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The Hyperbolic Length-Preserving Curvature Difference Flow of Plane Curves

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Abstract. In this paper, we study the curvature difference flow about two smooth strictly convex closed plane curves. The two initial curves have the same length, one of them is fixed and another is evolving. We show that, the flow is length-preserving and exists for a short time $[0, T]$. From the analysis of the solutions to the flow, we can't get any information about keeping convexity and explosion.

Keywords: Hyperbolic; Length-preserving; Curvature difference flow.

1. Introduction

In the past decades, the evolution of plane curves through curvature flow have been studied extensively, see B. Andrews [1] M. Grayson [6], M. Gage [4, 5], Chou and Zhu [2], etc. In their papers, the settings were similar to each other more or less. However, a very different and interesting setting was proposed, which has not appeared elsewhere before, we called it the Curvature Difference Flow. The specific process is shown in the following. Given two smooth embedded curves γ_1 and γ_2 in R^2, $\gamma_i(\varphi) : I \to \mathrm{R}^2, i = 1, 2$, where I is an interval either in R or S^1. Let γ_2 be fixed and γ_1 be evolving with the following curvature flow

$$\begin{cases} \frac{\partial \gamma_1}{\partial t}(\varphi, t) = [(k_1(\varphi, t) - k_2(\varphi)]\overrightarrow{N}_1(\varphi, t), \\ \gamma_1(\varphi, 0) = \gamma_1(\varphi), \varphi \in I \end{cases}$$

This type of flow is so attractive and interesting that it leads to a lot of questions.

A relative problem is implemented by Y.C. Lin and D.H. Tsai [8]. They considered two given strictly convex and closed curves γ_1 and γ_2 which are embedded in R^2, γ_2 is a fixed curve, while γ_1 is evolving with the flow

$$\begin{cases} \frac{\partial X}{\partial t}(\varphi,t) = (\frac{1}{k_1(\varphi,t)} - \frac{1}{k_2(\varphi,t)})\overrightarrow{N}_{1,out}(\varphi,t), \\ X(\varphi,0) = X_1(\varphi), \varphi \in S^1 \end{cases} \tag{1}$$

where $X_1(\varphi)$ is the parametrization of γ_1, $k_1(\varphi,t)$ and $\overrightarrow{N}_{1,out}(\varphi,t)$ is the curvature and the unit outward normal vector of $\gamma_{1,t}$, respectively, $k_2(\varphi,t)$ is the curvature of the fixed curve γ_2 at the unique point where its outward normal vector is $\overrightarrow{N}_{1,out}(\varphi,t)$.

They obtained the following conclusion: If the initial curves γ_1 and γ_2 have the same length, and γ_1 evolves according to (1), then the flow exists on $t \in [0,\infty)$. At the same time, the length of $\gamma_{1,t}$ is preserved and $\gamma_{1,t}$ will converge to γ_2 in C^∞ as $t \to \infty$ up to a translation, and the translation vector only depends on the given curves γ_1 and γ_2.

In addition to that, these years, many mathematicians studied the closed and convex evolving plane curves from another point of view: The hyperbolic versions of mean curvature flow. Their ideas were derived from a phenomenon: Melting crystals of helium. By studying on this phenomenon, Gurtin and Angenent [10] developed a hyperbolic theory for the evolution of plane curves. The hyperbolic mean curvature flows are such important in mathematics and application that they attracted many mathematicians' attention. For example, Kong, Liu and Wang [7] considered the following initial value problem about plane curves

$$\begin{cases} \frac{\partial^2 F}{\partial t^2}(u,t) = k(u,t)\overrightarrow{N}(u,t) - \nabla\rho(u,t), & \forall (u,t) \in S^1 \times [0,T), \\ F(u,0) = F_0(u), \\ \frac{\partial F}{\partial t}(u,0) = f(u)\overrightarrow{N}_0. \end{cases} \tag{2}$$

where k is the mean curvature of F, $\overrightarrow{N}$ is the unit inner normal vector of F. The function F_0 is a smooth strictly convex closed curve, $f(u)$ (which is non-negative) is the initial velocity of F_0, $\overrightarrow{N}_0$ is the unit inner normal vector of F_0. $\nabla\rho$ is defined by

$$\nabla\rho = \langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t} \rangle \overrightarrow{T},$$

where s is the arclength parameter, and $\overrightarrow{T}$ stands for the unit tangent vector of F.

