mathbb{C}$. The module $E$ is called an *(right) inner-product $\mathcal{A}$-module* if there exists a map $\langle \cdot, \cdot \rangle_{\mathcal{A}} : E \times E \to \mathcal{A}$ such that

(i) $\langle x, x \rangle_{\mathcal{A}} \geq 0$ for $x \in E$ and $\langle x, x \rangle_{\mathcal{A}} = 0$ only if $x = 0$,
(ii) $\langle x, ya \rangle_{\mathcal{A}} = \langle x, y \rangle_{\mathcal{A}} a$ for $x, y \in E$ and for $a \in \mathcal{A}$,
(iii) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^*$ for $x, y \in E$,
(iv) $\langle x, \mu y + \nu z \rangle_{\mathcal{A}} = \mu \langle x, y \rangle_{\mathcal{A}} + \nu \langle x, z \rangle_{\mathcal{A}}$ for $x, y, z \in E$ and for $\mu, \nu \in \mathbb{C}$.

An (right) inner-product $\mathcal{A}$-module $E$ is called *(right) Hilbert $\mathcal{A}$-module* or *(right) Hilbert $C^*$-module over $\mathcal{A}$* if it is complete with respect to the norm

$$\|x\| := \|\langle x, x \rangle_{\mathcal{A}}\|^{1/2} \text{ for } x \in E.$$

If there is no ambiguity, we simply write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_{\mathcal{A}}$. The notion of Hilbert $C^*$-module was introduced independently by Paschke in [27] and Rieffel in [31]. In [15], Kasparov used Hilbert $C^*$-modules as a tool to study a bivariate K-theory for $C^*$-algebras. Below we define the notion of $C^*$-correspondences which will play an important role in this article.

**Definition 3.1.** *Let $\mathcal{A}$ be a $C^*$-algebra. A right Hilbert $\mathcal{A}$-module $E$ is called a $C^*$-correspondence over $\mathcal{A}$ if there exists a $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}^a(E)$ where $\mathcal{B}^a(E)$ is the set of all adjointable operators on $E$, which gives a left action of $\mathcal{A}$ on $E$ as*

$$ay := \phi(a)y \text{ for all } a \in \mathcal{A}, y \in E.$$

*We use notation $(E, \mathcal{A}, \phi)$ for the $C^*$-correspondence and denote by $\mathcal{K}(E)$ the $C^*$-algebra generated by maps $\{\theta_{x,y} : x, y \in E\}$ defined by*

$$\theta_{x,y}(z) := x\langle y, z \rangle \text{ for } x, y, z \in E.$$

We recall the following definition of a certain type of action of a locally compact group on a $C^*$-correspondence from [9]:

**Definition 3.2.** *Let $(G, \alpha, \mathcal{A})$ be a $C^*$-dynamical system of a locally compact group $G$ and let $(E, \mathcal{A}, \phi)$ be a $C^*$-correspondence over $\mathcal{A}$. An $\alpha$-compatible action $\eta$ of $G$ on $E$ is defined as a group homomorphism from $G$ into the group of invertible linear transformations on $E$ such that*

(i) $\eta_g(\phi(a)x) = \alpha_g(a)\eta_g(x)$ *for* $a \in \mathcal{A},\ x \in E,\ g \in G$,
(ii) $\langle \eta_g(x), \eta_g(y) \rangle = \alpha_g(\langle x, y \rangle)$ *for* $x, y \in E,\ g \in G$,

*and $g \mapsto \eta_g(x)$ is continuous from $G$ into $E$ for each $x \in E$. We denote an $\alpha$-compatible action $\eta$ by $(\eta, \alpha)$. In this case, we define a $*$-isomorphism $Ad\eta_s : \mathcal{B}^a(E) \to \mathcal{B}^a(E)$ for each $s \in G$ by*

$$Ad\eta_s(T)(x) := \eta_s(T(\eta_{s^{-1}}(x)) \text{ for } T \in \mathcal{B}^a(E),\ x \in E. \tag{1}$$

Let $G$ be a locally compact group and $\triangle$ be the modular function of $G$. Let $\eta$ be an $\alpha$-compatible action of $G$ on the $C^*$-correspondence $(E, \mathcal{A}, \phi)$. Then the *crossed product* $E \times_\eta G$ (cf. [15, 7] and [9]) is a Hilbert $\mathcal{A} \times_\alpha G$-module and is defined as the completion of an inner-product $C_c(G, \mathcal{A})$-module $C_c(G, E)$ where the module action and the $C_c(G, \mathcal{A})$-valued inner-product are given by

$$l \cdot g(s) = \int_G l(t)\alpha_t(g(t^{-1}s))dt, \tag{2}$$

$$\langle l, m\rangle_{C_c(G,\mathcal{A})}(s) = \int_G \alpha_{t^{-1}}(\langle l(t), m(ts)\rangle_{\mathcal{A}})dt, \tag{3}$$

respectively for $g \in C_c(G, \mathcal{A})$ and $l, m \in C_c(G, E)$. For each $s \in G$ the $*$-isomorphism $Ad\eta_s$ defined by equation (1) satisfies $Ad\eta_s(\theta_{x,y}) = \theta_{\eta_s(x),\eta_s(y)}$ for $x, y \in E$, and we obtain a $C^*$-dynamical system $(G, Ad\eta, \mathcal{K}(E))$. From Definition 3.2(i) it follows that $\phi : \mathcal{A} \to M(\mathcal{K}(E))$ is equivariant, i.e.,

$$\phi(\alpha_s(a)) = Ad\eta_s(\phi(a)) \text{ for all } a \in \mathcal{A}, s \in G.$$

Indeed, using $Ad\eta_s$ we get another $*$-isomorphism $\Xi : \mathcal{K}(E \times_\eta G) \to \mathcal{K}(E) \times_{Ad\eta} G$ (cf. [15, Section 3.11] and [9, Section 2]) defined by

$$\Xi(\theta_{l,m})(s) := \int_G \theta_{l(r),Ad\eta_s(m(s^{-1}r))} \triangle (s^{-1}r)dr \text{ where } l, m \in C_c(G, E),\ s \in G.$$

From the fact that $\phi : \mathcal{A} \to M(\mathcal{K}(E))$ is equivariant we get an equivariant $*$-homomorphism $\chi : \mathcal{A} \times_\alpha G \to M(\mathcal{K}(E) \times_{Ad\eta} G)$ satisfying $\chi(f \otimes a) = f \otimes \phi(a)$ for $f \in C_c(G)$, $a \in \mathcal{A}$. We identify $\mathcal{K}(E \times_\eta G)$ with $\mathcal{K}(E) \times_{Ad\eta} G$ and treat $\chi$ and $\Xi^{-1} \circ \chi$ as same.

### 3.2. Rokhlin Property for Compatible Finite Group Actions on $C^*$-Correspondences

**Definition 3.3.** *Let $(G, \alpha, \mathcal{A})$ be a $C^*$-dynamical system of a finite group $G$ on $\mathcal{A}$ and let $(E, \mathcal{A}, \phi)$ be a $C^*$-correspondence. Let $(\eta, \alpha)$ be an $\alpha$-compatible action of $G$ on $E$. Then we say that $\eta$ has the Rokhlin property if for each $\epsilon > 0$, and finite subsets $S_1$ and $S_2$ of $E$ and $\mathcal{A}$ respectively, there exists $(a_g)_{g\in G} \subset \mathcal{A}$ consisting of mutually orthogonal positive contractions such that*

(i) $\|\sum_{g\in G} \phi(a_g)x - x\| < \epsilon$, $\|\sum_{g\in G} xa_g - x\| < \epsilon$, $\|\sum_{g\in G} a_g a - a\| < \epsilon$ *and* $\|\sum_{g\in G} aa_g - a\| < \epsilon$ *for all* $x \in S_1$, $a \in S_2$,

(ii) $\|\alpha_h(a_g) - a_{hg}\| < \epsilon$ *for* $h, g \in G$,

(iii) $\|xa_g - \phi(a_g)x\| < \epsilon$ *and* $\|a_g a - aa_g\| < \epsilon$ *for all* $x \in S_1$, $a \in S_2$ *and* $g \in G$.

The following example is based on the construction of an action of $\mathbb{Z}_2$ on $C_0(X)$ where $X$ is equipped with a homeomorphism of order 2 defined on it:

*Example 3.4.* Let $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ and the topology on $X$ be discrete. Define a map $\psi : X \to X$ by

$$\psi(1/(2n-1)) := 1/(2n),\ \psi(1/(2n)) := 1/(2n-1) \text{ for all } n \in \mathbb{N}.$$

Observe that $\psi$ is a homeomorphism of order 2. Thus we obtain an automorphism $\alpha : C_0(X) \to C_0(X)$ such that

$$\alpha(g)(x) := g(\psi^{-1}(x)) \text{ for each } x \in X,\ g \in C_0(X).$$

Indeed, $\alpha^2 = id_{C_0(X)}$ and this provides an action of $\mathbb{Z}_2$ on $C_0(X)$ which we denote by $\alpha$. Let $H$ be a Hilbert space and let $C_0(X, H)$ be the space of continuous $H$-valued functions on $X$ vanishing at infinity. The space $C_0(X, H)$ becomes a Hilbert $C_0(X)$-module where module action and inner product are defined as follow:

$$f \cdot g(x) := g(x)f(x);\ \langle f, f'\rangle(x) := \langle f(x), f'(x)\rangle$$

for all $f, f' \in C_0(X, H),\ g \in C_0(X)$. In fact, $C_0(X, H)$ becomes a $C^*$- correspondence over $C_0(X)$ with the left action $\phi$ defined by

$$\phi(g)f := f \cdot g \text{ for all } f \in C_0(X, H),\ g \in C_0(X).$$

Define $\eta : C_0(X, H) \to C_0(X, H)$ by

$$\eta(f)(x) := f(\psi^{-1}(x)) \text{ for all } x \in X,\ f \in C_0(X, H).$$

It follows that $\langle \eta(f), \eta(f')\rangle = \alpha(\langle f, f'\rangle)$ for all $f, f' \in C_0(X, H)$. Moreover, $\eta^2 = id_{C_0(X,H)}$ and hence we get an induced $\alpha$-compatible $\mathbb{Z}_2$ action on $(C_0(X, H), C_0(X), \phi)$, namely $(\eta, \alpha)$. The action $(\eta, \alpha)$ has the Rokhlin property in the sense of Definition 3.3: Let $a_n^{(0)}, a_n^{(1)} \in C_0(X)$ be the characteristic functions of the sets $\{1/(2k-1) : 1 \le k \le n\}$ and $\{1/(2k) : 1 \le k \le n\}$, respectively for each $n \in \mathbb{N}$. Note that these functions are continuous (because the given sets are open), $\alpha(a_n^{(0)}) = a_n^{(1)}$, and $(a_n^{(0)} + a_n^{(1)})_{n\in\mathbb{N}}$ is an approximate unit for $\mathrm{C}_0(X)$. It is clear that $a_n^{(0)}$ and $a_n^{(1)}$ are orthogonal. If $(e_n)_{n\in\mathbb{N}}$ is an approximate unit for a $C^*$-algebra $\mathcal{A}$ and $E$ is a Hilbert $\mathcal{A}$-module, then $(xe_n)_{n\in\mathbb{N}}$ converges to $x$ for each $x \in E$. Hence $(\eta, \alpha)$ has the Rokhlin property, for $C_0(X)$ is commutative.

For any subset $S$ of a $C^*$-algebra, we use symbol $S^*$ to denote the set $\{x^* : x \in S\}$.

*Example 3.5.* Let $l^2(\mathcal{A})$ be the direct sum of a countable number of copies of a $C^*$-algebra $\mathcal{A}$. The vector space $l^2(\mathcal{A})$ is known as the standard Hilbert $C^*$-module where the right $\mathcal{A}$-module action and the $\mathcal{A}$-valued inner-product is given by

$$(a_1, a_2, \ldots, a_n, \ldots)a := (a_1a, a_2a, \ldots, a_na, \ldots),$$

$$\langle (a_1, a_2, \ldots, a_n, \ldots), (a'_1, a'_2, \ldots, a'_n, \ldots)\rangle := \sum_{i=1}^{\infty} a_i^* a'_i$$

for all $a, a_1, a_1', a_2, a_2' \ldots \in \mathcal{A}$. It is easy to note that $(l^2(\mathcal{A}), \mathcal{A}, \phi)$ is a $C^*$-correspondence where the adjointable left action $\phi : \mathcal{A} \to \mathcal{B}^a(l^2(\mathcal{A}))$ is defined as

$$\phi(a)(a_1, a_2, \ldots, a_n, \ldots) = (aa_1, aa_2, \ldots, aa_n, \ldots) \text{ for all } a, a_1, a_2, \ldots \in \mathcal{A}.$$

Let $(G, \alpha, \mathcal{A})$ be a finite group action. Define $\eta : G \to Aut\ l^2(\mathcal{A})$ by

$$\eta_t(a_1, a_2, \ldots, a_n, \ldots) := (\alpha_t(a_1), \alpha_t(a_2), \ldots, \alpha_t(a_n), \ldots)$$

where $t \in G$ and $(a_1, a_2, \ldots, a_n, \ldots) \in l^2(\mathcal{A})$. It is clear that $\eta$ is an $\alpha$-compatible action of the group $G$ on $(l^2(\mathcal{A}), \mathcal{A}, \phi)$.

Next we show that if $\alpha$ has the Rokhlin property, then $\eta$ has the Rokhlin property as an $\alpha$-compatible action of the group $G$ on the $C^*$-correspondence $(l^2(\mathcal{A}), \mathcal{A}, \phi)$.

Let $\epsilon > 0$ and let $S_1 = \{(a_1^j, a_2^j, \ldots, a_n^j, \ldots) : j = 1, 2, \ldots, N\}$ and $S_2$ be finite subsets of $l^2(\mathcal{A})$ and $\mathcal{A}$ respectively. Then for each $j$, there exist positive integers $N^j$ such that $\|\sum_{n>N^j} a_n^{j*} a_n^j\|^{\frac{1}{2}} < \frac{\epsilon}{2(|G|^2+2|G|+1)}$. Fix $S_1' := \{a_n^j : n \leq N^j, 1 \leq j \leq N\}$ and let $K = (\max_j N^j) + 1$. Assume that $\alpha$ has the Rokhlin property for the finite set $S = S_1' \cup S_1'^* \cup S_2 \cup S_2^*$, i.e., we get Rokhlin elements $\{f_g : g \in G\}$ consist of mutually orthogonal positive contractions in $\mathcal{A}$ satisfying the following:

(i) $\|(\sum_{g \in G} f_g)a - a\| < \frac{\epsilon}{2K}$ for all $a \in S$,
(ii) $\|\alpha_h(f_g) - f_{hg}\| < \frac{\epsilon}{2K}$ for $h, g \in G$,
(iii) $\|[f_g, a]\| < \frac{\epsilon}{2K}$ for $g \in G$ and $a \in S$.

Now we check that the action $(\eta, \alpha)$ has the Rokhlin property w.r.t. $S_1$ and $S_2$, with Rokhlin elements $\{a_g : g \in G\}$ where $a_g := f_g$ for each $g \in G$. Note that

$$\begin{aligned}
&\left\|\sum_{g \in G} \phi(a_g)(a_1^j, a_2^j, \ldots, a_n^j, \ldots) - (a_1^j, a_2^j, \ldots, a_n^j, \ldots)\right\| \\
=&\left\|\left(\sum_{g \in G} f_g a_1^j - a_1^j, \sum_{g \in G} f_g a_2^j - a_2^j, \ldots, \sum_{g \in G} f_g a_n^j - a_n^j, \ldots\right)\right\| \\
<&\sum_{n=1}^{N^j} \frac{\epsilon}{2K} + \frac{\epsilon}{2} < \epsilon,
\end{aligned}$$

$$\begin{aligned}
&\|(a_1^j, a_2^j, \ldots, a_n^j, \ldots)a_g - \phi(a_g)(a_1^j, a_2^j, \ldots, a_n^j, \ldots)\| \\
=&\|(a_1^j a_g - a_g a_1^j, a_2^j a_g - a_g a_2^j, \ldots, a_n^j a_g - a_g a_n^j, \ldots)\| \\
<&\sum_{n=1}^{N^j} \|a_n^j f_g - f_g a_n^j\| + \frac{\epsilon}{2} \\
<&\sum_{n=1}^{N^j} \frac{\epsilon}{2K} + \frac{\epsilon}{2} < \epsilon \quad \text{for all } (a_1^j, a_2^j, \ldots, a_n^j, \ldots) \in S_1 \text{ and } g \in G.
\end{aligned}$$

This verifies some conditions of the definition of the Rokhlin property for $(\eta, \alpha)$. It is easy to check the other conditions of the definition of the Rokhlin property also holds for $(\eta, \alpha)$.