From (2), the authors derived a beautiful hyperbolic Monge-Ampère equation of the supported function and arrived at conclusions as follows.

Theorem 1.1. (Local existences and uniqueness) *Suppose that* F_0 *is a smooth strictly convex closed curve, then there exist a positive* T *and a family of smooth strictly convex closed curves* $F(\cdot,t)$*, which satisfy* (2) *in the time interval* $[0,T_{max}]$*, provided that* $f(u)$ *is a smooth function on* S^1*.*

Theorem 1.2. *Suppose that F_0 is a smooth strictly convex closed curve, then there exist a class of initial velocities such that the solution of* (2) *with F_0 and $f(u)$ as its initial curve and initial velocity respectively, and they exist only at a finite time interval $[0, T_{max}]$. Moreover, as $t \to T_{max}$, one of the following must be true:*

- (i) *the solution $F(\cdot, t)$ converge to a point, i.e., the curvature of the limit curve becomes unbounded as $t \to T_{max}$;*
- (ii) *the curvature k of the curve is discontinuous as $t \to T_{max}$, so that the solution converges to a piecewise smooth curve.*

Inspired by [7], we consider a relative problem, assume that γ_1 and γ_2 are given strictly convex closed curves in R^2, γ_2 is fixed and γ_1 evolves with the following equation

$$\begin{cases} \frac{\partial^2 F}{\partial t^2}(u,t) = (\frac{1}{k_1(u,t)} - \frac{1}{k_2(u)})\overrightarrow{N}_{1,out}(u,t) - \nabla\rho(u,t), \quad \forall (u,t) \in S^1 \times [0,T), \\ F(u,0) = F_0(u), \\ \frac{\partial F}{\partial t}(u,0) = f(u)\overrightarrow{N}_0. \end{cases} \tag{3}$$

where $\nabla\rho = \langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t} \rangle \overrightarrow{T}$. For the sake of discussion, we set the initial velocity $f(u) \equiv 0$ for all $u \in S^1$.

2. The Basic Knowledge

Here we use the arc-length parameter s which is defined by

$$s(u,t) = \int_0^u v(x,t) \mathrm{d}x,$$

where $v(u,t) = \sqrt{x_u^2 + y_u^2}$. The element of arc-length is: $\mathrm{d}s = v\mathrm{d}u$, and the operator $\frac{\partial}{\partial s}$ is given in terms of u by $\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}$.

As [7], we can choose a suitable parametrization for convenience, and the following notion [10, 9] is needed.

Definition 2.1. *A curve $F : S^1 \times [0,T) \longrightarrow \mathrm{R}^2$ evolves normally if $\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial u} \rangle = 0$ for all $(u,t) \in S^1 \times [0,T)$.*

Lemma 2.2. *If the evolving curve ℓ is closed, then there is a parameter change ϕ for ℓ, such that $\ell \circ \phi$ is a normally evolving curve.*

The lemma ensures that we can limit our attention to curves which evolve normally within a large class of time-dependent curves without essential loss of generality.

Remark 2.3. For the initial value problem (3), if the initial velocity filed is normal to the curve at the beginning, then the property will be preserved during the evolution.