In the above two examples the left actions in the $C^*$-correspondences are non- degenerate. When the left actions are non-degenerate, Definition 3.3 can be simplified as stated in the following prop:

**Proposition 3.6.** *Let $(G, \alpha, \mathcal{A})$ be a $C^*$-dynamical system of a finite group $G$ on $\mathcal{A}$ and let $(E, \mathcal{A}, \phi)$ be a $C^*$-correspondence. Suppose that $(\eta, \alpha)$ is an $\alpha$-compatible action of $G$ on $E$ and $\phi$ is non-degenerate, that is, $\overline{\text{span}\{\phi(\mathcal{A})E\}}$ is dense in $E$* (see [16, Definition 1.4]). *Then the action $\eta$ has the Rokhlin property if and only if for each $\varepsilon > 0$, and finite subsets $F_1$ and $F_2$ of $E$ and $\mathcal{A}$, respectively, there exists $(a_g)_{g\in G} \subset \mathcal{A}$ consisting of mutually orthogonal positive contractions such that*

(i) $\|\sum_{g\in G} a_g a - a\| < \varepsilon$ *for all* $a \in F_1$
(ii) $\|\alpha_h(a_g) - a_{hg}\| < \varepsilon$ *for* $h, g \in G$,
(iii) $\|x a_g - \phi(a_g)x\| < \varepsilon$ *and* $\|a_g a - a a_g\| < \varepsilon$ *for all* $x \in F_1$, $a \in F_2$ *and* $g \in G$.

*Proof.* It is enough to show the "only if" part. Let $\epsilon > 0$, and $S_1$ and $S_2$ be finite subsets of $E$ and $\mathcal{A}$, respectively. Without loss of generality we assume that $S_1 = \{x\}$. Since $\phi$ is non-degenerate, there exist $\{a_i\}_{i=1}^n \subset \mathcal{A}$ and $\{x_i\}_{i=1}^n \subset E$ such that $\|x - \sum_{i=1}^n \phi(a_i)x_i\| < \varepsilon$. Set $F_1 = S_1 \cup \{x_i\}_{i=1}^n$ and $F_2 = S_2 \cup S_2^* \cup \{a_i\}_{i=1}^n$, and $\varepsilon = \frac{\epsilon}{4n|G|M}$, where $M = \max_{1\le i\le n}\{1, \|x_i\|\}$. So there exists $\{a_g\}_{g\in G}$ which satisfy the conditions (i)–(iii) in the above statement. We have

$$\begin{aligned}
\|\sum_{g\in G}\phi(a_g)x - x\| &\le \|\sum_{g\in G}\phi(a_g)(x - \sum_{i=1}^n \phi(a_i)x_i)\| + \|\sum_{g\in G}\sum_{i=1}^n \phi(a_g)\phi(a_i)x_i - x\| \\
&\le |G|\varepsilon + \|\sum_{i=1}^n \phi(\sum_{g\in G} a_g a_i - a_i)x_i\| + \|\sum_{i=1}^n \phi(a_i)x_i - x\| \\
&\le |G|\varepsilon + nM\varepsilon + \varepsilon < \epsilon.
\end{aligned}$$

Thus we obtain the first estimate in Definition 3.3 (1). Similarly, we have

$$\begin{aligned}
\|\sum_{g\in G} x a_g - x\| \le &\|(x - \sum_i \phi(a_i)x_i)\sum_{g\in G} a_g\| + \|\sum_{i=1}^n \phi(a_i)x_i \sum_{g\in G} a_g - x\| \\
\le &|G|\varepsilon + \|\sum_{i=1}^n \phi(a_i)\sum_{g\in G}(x_i a_g - \phi(a_g)x_i)\| \\
&+ \|\sum_{i=1}^n \phi(a_i \sum_g a_g)x_i - x\| \\
\le &|G|\varepsilon + \|\sum_{i=1}^n \phi(a_i)\sum_{g\in G}(x_i a_g - \phi(a_g)x_i)\|
\end{aligned}$$

$$+ \| \sum_{i=1}^{n} \phi(\sum_{g} a_i a_g - a_i) x_i \| + \| \sum_{i=1}^{n} \phi(a_i) x_i - x \|$$
$$\leq |G|\varepsilon + n|G|M\varepsilon + (nM\varepsilon + \varepsilon) < \epsilon.$$

This computation gives the second estimate in Definition 3.3 (1).

Since $\| \sum_{g\in G} a_g a^* - a^* \| < \varepsilon$ for all $a \in S_2$, we get

$$\| \sum_{a\in G} a a_g - a \| = \| (\sum_{g\in G} a_g a^* - a^*)^* \| < \varepsilon < \epsilon.$$

■

### 3.3. Rokhlin Property for Induced Actions on Cuntz-Pimsner Algebras

For a $C^*$-correspondence, Katsura in [16] introduced the following associated $C^*$-algebra:

**Definition 3.7.** *Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras, and $(E, \mathcal{A}, \phi)$ be a $C^*$-correspondence over $\mathcal{A}$.*

(i) *A pair $(\pi, \Psi)$ is called covariant representation of $(E, \mathcal{A}, \phi)$ on $\mathcal{B}$ if $\pi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism and $\Psi : E \to \mathcal{B}$ is a bounded linear map satisfying*

  (a) $\Psi(x)^*\Psi(y) = \pi(\langle x, y\rangle)$ *for all $x, y \in E$,*

  (b) $\pi(a)\Psi(x) = \Psi(\phi(a)x)$ *for all $a \in \mathcal{A}$ and $x \in E$,*

  (c) $\pi(b) = \Pi_\Psi(\phi(b))$ *for all $b \in \mathcal{J}_E$ where*

$$\mathcal{J}_E := \phi^{-1}(\mathcal{K}(E)) \cap (ker\phi)^{\perp}$$

  *and $\Pi_\Psi : \mathcal{K}(E) \to \mathcal{B}$ is a $*$-homomorphism defined by*

$$\Pi_\Psi(\theta_{x,y}) := \Psi(x)\Psi(y)^* \text{ for } x, y \in E.$$

  *The notation $C^*(\pi, \Psi)$ denotes the $C^*$-algebra generated by the images of mappings $\pi$ and $\Psi$ in $\mathcal{B}$.*

(ii) *A covariant representation $(\pi_U, \Psi_U)$ of a $C^*$-correspondence $(E, \mathcal{A}, \phi)$ is said to be universal if for any covariant representation $(\pi, \Psi)$ of $(E, \mathcal{A}, \phi)$ on $\mathcal{B}$, there exists a natural surjection $\psi : C^*(\pi_U, \Psi_U) \to C^*(\pi, \Psi)$ such that $\pi = \psi \circ \pi_U$ and $\Psi = \psi \circ \Psi_U$. We denote the $C^*$-algebra $C^*(\pi_U, \Psi_U)$ by $\mathcal{O}_E$.*

In [9, Lemma 2.6] Hao and Ng proved that each action $(\eta, \alpha)$ of a locally compact group $G$ on $(E, \mathcal{A}, \phi)$ induces a $C^*$-dynamical system $(G, \gamma, \mathcal{O}_E)$ such that $\gamma_s(\pi_U(a)) = \pi_U(\alpha_s(a))$ and $\gamma_s(\Psi_U(x)) = \Psi_U(\eta_s(x))$ for all $a \in \mathcal{A}$, $x \in E$ and $s \in G$. The theorem below shows that Definition 3.3 is the natural choice for Rokhlin property.

**Theorem 3.8.** *Let $(\eta, \alpha)$ be an action of a finite group $G$ on a $C^*$-correspondence $(E, \mathcal{A}, \phi)$. Then, the following statements are equivalent:*

(i) *The action* $(\eta, \alpha)$ *has the Rokhlin property.*
(ii) *The induced action* $\gamma$ *of* $G$ *on* $\mathcal{O}_E$ *as mentioned above has the Rokhlin property with Rokhlin elements from* $\pi_U(\mathcal{A})$.

*Proof.* Assume that the action $(\eta, \alpha)$ has the Rokhlin property. Let $\epsilon > 0$ and let $S = \{b_1, b_2, \dots, b_n\}$ be any finite subset of $\mathcal{O}_E$. For each $1 \le j \le n$ there exist finite sets $\{x_j^l\}_{1 \le l \le l_j} \subset E$ and $\{a_j^m\}_{1 \le m \le m_j} \subset \mathcal{A}$ such that $\|b_j - p_j(\Psi_U(x_j^l), \pi_U(a_j^m))\| < \frac{\epsilon}{3|G|}$ where

$$p_j(\Psi_U(x_j^l), \pi_U(a_j^m)) = \sum_{i=1}^{n_j} \lambda_{j,i} u_{j,i,1} u_{j,i,2} \dots u_{j,i,k_{j,i}}$$

is a finite linear combination of words $u_{j,i,1} u_{j,i,2} \dots u_{j,i,k_{j,i}}$ with each $u_{j,i,l}$ belonging to the set

$$\{\Psi_U(x_j^l), \Psi_U(x_j^l)^*, \pi_U(a_j^m), \pi_U(a_j^m)^* : 1 \le l \le l_j, 1 \le m \le m_j, 1 \le j \le n\}.$$

Let $S_1 = \{x_j^l\}_{l,j}$, $S_2 = \{a_j^m\}_{m,j}$ and $K_j = \sum_{i=1}^{n_j} |\lambda_{j,i}| \|u_{j,i,1}\| \|u_{j,i,2}\| \dots \|u_{j,i,k_{j,i}}\|$. Set $K := 3\left(\max_{1 \le j \le n} \max_{1 \le i \le n_j} k_{j,i}\right)\left(\max_{1 \le j \le n} K_j\right)$. Since $(\eta, \alpha)$ has the Rokhlin property, there exists $(a_g)_{g \in G} \subset \mathcal{A}$ consisting of mutually orthogonal positive contractions such that

(a) $\|\sum_{g \in G} \phi(a_g)x - x\| < \frac{\epsilon}{K}$, $\|\sum_{g \in G} x a_g - x\| < \frac{\epsilon}{K}$, $\|\sum_{g \in G} a_g a - a\| < \frac{\epsilon}{K}$ and $\|\sum_{g \in G} a a_g - a\| < \frac{\epsilon}{K}$ for all $x \in S_1$, $a \in S_2$,
(b) $\|\alpha_h(a_g) - a_{hg}\| < \frac{\epsilon}{K}$ for $h, g \in G$,
(c) $\|x a_g - \phi(a_g)x\| < \frac{\epsilon}{K}$ and $\|a_g a - a a_g\| < \frac{\epsilon}{K}$ for all $x \in S_1$, $a \in S_2$ and $g \in G$.

We show that $\gamma$ has the Rokhlin property with respect to the finite set $S$. For each $g \in G$, let $f_g := \pi_U(a_g)$. For each $g \in G$, we have $\|f_g\| \le 1$, and $f_g$'s are mutually orthogonal positive contractions. Further

(a) For each $1 \le j \le n$,

$$\begin{aligned} &\|\sum_{g \in G} f_g b_j - b_j\| \\ <\epsilon \le &\|\sum_{g \in G} \pi_U(a_g) b_j - \sum_{g \in G} \pi_U(a_g) p_j(\Psi_U(x_j^l), \pi_U(a_j^m))\| \\ &+ \|\sum_{g \in G} \pi_U(a_g) p_j(\Psi_U(x_j^l), \pi_U(a_j^m)) - p_j(\Psi_U(x_j^l), \pi_U(a_j^m))\| \\ &+ \|p_j(\Psi_U(x_j^l), \pi_U(a_j^m)) - b_j\| < \epsilon. \end{aligned}$$

(b) For $h, g \in G$ we have

$$\|\gamma_h(\pi_U(a_g)) - \pi_U(a_{hg})\| = \|\pi_U(\alpha_h(a_g)) - \pi_U(a_{hg})\| < \epsilon.$$

(c) For $1 \leq j \leq n$ we get

$$\begin{aligned}&\|\pi_U(a_g)b_j - b_j\pi_U(a_g)\|\\ =&\|\pi_U(a_g)b_j - \pi_U(a_g)p_j(\Psi_U(x_j^l), \pi_U(a_j^m))\|\\ &+ \|\pi_U(a_g)p_j(\Psi_U(x_j^l), \pi_U(a_j^m)) - p_j(\Psi_U(x_j^l), \pi_U(a_j^m))\pi_U(a_g)\|\\ &+ \|p_j(\Psi_U(x_j^l), \pi_U(a_j^m))\pi_U(a_g) - b_j\pi_U(a_g)\| < \epsilon.\end{aligned}$$

Thus $\gamma$ has the Rokhlin property with Rokhlin elements from the $C^*$-algebra $\pi_U(\mathcal{A})$.

Conversely, assume that the statement (b) of the theorem holds. Let $S_1$ and $S_2$ be finite subsets of $E$ and $\mathcal{A}$ respectively. Fix $\epsilon > 0$ and

$$S := \{\Psi_U(x), \pi_U(\langle x, x\rangle), \pi_U(a), \pi_U(a^*) : x \in S_1, a \in S_2\}.$$

Since $\gamma$ has the Rokhlin property with Rokhlin elements from the $C^*$-algebra $\pi_U(\mathcal{A})$, there exist mutually orthogonal positive contractions $(\pi_U(f_g))_{g\in G} \subset \pi_U(\mathcal{A})$ satisfying

(a) $\|(\sum_{g\in G} \pi_U(f_g))a - a\| < \epsilon'$ for all $a \in S$,
(b) $\|\alpha_h(\pi_U(f_g)) - \pi_U(f_{hg})\| < \epsilon'$ for $h, g \in G$,
(c) $\|[\pi_U(f_g), a]\| < \epsilon'$ for $g \in G$ and $a \in S$,

where $\epsilon' = min\{\epsilon, \frac{\epsilon^2}{|G|+1}\}$. It can be checked that $(\eta, \alpha)$ has the Rokhlin property with Rokhlin elements $(f_g)_{g\in G}$. ■

### 3.4. Applications of Our Characterization

Katsura obtained several results about the nuclearity of the $C^*$-algebra $\mathcal{O}_E$ associated to a $C^*$-correspondence $E$ in [18]. We discuss permanence properties of the nuclearity and several other notions for $\mathcal{O}_E$, with respect to the crossed product $E \times_\eta G$ of a $C^*$-correspondence $E$ for some action $(\eta, \alpha)$ of a finite group $G$ with Rokhlin property. The nuclear dimension of $\mathcal{O}_E$ is estimated recently in [5, Corollary 5.22].

**Corollary 3.9.** *Assume $(\eta, \alpha)$ to be an action of a finite group $G$ on a $C^*$-correspondence $(E, \mathcal{A}, \phi)$. If $(\eta, \alpha)$ has the Rokhlin property and if $\mathcal{O}_E$ belongs to any one of the classes, say $\mathfrak{C}$, listed in Theorem 2.3, then $\mathcal{O}_{E\times_\eta G}$ also belongs to the same class $\mathfrak{C}$.*

A *directed graph* $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ consists of a countable vertex set $\mathcal{E}^0$, and a countable edge set $\mathcal{E}^1$, along with maps $r, s : \mathcal{E}^1 \to \mathcal{E}^0$ describing the range and the source of edges. We also assume that the directed graph is always *row finite*, i.e., for every vertex $v \in \mathcal{E}_0$, the set $s^{-1}(v)$ is a finite subset of $\mathcal{E}_1$. Let $\mathcal{A}$ denote the $C^*$-algebra $C_0(\mathcal{E}^0)$. A *graph $C^*$-algebra* of the directed graph $\mathcal{E}$ (cf.