In fact, due to $\frac{\partial^2 F}{\partial t^2} = (\frac{1}{k_1(u,t)} - \frac{1}{k_2(u)})\overrightarrow{N}_{1,out}(u,t) - \nabla\rho(u,t)$ and $\mathrm{d}s = v\mathrm{d}u$, we have

$$\begin{aligned}&\frac{\partial}{\partial t}\langle\frac{\partial F}{\partial t},\frac{\partial F}{\partial u}\rangle = \langle\frac{\partial^2 F}{\partial t^2},\frac{\partial F}{\partial u}\rangle + \langle\frac{\partial F}{\partial t},\frac{\partial^2 F}{\partial t\partial u}\rangle\\ &= \langle(\frac{1}{k_1(u,t)} - \frac{1}{k_2(u)})\overrightarrow{N}_{1,out}(u,t) - \nabla\rho(u,t), v\frac{\partial F}{\partial s}\rangle + \langle\frac{\partial F}{\partial t}, v\frac{\partial^2 F}{\partial s\partial t}\rangle\\ &= -|\nabla\rho|v + v|\nabla\rho| = 0.\end{aligned}$$

from the above, we know $\langle\frac{\partial F}{\partial t},\frac{\partial F}{\partial u}\rangle$ is time-invariant, the velocity field of the initial curve is $f(u)\overrightarrow{N}_0$, and $\langle f(u)\overrightarrow{N}_0, \frac{\partial F}{\partial u}(u,0)\rangle = 0$, so $\langle\frac{\partial F}{\partial t},\frac{\partial F}{\partial u}\rangle = 0$ for all $(u,t) \in S^1 \times [0,T)$. That is to say, the flow (3) is a normal flow and naturally the following flow

$$\begin{cases} \frac{\partial F}{\partial t} = \sigma\overrightarrow{N}_{1,out}(u,t), \\ F(u,0) = F_0(u), \end{cases} \tag{4}$$

where $\sigma = \int_0^t(\frac{1}{k_1(u,\xi)} - \frac{1}{k_2(u)})\mathrm{d}\xi$. Then $\frac{\partial\sigma}{\partial t} = \frac{1}{k_1(u,t)} - \frac{1}{k_2(u)}$, $\sigma\frac{\partial\sigma}{\partial s} = \langle\frac{\partial^2 F}{\partial s\partial t},\frac{\partial F}{\partial t}\rangle$. As we know, s is a function of u and t, so $\frac{\partial}{\partial s}$ is not a partial derivative, which makes us change the parameters u and t to new parameters θ and τ, θ is the outward normal angle and τ is the new time parameter. Now we can use the normal angle to parameter the convex curves $F(u,t): S^1 \times [0,T) \longrightarrow \mathrm{R}^2$, i.e.,

$$\widetilde{F}(\theta,\tau) = F(u(\theta,\tau), t(\theta,\tau)), \text{ where } t(\theta,\tau) = \tau.$$

Set

$$\overrightarrow{N}_{1,out} = (\cos\theta, \sin\theta), \overrightarrow{T} = (-\sin\theta, \cos\theta),$$

and from now on, we will write $\overrightarrow{N}$ instead of $\overrightarrow{N}_{1,out}$ for the sake of simplification. By Frenet formular

$$\frac{\partial\overrightarrow{T}}{\partial s} = -k\overrightarrow{N}, \frac{\partial\overrightarrow{N}}{\partial s} = k\overrightarrow{T}$$

which implies that

$$\frac{\partial\theta}{\partial s} = k,$$

and

$$\frac{\partial\overrightarrow{N}}{\partial t} = \frac{\partial\theta}{\partial t}\overrightarrow{T}, \frac{\partial\overrightarrow{T}}{\partial t} = -\frac{\partial\theta}{\partial t}\overrightarrow{N}.$$

Since $\overrightarrow{T} = \frac{\partial F}{\partial s}$, we have $\frac{\partial\overrightarrow{T}}{\partial t} = \frac{\partial}{\partial t}(\frac{\partial F}{\partial s})$. In order to changing the partials, we also need

$$\frac{\partial v}{\partial t} = k\sigma v, \quad \frac{\partial^2}{\partial t\partial s} = -k\sigma\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s\partial t}.$$

Thus we have

$$\frac{\partial \overrightarrow{T}}{\partial t} = \frac{\partial \sigma}{\partial s}\overrightarrow{N}.$$

By $\frac{\partial}{\partial t}\langle \overrightarrow{N}, \overrightarrow{T}\rangle = 0$, we know that $\frac{\partial \overrightarrow{N}}{\partial t} = -\frac{\partial \sigma}{\partial s}\overrightarrow{T}$, and so $\frac{\partial \theta}{\partial t} = -\frac{\partial \sigma}{\partial s}$.