[21]) is a universal $C^*$-algebra generated by partial isometries $\{S_e : e \in \mathcal{E}^1\}$ and projections $\{P_v : v \in \mathcal{E}^0\}$ such that

$$S_e{}^* S_e = P_{r(e)} = \sum_{s(f)=r(e)} S_f S_f{}^* \text{ for all } e \in \mathcal{E}^1.$$

Since the graph is row finite, the summation is finite. We use the symbol $C^*(\mathcal{E})$ to denote the graph $C^*$-algebra of a directed graph $\mathcal{E}$. The vector space $C_c(\mathcal{E}^1)$ becomes an inner-product $\mathcal{A}$-module with the following inner-product and module action:

$$\langle f, g\rangle(v) := \sum_{e \in r^{-1}(v)} \overline{f(e)} g(e) \text{ for each } v \in \mathcal{E}^0;$$
$$(fh)(e) := f(e)h(r(e)) \text{ for all } e \in \mathcal{E}^1;$$

where $f, g \in C_c(\mathcal{E}^1)$ and $h \in \mathcal{A}$. Let $E(\mathcal{E})$ denote the completion of the inner-product module $C_c(\mathcal{E}^1)$. Define $\phi : \mathcal{A} \to \mathcal{B}^a(E(\mathcal{E}))$ by

$$\phi(h)f(e) := h(s(e))f(e) \text{ for each } e \in \mathcal{E}^1; f \in C_c(\mathcal{E}^1); h \in \mathcal{A}.$$

Thus $(E(\mathcal{E}), \mathcal{A}, \phi)$ is a $C^*$-correspondence and the graph $C^*$-algebra $C^*(\mathcal{E})$ of a directed graph $\mathcal{E}$ is always isomorphic to $\mathcal{O}_{E(\mathcal{E})}$ (cf. [16, Proposition 3.10]).

**Definition 3.10.** *Let $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ be a directed graph and let $c$ from $\mathcal{E}^1$ to a countable group $G$ be a mapping. The skew-product graph is the graph $\mathcal{E}(c) = (G \times \mathcal{E}^0, G \times \mathcal{E}^1, r', s')$ where $r'(g, e) := (gc(e), r(e))$ and $s'(g, e) := (g, s(e))$ for all $g \in G$; $e \in \mathcal{E}^1$.*

For a given countable abelian group $G$ and a function $c : \mathcal{E}^1 \to G$, we can define an action $\gamma^c$ of $\widehat{G}$ on $C^*(\mathcal{E})$ (cf. [21, Cor. 2.5]) by

$$\gamma^c_\chi(S_e) := \langle \chi, c(e)\rangle S_e \text{ for each } \chi \in \widehat{G},\ e \in \mathcal{E}^1.$$

Let $\alpha$ be the trivial action of $\widehat{G}$ on $\mathcal{A}$ and let $\eta$ be an action of $\widehat{G}$ on $E(\mathcal{E})$ defined by

$$\eta_\chi(f)(e) := \langle \chi, c(e)\rangle f(e) \text{ for each } \chi \in \widehat{G}, e \in \mathcal{E}^1, f \in C_c(\mathcal{E}^1).$$

From [9, Cor. 2.11], it follows that $\gamma^c$ coincides with the action of $G$ on $\mathcal{O}_{E(\mathcal{E})}$ induced by the action $(\eta, \alpha)$, and we also get $C^*(\mathcal{E}(c)) \cong C^*(\mathcal{E}) \times_{\gamma^c} \widehat{G}$.

**Proposition 3.11.** *Let $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ be a directed graph, $G$ be a finite abelian group and $c : \mathcal{E}^1 \to G$ be a function. Let $(\eta, \alpha)$ be the action of $\widehat{G}$ on the $C^*$-correspondence $E(\mathcal{E})$ defined in the previous paragraph. Then $(\eta, \alpha)$ does not have the Rokhlin property.*

*Proof.* Let $(f_s)_{s\in G}$ be a subset of $\mathcal{A}$ consisting of orthogonal positive contractions. Let $\epsilon > 0$ be given. Since $\alpha$ is the trivial action of $\widehat{G}$ on $\mathcal{A}$, we have

$$\begin{aligned}\|\alpha_g(f_s) - f_{gs}\| &= \|f_s - f_{gs}\| = \|f_s + f_{gs}\| \\ &\geq \max\{\|f_s\|, \|f_{gs}\|\} \text{ for all } s, g \in \widehat{G},\end{aligned}$$

$$\|a\| \leq \|\sum_{s\in\widehat{G}} f_s a - a\| + \|\sum_{s\in\widehat{G}} f_s a\| < \epsilon + \sum_{s\in\widehat{G}} \|f_s\|\|a\|.$$

Thus $\alpha$ does not have the Rokhlin property. It follows that the condition (2) in Definition 3.3 is not satisfied, and hence the $\alpha$-compatible action $(\eta, \alpha)$ does not have the Rokhlin property. ∎

**Corollary 3.12.** *Let $\mathcal{E} = (\mathcal{E}^0, \mathcal{E}^1, r, s)$ be a directed graph, $G$ be a finite abelian group and $c : \mathcal{E}^1 \to G$ be a function. Let $(\eta, \alpha)$ be the action of $\widehat{G}$ on the $C^*$-correspondence $E(\mathcal{E})$ as in Proposition 3.11. Then the induced action $\gamma^c$ of $\widehat{G}$ on $C^*(\mathcal{E})$ does not have the Rokhlin property with Rokhlin elements from $\pi_U(\mathcal{A})$.*

## 4. Rokhlin Property for Group Actions on Hilbert Bimodules

Analogous to a right Hilbert $\mathcal{A}$-module, a *left Hilbert $\mathcal{A}$-module* is defined as a left $\mathcal{A}$-module with the positive definite form ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle : E \times E \to \mathcal{A}$ which is, conjugate-linear in the second variable, linear in the first variable and satifies

$${}_{\mathcal{A}}\langle ax, y\rangle = a\,{}_{\mathcal{A}}\langle x, y\rangle \text{ for } x, y \in E, a \in \mathcal{A}.$$

**Definition 4.1.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$-algebras. A left Hilbert $\mathcal{B}$-module $E$ is called Hilbert $\mathcal{B}$-$\mathcal{A}$ bimodule if it is also a right Hilbert $\mathcal{A}$-module satisfying*

$${}_{\mathcal{B}}\langle x, y\rangle z = x\langle y, z\rangle_{\mathcal{A}} \text{ for } x, y, z \in E.$$

On a Hilbert bimodule we consider actions of a locally compact group similar to those introduced in Definition 3.2.

**Definition 4.2.** *Let $(G, \alpha, \mathcal{A})$ and $(G, \beta, \mathcal{B})$ be $C^*$-dynamical systems of a locally compact group $G$ and let $E$ be a $\mathcal{B}$-$\mathcal{A}$ Hilbert bimodule. A $\beta$-*compatible *action (respectively an $\alpha$-*compatible action*) $\eta$ of $G$ on $E$ is defined as a group homomorphism from $G$ into the group of invertible linear transformations on $E$ such that*

(i) $\eta_g(bx) = \beta_g(b)\eta_g(x)$ *(respectively* $\eta_g(xa) = \eta_g(x)\alpha_g(a)$*),*
(ii) ${}_{\mathcal{B}}\langle\eta_g(x), \eta_g(y)\rangle = \beta_g({}_{\mathcal{B}}\langle x, y\rangle)$ *(respectively* $\langle\eta_g(x), \eta_g(y)\rangle_{\mathcal{A}} = \alpha_g(\langle x, y\rangle_{\mathcal{A}})$*)*

*for $a \in \mathcal{A},\ b \in \mathcal{B},\ x, y \in E,\ g \in G$; and $g \mapsto \eta_g(x)$ is continuous from $G$ into $E$ for each $x \in E$. The combination of these two compatibility conditions will be simply called $(\beta, \alpha)$-compatibility.*

Consider a $(\beta, \alpha)$-compatible action $\eta$ of a locally compact group $G$ on the $\mathcal{B}$-$\mathcal{A}$ Hilbert bimodule $E$. The *crossed product bimodule* $E \times_\eta G$ (cf. [15] and [7]) is an $\mathcal{B} \times_\beta G$-$\mathcal{A} \times_\alpha G$ Hilbert bimodule obtained by completion of $C_c(G, E)$ where the module actions and the $C^*$-valued inner products are given by

$$\begin{aligned}(lg)(s) &= \int_G l(t)\alpha_t(g(t^{-1}s))dt,\\ (fm)(s) &= \int_G f(t)\eta_t(m(t^{-1}s))dt,\\ \langle l, m\rangle_{\mathcal{A}\times_\alpha G}(s) &= \int_G \alpha_{t^{-1}}(\langle l(t), m(ts)\rangle_{\mathcal{A}})dt,\\ {}_{\mathcal{B}\times_\beta G}\langle l, m\rangle(s) &= \int_G {}_{\mathcal{B}}\langle l(st^{-1}), \eta_s(m(t^{-1}))\rangle dt\end{aligned}$$

for all $f \in C_c(G, \mathcal{B})$, $g \in C_c(G, \mathcal{A})$ and $l, m \in C_c(G, E)$.