From the above formulas, we know that $\overrightarrow{T}$ and $\overrightarrow{N}$ are independent of τ. In fact, owing to

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial \theta}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t} = 0,$$

we have

$$\frac{\partial \theta}{\partial t} = -\frac{\partial \theta}{\partial u}\frac{\partial u}{\partial \tau} = -\frac{\partial \theta}{\partial s}\frac{\partial \theta}{\partial s}\frac{\partial u}{\partial \tau}\frac{\partial u}{\partial \tau} = -kv\frac{\partial u}{\partial \tau}.$$

Then

$$\frac{\partial \overrightarrow{T}}{\partial \tau} = \frac{\partial \overrightarrow{T}}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial \overrightarrow{T}}{\partial t} = -(kv\frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t})\overrightarrow{N} = 0,$$
$$\frac{\partial \overrightarrow{N}}{\partial \tau} = \frac{\partial \overrightarrow{N}}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial \overrightarrow{N}}{\partial t} = (kv\frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t})\overrightarrow{T} = 0,$$

and

$$\frac{\partial \widetilde{F}}{\partial \tau} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial F}{\partial t}, \frac{\partial^2 \widetilde{F}}{\partial \tau^2} = \frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 F}{\partial u^2}(\frac{\partial u}{\partial \tau})^2 + 2\frac{\partial^2 F}{\partial u \partial t^2}\frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}.$$

Since both of γ_1 and γ_2 are smooth convex closed curves, we can use the outward normal angle θ to parameterize them simultaneously, and the support function of $\gamma_{1,t}$ is given by

$$u_1(\theta, \tau) = \langle \widetilde{F}(\theta, \tau), \overrightarrow{N}\rangle = \langle \widetilde{F}(\theta, \tau), (\cos\theta, \sin\theta)\rangle.$$

We have already had the conclusion that if $u_{\theta\theta} + u > 0$ for all $\theta \in S^1$, then the relationship between the position vector and the support function is one-to-one correspondence. It immediately follows that we can use one to express the other one: $F(\theta, t) = u(\theta, t)(\cos\theta, \sin\theta) + u_\theta(\theta, t)(-\sin\theta, \cos\theta)$. As a result, all geometric properties of the curve $F(\theta, t)$ can be expressed by the support function u. In particular, the curvature can be written as

$$k = \frac{1}{u_{\theta\theta} + u}.$$

In the following, the relationship of u and its derivatives about θ and τ will be deduced, we will replace τ by t for simplicity. Since

$$u_1(\theta, t) = \langle \widetilde{F}(\theta, t), \overrightarrow{N}\rangle,$$

we have

$$\frac{\partial u_1}{\partial t} = \langle \frac{\partial \widetilde{F}}{\partial t}, \overrightarrow{N}\rangle = \widetilde{\sigma} = \int_0^t [\frac{1}{k_1(\theta, \xi)} - \frac{1}{k_2(\theta)}]\mathrm{d}\xi,$$
$$\frac{\partial^2 u_1}{\partial t^2} = \frac{1}{k_1(\theta, t)} - \frac{1}{k_2(\theta)} = (u_1 - u_2)_{\theta\theta} + (u_1 - u_2),$$

i.e., the support function of $\gamma_{1,t}$ satisfies

$$\frac{\partial^2}{\partial t^2}(u_1 - u_2) = (u_1 - u_2)_{\theta\theta} + (u_1 - u_2), \tag{5}$$

and the initial value condition is $u_1(\theta, 0) = u_0(\theta)$, $\frac{\partial u_1}{\partial t}(\theta, 0) = 0$, where $u_0(\theta)$ is the support function of the initial curve $\gamma_1(\theta)$. Obviously, it is a second-order partial differential equation with the initial value condition. Let us introduce a definition from [3], a symbol L denotes a second-order partial differential operator for each time t, having the non-divergence form

$$\mathrm{L}u = -\sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u$$

for given coefficients a^{ij},b^i,c $(i, j = 1, 2, \cdots, n)$.