If $E$ is a (right) Hilbert $\mathcal{A}$-module, then $E$ is a $\mathcal{K}(E)$-$\mathcal{A}$ Hilbert bimodule with respect to ${}_{\mathcal{K}(E)}\langle x, y\rangle = \theta_{x,y}$ for all $x, y \in E$. Moreover, we can associate a $C^*$-algebra called the *linking algebra*, defined by

$$\mathfrak{L}_E := \begin{pmatrix}\mathcal{K}(E) & E\\ E^* & \mathcal{A}\end{pmatrix} \subset \mathcal{K}(E \oplus \mathcal{A}) \tag{4}$$

(cf. [30, p. 50]), to each (right) Hilbert $\mathcal{A}$-module $E$. Let $(G, \alpha, \mathcal{A})$ be a $C^*$-dynamical system and $\eta$ be an $\alpha$-compatible action of $G$ on $E$. For each $s \in G$, let us define $Ad\eta_s(t) := \eta_s t \eta_{s^{-1}}$ for $t \in \mathcal{K}(E)$ where $\eta_s^*(x^*) := \eta_s(x)^*$ for $x \in E$. Indeed, $\eta$ is also an $(Ad\eta, \alpha)$-compatible action and we get the *induced action* $\theta$ of $G$ on $\mathfrak{L}_E$ (cf. [7, 22]) defined by

$$\theta_s\begin{pmatrix}t & x\\ y^* & a\end{pmatrix} := \begin{pmatrix}Ad\eta_s t & \eta_s(x)\\ \eta_s^*(y^*) & \alpha_s(a)\end{pmatrix}$$

for all $s \in G$, $t \in \mathcal{K}(E)$, $a \in \mathcal{A}$ and $x, y \in E$. We denote this $C^*$-dynamical system by $(G, \theta, \mathfrak{L}_E)$.

### 4.1. Rokhlin Property for Induced Finite Group Actions on Linking Algebras

Analogous to Definition 3.3 the Rokhlin property for finite group actions on Hilbert bimodules is defined as follows:

**Definition 4.3.** *Let $(G, \alpha, \mathcal{A})$ and $(G, \beta, \mathcal{B})$ be $C^*$-dynamical systems of a finite group $G$ and let $E$ be a $\mathcal{B}$-$\mathcal{A}$ Hilbert bimodule. Assume $\eta$ to be a $(\beta, \alpha)$-compatible* *action of $G$ on $E$. We say that $\eta$ has the Rokhlin property if for each $\epsilon > 0$, finite subsets $S_1$ and $S_2$ of $E$, and finite subsets $S_3$ and $S_4$ of $\mathcal{B}$ and $\mathcal{A}$ respectively, there are sets $(a_g)_{g\in G} \subset \mathcal{A}$ and $(b_g)_{g\in G} \subset \mathcal{B}$ consisting of mutually orthogonal positive contractions such that*

(i) $\|\sum_{g\in G} a_g u - u\| < \epsilon$ *for all* $u \in S_2^* \cup S_4$, $\|\sum_{g\in G} b_g v - v\| < \epsilon$ *for all* $v \in S_1 \cup S_3$.
(ii) $\|\alpha_h(a_g) - a_{hg}\| < \epsilon$ *and* $\|\beta_h(b_g) - b_{hg}\| < \epsilon$ *for* $h, g \in G$,
(iii) $\|xa_g - b_g x\| < \epsilon$, $\|tb_g - b_g t\| < \epsilon$ *and* $\|a_g a - aa_g\| < \epsilon$, $\|a_g y^* - y^* b_g\| < \epsilon$ *for all* $x \in S_1$, $y \in S_2$, $t \in S_3$, $a \in S_4$ *and* $g \in G$.

Following theorem justifies the choice of this version of Rokhlin property for group actions on a bimodule:

**Theorem 4.4.** *Let $E$ be a Hilbert $\mathcal{A}$-module where $\mathcal{A}$ is $C^*$-algebra. Suppose that $\alpha : G \to Aut(\mathcal{A})$ is an action of a finite group $G$ and $\eta$ is an $\alpha$-compatible action of $G$ on $E$. Then the following statements are equivalent:*

(i) *$\eta$ has the Rokhlin property as an $(Ad\eta, \alpha)$-compatible action.*
(ii) *The action $\theta$ of $G$ on $\mathfrak{L}_E$ induced by $\eta$ has the Rokhlin property with Rokhlin elements coming from the $C^*$-subalgebra $\begin{pmatrix} \mathcal{K}(E) & 0 \\ 0 & \mathcal{A} \end{pmatrix}$ of $\mathfrak{L}_E$.*

*Proof.* Let $\epsilon > 0$ be given and $S = \left\{ \begin{pmatrix} t_i & x_i \\ y_i^* & a_i' \end{pmatrix} : i = 1, 2, \ldots, n \right\}$ be any finite subset of $\mathfrak{L}_E$. Consider $S_1 = \{x_1, x_2, \ldots, x_n\}$, $S_2 = \{y_1, y_2, \ldots, y_n\}$, $S_3 = \{t_1, t_2, \ldots, t_n\}$ and $S_4 = \{a_1', a_2', \ldots, a_n'\}$. Suppose $\eta$ has the Rokhlin property as an $(Ad\eta, \alpha)$-compatible action. So, there are sets $(a_g)_{g\in G} \subset \mathcal{A}$ and $(b_g)_{g\in G} \subset \mathcal{K}(E)$ consisting of mutually orthogonal positive contractions such that

(a) $\|\sum_{g\in G} a_g u - u\| < \frac{\epsilon}{4}$ for all $u \in S_2^* \cup S_4$, $\|\sum_{g\in G} b_g v - v\| < \frac{\epsilon}{4}$ for all $v \in S_1 \cup S_3$.
(b) $\|\alpha_h(a_g) - a_{hg}\| < \frac{\epsilon}{4}$ and $\|Ad\eta_h(b_g) - b_{hg}\| < \frac{\epsilon}{4}$ for $h, g \in G$,
(c) $\|xa_g - b_g x\| < \frac{\epsilon}{4}$, $\|tb_g - b_g t\| < \frac{\epsilon}{4}$ and $\|a_g a - aa_g\| < \frac{\epsilon}{4}$, $\|a_g y^* - y^* b_g\| < \frac{\epsilon}{4}$ for all $x \in S_1$, $y \in S_2$, $t \in S_3$, $a \in S_4$ and $g \in G$.

We prove that the action $\theta$ of $G$ on $\mathfrak{L}_E$ induced by $\eta$ has the Rokhlin property by showing that for the above set $S$, the Rokhlin elements are $(f_g)_{g\in G}$ where $f_g := \begin{pmatrix} b_g & 0 \\ 0 & a_g \end{pmatrix}$. For each $g \in G$, $\|f_g\| = \sup_{\|(x,a)\|\le 1} \left\| \begin{pmatrix} b_g & 0 \\ 0 & a_g \end{pmatrix} \begin{pmatrix} x \\ a \end{pmatrix} \right\| \le 1$. Further for $g, h \in G$ with $g \ne h$ we get

$$f_g f_h = \begin{pmatrix} b_g & 0 \\ 0 & a_g \end{pmatrix} \begin{pmatrix} b_h & 0 \\ 0 & a_h \end{pmatrix} = 0.$$

Then, it is easy to verify conditions (1)-(3) of Definition 2.1.

Conversely, let $\epsilon > 0$ and let $S_1,\ S_2$ be finite subsets of $E$. Let $S_3$ and $S_4$ be finite subsets of $\mathcal{K}(E)$ and $\mathcal{A}$, respectively. Choose

$$S = \left\{ \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} : x \in S_1, y \in S_2, t \in S_3, a \in S_4 \right\}.$$

Suppose $\theta$ has the Rokhlin property with Rokhlin elements $\left\{ \begin{pmatrix} b_g & 0 \\ 0 & a_g \end{pmatrix} : g \in G \right\}$ $\subset \begin{pmatrix} \mathcal{K}(E) & 0 \\ 0 & \mathcal{A} \end{pmatrix} \subset \mathfrak{L}_E$ with respect to the set $S$ and $\epsilon > 0$. Then it is easy to verify that $\eta$ has the Rokhlin property as an $(Ad\eta, \alpha)$-compatible action with the positive contractions $(a_g)_{g\in G} \subset \mathcal{A}$ and $(b_g)_{g\in G} \subset \mathcal{K}(E)$ with respect to the sets $S_1,\ S_2,\ S_3$ and $S_4$ on noting that the steps and arguments of the first part of this proof can be carried out in the reverse order also. ■

To obtain new examples of $C^*$-algebras for which the nuclear dimension is at most $n$, choose a Hilbert $\mathcal{A}$-module $E$ where $\mathcal{A}$ has nuclear dimension at most $n$. Because from the fact that $\mathcal{A}$ is a full hereditary $C^*$-subalgebra of $\mathfrak{L}_E$ it follows that the nuclear dimension of $C^*$-algebra $\mathfrak{L}_E$ is at most $n$ (see [33, Cor. 2.8]).

**Corollary 4.5.** *Let $(G, \alpha, \mathcal{A})$ be a $C^*$-dynamical system where $G$ is finite group and $n \in \mathbb{N} \cup \{0\}$. Let $E$ be a Hilbert $\mathcal{A}$-module. Assume $\eta$ to be an action of $G$ on $E$ which has the Rokhlin property as an $(Ad\eta, \alpha)$-compatible action and $\mathcal{A}$ belongs to any one of the classes, say $\mathcal{C}$, listed in Theorem 2.3, then $\mathfrak{L}_{E\times_\eta G}$ also belongs to the same class $\mathcal{C}$.*

### 4.2. Rokhlin Property for Induced Actions of $\mathbb{Z}$ on Linking Algebras

Next we investigate the case where action is of the infinite discrete group $\mathbb{Z}$ over a Hilbert bimodule $E$. If $\alpha$ is an action of of $\mathbb{Z}$ on a $C^*$-algebra $\mathcal{A}$, then $\alpha_n = \alpha_1^n$ for all $n \in \mathbb{N}$ and so we use notation $\alpha$ for $\alpha_1$. Similarly for any action $\eta$ of $\mathbb{Z}$ on $E$, we write $\eta$ for $\eta_1$. We first recall the definition of the Rokhlin property for the automorphisms on $C^*$-algebras from [3].

**Definition 4.6.** *Let $\mathcal{A}$ be a $C^*$-algebra and $\alpha \in Aut(\mathcal{A})$. We say that $\alpha$ has the Rokhlin property if for any positive integer $p$, any finite set $S \subset \mathcal{A}$, and any $\epsilon > 0$, there are mutually orthogonal positive contractions $e_{0,0}, \ldots, e_{0,p-1}$, $e_{1,0}, \ldots, e_{1,p}$ such that*

(i) $\left\| \left( \sum_{r=0}^{1} \sum_{j=0}^{p-1+r} e_{r,j} \right) a - a \right\| < \epsilon$ *for all* $a \in S$,
(ii) $\|[e_{r,j}, a]\| < \epsilon$ *for all* $r, j$ *and* $a \in S$,
(iii) $\|\alpha(e_{r,j})a - e_{r,j+1}a\| < \epsilon$ *for all* $a \in S$, $r = 0, 1$ *and* $j = 0, 1, \ldots, p-2+r$,
(iv) $\|\alpha(e_{0,p-1} + e_{1,p})a - (e_{0,0} + e_{1,0})a\| < \epsilon$ *for all* $a \in S$.

*We call elements $e_{0,0}, \ldots, e_{0,p-1}, e_{1,0}, \ldots, e_{1,p}$ the Rokhlin elements for $\alpha$.*

**Definition 4.7.** *Let $(\mathbb{Z}, \alpha, \mathcal{A})$ and $(\mathbb{Z}, \beta, \mathcal{B})$ be $C^*$-dynamical systems. Suppose that $E$ is a Hilbert $\mathcal{B}$-$\mathcal{A}$ bimodule and $\eta$ is a $(\beta,\alpha)$-compatible automorphism of $E$. We say that $\eta$ has the Rokhlin property if for any $\epsilon > 0$, any positive integer $p$, any finite subsets $S_1$ and $S_2$ of $E$, and any finite subsets $S_3$ and $S_4$ of $\mathcal{B}$ and $\mathcal{A}$ respectively, there are sets consisting of mutually orthogonal positive contractions $\{a_{0,0}, \ldots, a_{0,p-1}, a_{1,0}, \ldots, a_{1,p}\} \subset \mathcal{A}$ and $\{b_{0,0}, \ldots, b_{0,p-1}, b_{1,0}, \ldots, b_{1,p}\} \subset \mathcal{B}$ such that*

(i) $b_{i,j} b_{i',j'} = 0$ *and* $a_{i,j} a_{i',j'} = 0$ *if* $(i, j) \neq (i', j')$.

(ii) $\left\|\sum_{r=0}^{1} \sum_{j=0}^{p-1+r} b_{r,j} v - v\right\| < \epsilon$, $\left\|\sum_{r=0}^{1} \sum_{j=0}^{p-1+r} a_{r,j} u - u\right\| < \epsilon$ *for all* $v \in S_1 \cup S_3$, $u \in S_2^* \cup S_4$.

(iii) $\|x a_{r,j} - b_{r,j} x\| < \epsilon$, $\|a_{r,j} y^* - y^* b_{r,j}\| < \epsilon$, $\|b b_{r,j} - b_{r,j} b\| < \epsilon$, $\|a_{r,j} a - a a_{r,j}\| < \epsilon$ *for all* $x \in S_1, y \in S_2, b \in S_3, a \in S_4$ *and for all* $r, j$.

(iv) $\|\beta(b_{r,j}) v - b_{r,j+1} v\| < \epsilon$ , $\|\alpha(a_{r,j}) u - a_{r,j+1} u\| < \epsilon$ *for all* $v \in S_1 \cup S_3$, $u \in S_2^* \cup S_4$; $r = 0, 1$ *and* $j = 0, \ldots, p-2+r$.

(v) $\|\beta(b_{0,p-1} + b_{1,p}) v - (b_{0,0} + b_{1,0}) v\| < \epsilon$, $\|\alpha(a_{0,p-1} + a_{1,p}) u - (a_{0,0} + a_{1,0}) u\| < \epsilon$ *for all* $v \in S_1 \cup S_3$, $u \in S_2^* \cup S_4$.