Definition 2.4. *We say the partial differential operator* $\frac{\partial^2}{\partial t^2} + \mathrm{L}$ *is (uniformly) hyperbolic if there exists a constant* $\theta > 0$ *such that* $\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$ *for all* $(x,t) \in U_T, \xi \in R^n$.

In our case, $U_T = S^1 \times \mathrm{R}$, $\xi \in \mathrm{R}$ and $a^{ij} = 1$, $b^i = 0$, $c = 1$ $(i, j = 1, 2, \cdots, n)$. Thus, (5) is a second-order (uniformly) hyperbolic equation. The standard theory of second-order hyperbolic equations [3] ensures the existence and uniqueness of a family of smooth convex closed solutions $u_1(\theta, t) : S^1 \times [0, T) \to \mathrm{R}^2$ to (5) for some short time $T > 0$. From the one-to-one correspondence property, we can obtain the main result of this paper.

Theorem 2.5. (Local existences and uniqueness) *Suppose that* $\gamma_1(\theta)$ *and* $\gamma_2(\theta)$ *are smooth strictly convex closed plane curves. Then there exist a positive* T *and a family of strictly convex closed curves* $\gamma_1(\theta, t)$ *(where* $t \in [0, T)$*), such that* $\gamma_1(\theta, t)$ *satisfy* (3) *or* (4).

3. Examples and Evolutions

In this section, we try to understand the evolution behavior of flow. Let L denote the perimeter of a plane curve and A denote the area enclosed by a plane curve.

Remark 3.1. Assume $\gamma_1(\theta)$ and $\gamma_2(\theta)$ are initial smooth strictly convex closed plane curves, $\gamma_1(\theta, t)$ (denoted by $\gamma_{1,t}$) evolves by (3) and $\gamma_2(\theta)$ is the steady one. If $L(\gamma_1(\theta)) > L(\gamma_2(\theta))$, then $\gamma_{1,t}$ will expand for $t \in [0, \infty)$, and as $t \to \infty$, $\min_{\theta \in S^1} u_1(\theta, t) \to \infty$; if $L(\gamma_1(\theta)) < L(\gamma_2(\theta))$, then $\gamma_{1,t}$ will converge to a point in a finite time $[0, T^*]$, and $\max_{\theta \in S^1} u_1(\theta, t) \to 0$ as $t \to T^*$.

Example 3.2. Consider $\gamma_1(\theta, t)$ to be a family of round circles with the radius $R_1(t)$ centered at the origin, $\gamma_2(\theta)$ is a round circle with radius R_2. Then (4) gives

$$\begin{cases} \frac{d^2R_1(t)}{dt^2} = R_1(t) - R_2, \\ R_1(0) = R_1, R_{1t}(0) = 0. \end{cases}$$

By solving the above equation, we have $R_1(t) = R_2 + \frac{R_1 - R_2}{2}(e^t + e^{-t})$.

If $L(\gamma_1(\theta, 0)) > L(\gamma_2(\theta))$, or $R_1 > R_2$ equivalently, then as $t \to \infty$, we have $R_1(t) \to \infty$; if $L(\gamma_1(\theta, 0)) < L(\gamma_2(\theta))$, or $R_1 < R_2$ equivalently, then there exists a finite time

$$T^* = \ln[\frac{R_2}{R_2 - R_1} + \sqrt{(\frac{R_2}{R_2 - R_1})^2 - 1}]$$

such that $R_1(T^*) = 0$.