Notion similar to Definition 4.7 was introduced in [5, Definition 3.1]. The following observation is a justification for the choice of the above definition of Rokhlin property for actions of $\mathbb{Z}$ on a bimodule:

**Theorem 4.8.** *Suppose $(\mathbb{Z}, \alpha, \mathcal{A})$ is a $C^*$-dynamical system. Assume $E$ to be a Hilbert $\mathcal{A}$-module and $\eta$ to be an automorphism on $E$. The following statements are equivalent:*

(i) *$\eta$ has the Rokhlin property as an $(Ad\eta,\alpha)$-compatible automorphism.*

(ii) *The automorphism $\theta$ in $Aut(\mathfrak{L}_E)$ induced by $\eta$ has the Rokhlin property with Rokhlin elements coming from the $C^*$-subalgebra $\begin{pmatrix} \mathcal{K}(E) & 0 \\ 0 & \mathcal{A} \end{pmatrix}$ of $\mathfrak{L}_E$.*

*Proof.* Let $\epsilon > 0$ be given and $S = \left\{\begin{pmatrix} t_i & x_i \\ y_i^* & a_i \end{pmatrix} : i = 1, 2, \ldots, n\right\}$ be any finite subset of $\mathfrak{L}_E$. Consider $S_1 = \{x_1, x_2, \ldots, x_n\}$, $S_2 = \{y_1, y_2, \ldots, y_n\}$, $S_3 = \{t_1, t_2, \ldots, t_n\}$ and $S_4 = \{a_1, a_2, \ldots, a_n\}$. Suppose $\eta$ has the Rokhlin property. So, there are sets $\{a_{0,0}, \ldots, a_{0,p-1}, a_{1,0}, \ldots, a_{1,p}\} \subset \mathcal{A}$ and $\{b_{0,0}, \ldots, b_{0,p-1}, b_{1,0}, \ldots, b_{1,p}\} \subset \mathcal{K}(E)$ consisting of mutually orthogonal positive contractions such that

(a) $b_{i,j} b_{i',j'} = 0$ and $a_{i,j} a_{i',j'} = 0$ if $(i, j) \neq (i', j')$.

(b) $\left\|\sum_{r=0}^{1} \sum_{j=0}^{p-1+r} b_{r,j} v - v\right\| < \frac{\epsilon}{4}$, $\left\|\sum_{r=0}^{1} \sum_{j=0}^{p-1+r} a_{r,j} u - u\right\| < \frac{\epsilon}{4}$ for all $v \in S_1 \cup S_3$, $u \in S_2^* \cup S_4$.

(c) $\|x a_{r,j} - b_{r,j} x\| < \frac{\epsilon}{4}$, $\|a_{r,j} y^* - y^* b_{r,j}\| < \frac{\epsilon}{4}$, $\|b b_{r,j} - b_{r,j} b\| < \frac{\epsilon}{4}$, $\|a_{r,j} a - a a_{r,j}\| < \frac{\epsilon}{4}$ for all $x \in S_1, y \in S_2, b \in S_3, a \in S_4$ and for all $r, j$.

(d) $\|Ad\eta(b_{r,j}) v - b_{r,j+1} v\| < \frac{\epsilon}{4}$ , $\|\alpha(a_{r,j}) u - a_{r,j+1} u\| < \frac{\epsilon}{4}$ for all $v \in S_1 \cup S_3$, $u \in S_2^* \cup S_4$; $r = 0, 1$ and $j = 0, \ldots, p-2+r$.

(e) $\|Ad\eta(b_{0,p-1}+b_{1,p})v-(b_{0,0}+b_{1,0})v\|<\frac{\epsilon}{4}$, $\|\alpha(a_{0,p-1}+a_{1,p})u-(a_{0,0}+a_{1,0})u\|<\frac{\epsilon}{4}$ for all $v\in S_1\cup S_3$, $u\in S_2^*\cup S_4$.

We verify that the action $\theta$ of $G$ on $\mathfrak{L}_E$ induced by $\eta$ has the Rokhlin property as an $(\mathrm{Ad}\eta,\alpha)$-compatible automorphism by showing that for the above defined set $S$, the Rokhlin elements are $e_{0,0},\ldots,e_{0,p-1},e_{1,0},\ldots,e_{1,p}$ where $e_{i,j}:=\begin{pmatrix} b_{i,j} & 0\\ 0 & a_{i,j}\end{pmatrix}$. For each $(i,j)$,

$$\|e_{i,j}\|=\sup_{\|(x,a)\|\le 1}\left\|\begin{pmatrix} b_{i,j} & 0\\ 0 & a_{i,j}\end{pmatrix}\begin{pmatrix} x\\ a\end{pmatrix}\right\|\le 1.$$

For $(i,j)\neq(i',j')$ we have

$$e_{i,j}e_{i',j'}=\begin{pmatrix} b_{i,j} & 0\\ 0 & a_{i,j}\end{pmatrix}\begin{pmatrix} b_{i',j'} & 0\\ 0 & a_{i',j'}\end{pmatrix}=\begin{pmatrix} b_{i,j}b_{i',j'} & 0\\ 0 & a_{i,j}a_{i',j'}\end{pmatrix}=0.$$

Then, it is easy to verify conditions (1)-(4) of Definition 4.6.

Conversely, let $S_1\cup S_2$, $S_3$ and $S_4$ be finite subsets of $E$, $\mathcal{K}(E)$ and $\mathcal{A}$, respectively. Let

$$S=\left\{\begin{pmatrix} t & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & x\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & 0\\ y^* & 0\end{pmatrix},\begin{pmatrix} 0 & 0\\ 0 & a\end{pmatrix}: x\in S_1, y\in S_2, t\in S_3, a\in S_4\right\}.$$

Suppose $\theta$ has the Rokhlin property with Rokhlin elements $\begin{pmatrix} b_{0,0} & 0\\ 0 & a_{0,0}\end{pmatrix}$, $\begin{pmatrix} b_{0,1} & 0\\ 0 & a_{0,1}\end{pmatrix},\ldots,\begin{pmatrix} b_{0,p-1} & 0\\ 0 & a_{0,p-1}\end{pmatrix}$, $\begin{pmatrix} b_{1,0} & 0\\ 0 & a_{1,0}\end{pmatrix},\begin{pmatrix} b_{1,1} & 0\\ 0 & a_{1,1}\end{pmatrix},\ldots,$ $\begin{pmatrix} b_{1,p} & 0\\ 0 & a_{1,p}\end{pmatrix}$ coming from the $C^*$-algebra $\begin{pmatrix} \mathcal{K}(E) & 0\\ 0 & \mathcal{A}\end{pmatrix}\subset\mathfrak{L}_E$ with respect to the set $S$ and any $\epsilon>0$. Then it is easy to check that $\eta$ has the Rokhlin property as an $(Ad\eta,\alpha)$-compatible action with the positive contractions $\{a_{0,0},\ldots,a_{0,p-1},a_{1,0},\ldots,a_{1,p}\}\subset\mathcal{A}$ and $\{b_{0,0},\ldots,b_{0,p-1},b_{1,0},\ldots,b_{1,p}\}\subset\mathcal{K}(E)$ with respect to the sets $S_1\cup S_2$, $S_3$ and $S_4$ on observing that the steps and arguments of the first part of this proof can be carried out in the reverse order also. ■

We recall the definition of $\mathcal{D}$-absorbing (cf. [11]).

**Definition 4.9.** *A separable, unital $C^*$-algebra $\mathcal{D}\not\cong\mathbb{C}$ is strongly self-absorbing if there exists an isomorphism $\varphi:\mathcal{D}\to\mathcal{D}\otimes\mathcal{D}$ such that $\varphi$ and $id_{\mathcal{D}}\otimes 1_{\mathcal{D}}$ are approximately unitarily equivalent $*$-homomorphisms. If $\mathcal{D}$ is a strongly self-absorbing $C^*$-algebra, we say that a $C^*$-algebra $\mathcal{A}$ is $\mathcal{D}$-absorbing if $\mathcal{A}\cong\mathcal{A}\otimes\mathcal{D}$.*

In the following we observe a permanence property of the $\mathcal{D}$-absorbing property with respect to the crossed product $E\times_\eta\mathbb{Z}$ of a bimodule $E$ for an action $\eta$ which is $(\mathrm{Ad}\eta,\alpha)$-compatible and has Rokhlin property:

**Corollary 4.10.** *Assume* $(\mathbb{Z}, \alpha, \mathcal{A})$ *to be a* $C^*$*-dynamical system. Let* $\eta$ *be an* $\alpha$*-compatible automorphism on a Hilbert* $\mathcal{A}$*-module* $E$ *and let* $\mathcal{D}$ *be a strongly self-absorbing* $C^*$*-algebra. If* $\eta \in Aut(E)$ *has the Rokhlin property as an* $(Ad\eta, \alpha)$*-compatible action and if* $\mathcal{A}$ *is separable and* $\mathcal{D}$*-absorbing, then* $\mathfrak{L}_{E \times_\eta \mathbb{Z}}$ *is* $\mathcal{D}$*-absorbing.*

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