Lemma 3.3. *Let $L_1(t), A_1(t), k_1(\theta, t)$ be the perimeter, enclosed area, and curvature of $\gamma_1(\theta, t)$, respectively. Then we have*

$$\begin{aligned} \frac{\mathrm{d}L_1(t)}{\mathrm{d}t} &= \int_0^{2\pi} \int_0^t [\frac{1}{k_1(\theta, \xi)} - \frac{1}{k_2(\theta)}] \mathrm{d}\xi \mathrm{d}\theta, \\ \frac{\mathrm{d}A_1}{\mathrm{d}t} &= \int_0^{2\pi} \frac{1}{k_1(\theta, t)} \int_0^t [\frac{1}{k_1(\theta, \xi)} - \frac{1}{k_2(\theta)}] \mathrm{d}\xi \mathrm{d}\theta, \\ \frac{\partial k_1(\theta, t)}{\partial t} &= -k_1^2 \{ \int_0^t [\frac{1}{k_1(\theta, \xi)} - \frac{1}{k_2(\theta)}]_{\theta\theta} \mathrm{d}\xi + \int_0^t [\frac{1}{k_1(\theta, \xi)} - \frac{1}{k_2(\theta)}] \mathrm{d}\xi \}. \end{aligned}$$

Proof. Since

$$\begin{aligned} L_1(t) &= \int_{S^1} \mathrm{d}s = \int_{S^1} \frac{1}{k_1} \mathrm{d}\theta = \int_{S^1} v(\theta, t) \mathrm{d}\theta, \\ A_1(t) &= \frac{1}{2} \int_{S^1} \langle F, v\overrightarrow{N} \rangle \mathrm{d}\theta = -\frac{1}{2} \int_{S^1} \frac{u_1}{k} \mathrm{d}\theta, \end{aligned}$$

we have

$$\begin{aligned} \frac{\mathrm{d}L_1(t)}{\mathrm{d}t} &= \int_{S^1} \frac{\partial v(\theta, t)}{\partial t} \mathrm{d}\theta = \int_{S^1} k\sigma v \mathrm{d}\theta = \int_{S^1} \sigma \mathrm{d}\theta \\ &= \int_{S^1} \int_0^t [\frac{1}{k_1(\theta, \xi)} - \frac{1}{k_2(\theta)}] \mathrm{d}\xi \mathrm{d}\theta, \\ \frac{\mathrm{d}A_1(t)}{\mathrm{d}t} &= -\frac{1}{2} \int_0^{2\pi} \frac{u_{1t}k_1 - u_1 k_{1t}}{k_1^2} \mathrm{d}\theta = -\int_0^{2\pi} \frac{u_{1t}}{k_1} \mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{1}{k_1(\theta, t)} \int_0^t [\frac{1}{k_1(\theta, \xi)} - \frac{1}{k_2(\theta)}] \mathrm{d}\xi \mathrm{d}\theta. \end{aligned}$$

By using $k_1 = \frac{1}{u_{1\theta\theta} + u_1}$, we derive that

$$\begin{aligned} k_{1t} &= \frac{\partial k_1}{\partial t} = -\frac{u_{1\theta\theta t} + u_{1t}}{(u_{1\theta\theta} + u_1)^2} = -k_1^2(u_{1\theta\theta t} + u_{1t}) = -k_1^2(\sigma_{\theta\theta} + \sigma) \\ &= -k_1^2 (\int_0^t [\frac{1}{k_1(\theta, \xi)} - \frac{1}{k_2(\theta)}]_{\theta\theta} \mathrm{d}\xi + \int_0^t [\frac{1}{k_1(\theta, \xi)} - \frac{1}{k_2(\theta)}] \mathrm{d}\xi). \end{aligned}$$

■

Theorem 3.4. *If the initial curves γ_1 and γ_2 satisfy*

$$\int_{S^1} u_1(\theta)\mathrm{d}\theta = \int_{S^1} u_2(\theta)\mathrm{d}\theta, \tag{6}$$

that means $L(\gamma_1) = L(\gamma_2)$, then we have $\int_{S^1} u_1(\theta, t)\mathrm{d}\theta = \int_{S^1} u_2(\theta)\mathrm{d}\theta$, i.e.,

$$L_1(t) \doteq L(\gamma_{1,t}) = L(\gamma_2) \ for \ all \ t \in [0, \infty),$$

the flow (4) *is length-preserving.*

Since $L = \int_{S^1} \frac{1}{k}\mathrm{d}\theta = \int_{S^1} u\mathrm{d}\theta$, we can derive the parallel theorem of the above theorem.

Theorem 3.5. *If the initial curves γ_1 and γ_2 satisfy*

$$\int_{S^1} \frac{1}{k_1(\theta)}\mathrm{d}\theta = \int_{S^1} \frac{1}{k_2(\theta)}\mathrm{d}\theta, \tag{7}$$

that means $L(\gamma_1) = L(\gamma_2)$, then we have $\int_{S^1} \frac{1}{k_1(\theta,t)}\mathrm{d}\theta = \int_{S^1} \frac{1}{k_2(\theta)}\mathrm{d}\theta$, i.e.,

$$L_1(t) = L(\gamma_2) \ for \ all \ t \in [0, \infty),$$

and the flow (4) *is length-preserving.*

Proof. By Lemma 3.3

$$\frac{dL_1(t)}{\mathrm{d}t} = \int_0^{2\pi} \int_0^t [\frac{1}{k_1(\theta, \xi)} - \frac{1}{k_2(\theta)}]\mathrm{d}\xi\mathrm{d}\theta,$$

we have

$$\frac{\mathrm{d}^2 L_1(t)}{\mathrm{d}t^2} = \int_0^{2\pi} [\frac{1}{k_1(\theta, t)} - \frac{1}{k_2(\theta)}]\mathrm{d}\theta = L_1(t) - L(\gamma_2).$$

It implies that

$$\int_0^{2\pi} [\frac{1}{k_1(\theta, t)} - \frac{1}{k_2(\theta)}]\mathrm{d}\theta = c_1 e^t + c_2 e^{-t}.$$

When $t = 0$, owing to the initial value condition, we have

$$c_1 + c_2 = \int_0^{2\pi} [\frac{1}{k_1(\theta)} - \frac{1}{k_2(\theta)}]\mathrm{d}\theta, \quad \frac{\mathrm{d}L_1(t)}{\mathrm{d}t}|_{t=0} = 0 = c_1 - c_2.$$

Then, the solution is $\int_0^{2\pi} [\frac{1}{k_1(\theta,t)} - \frac{1}{k_2(\theta)}]\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} [\frac{1}{k_1(\theta)} - \frac{1}{k_2(\theta)}]\mathrm{d}\theta(e^t + e^{-t})$, i.e.,

$$L_1(t) - L(\gamma_2) = \frac{1}{2}[L(\gamma_1) - L(\gamma_2)](e^t + e^{-t}).$$

The proof of Theorem 3.4 is completed. ∎

From the above proof, we can derive a result which is similar with Remark 3.1.

Remark 3.6. The area $A_1(t)$ enclosed by the closed curve $\gamma_1(\theta,t)$ is not preserved.

In fact, suppose the two initial curves satisfy

$$\int_0^{2\pi}\frac{u_1(\theta)}{k_1(\theta)}\mathrm{d}\theta=\int_0^{2\pi}\frac{u_2(\theta)}{k_2(\theta)}\mathrm{d}\theta,\quad i.e.,\quad A(\gamma_1)=A(\gamma_2),$$

and we have known that

$$\frac{\mathrm{d}A_1(t)}{\mathrm{d}t}=-\frac{1}{2}\int_0^{2\pi}\frac{u_{1t}k_1-u_1k_{1t}}{k_1^2}\mathrm{d}\theta=-\int_0^{2\pi}\frac{u_{1t}}{k_1}\mathrm{d}\theta.$$

Then

$$\begin{aligned}\frac{d^2A_1(t)}{dt^2}&=-\int_0^{2\pi}\{\frac{1}{k_1(\theta,t)}[\frac{1}{k_1(\theta,t)}-\frac{1}{k_2(\theta)}]-\frac{k_{1t}}{k_1^2}\int_0^t[\frac{1}{k_1(\theta,\xi)}-\frac{1}{k_2(\theta)}]\mathrm{d}\xi\}\mathrm{d}\theta\\&=\alpha(t)+\beta(t),\end{aligned}$$

where

$$\begin{aligned}\alpha(t)&=-\int_0^{2\pi}\frac{1}{k_1(\theta,t)}[\frac{1}{k_1(\theta,t)}-\frac{1}{k_2(\theta)}]\mathrm{d}\theta,\\ \beta(t)&=-\int_0^{2\pi}\int_0^t[\frac{1}{k_1(\theta,\xi)}-\frac{1}{k_2(\theta)}]\mathrm{d}\xi\cdot\int_0^t\{[\frac{1}{k_1(\theta,\xi)}-\frac{1}{k_2(\theta)}]_{\theta\theta}\\&\qquad+[\frac{1}{k_1(\theta,\xi)}-\frac{1}{k_2(\theta)}]\}\mathrm{d}\xi\mathrm{d}\theta.\end{aligned}$$

When $t=0$, we have $\frac{\mathrm{d}A_1(t)}{\mathrm{d}t}\mid_{t=0}=0$, but

$$\frac{\mathrm{d}^2A_1(t)}{\mathrm{d}t^2}\mid_{t=0}=-\int_0^{2\pi}\frac{1}{k_1(\theta)}[\frac{1}{k_1(\theta)}-\frac{1}{k_2(\theta)}]\mathrm{d}\theta\neq 0.$$

Thus, there exists a small time interval $(0,\varepsilon)$ such that $\frac{\mathrm{d}A_1(t)}{\mathrm{d}t}\neq 0$, and the flow is not area-preserving.

By the evolving equation of $k_1(\theta,t)$, we have

$$\frac{\partial^2}{\partial t^2}[\frac{1}{k_1(\theta,t)}-\frac{1}{k_2(\theta)}]=[\frac{1}{k_1(\theta,t)}-\frac{1}{k_2(\theta)}]_{\theta\theta}+[\frac{1}{k_1(\theta,t)}-\frac{1}{k_2(\theta)}].$$

Let $z(\theta,t)=\frac{1}{k_1(\theta,t)}-\frac{1}{k_2(\theta)}$. Then we just have to focus on the following equation

$$\frac{\partial^2 z}{\partial t^2}(\theta,t)=z_{\theta\theta}(\theta,t)+z(\theta,t),$$

where $z(\theta,t)$ is a 2π-periodic function about θ on S^1, and it satisfies

$$\int_{S^1} z(\theta,t)\mathrm{d}\theta = 0.$$

It is a second-order hyperbolic equation identified with (5), and we can give analysis on the existence, uniqueness and regularity about it. Furthermore, we have known that $z(\theta,t) = \frac{1}{k_1(\theta,t)} - \frac{1}{k_2(\theta)}$ is a 2π-periodic function about θ on S^1, so we try to use Fourier series to study the equation.

Assume the Fourier series expansion of $z(\theta,t) = \frac{1}{k_1(\theta,t)} - \frac{1}{k_2(\theta)}$ is

$$z(\theta,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty}(a_n(t)\cos n\theta + b_n(t)\sin n\theta).$$

Thus

$$\begin{aligned} z_{tt} &= \frac{a_0''(t)}{2} + \sum_{n=1}^{\infty}(a_n''(t)\cos n\theta + b_n''(t)\sin n\theta), \\ z_{\theta\theta} &= \sum_{n=1}^{\infty}(-n^2)(a_n(t)\cos n\theta + b_n(t)\sin n\theta). \end{aligned}$$

Comparing coefficients, we have

$$a_0''(t) = a_0(t), \quad a_n''(t) = (1-n^2)a_n(t), \quad b_n''(t) = (1-n^2)b_n(t). \quad (n = 1, 2, \cdots)$$

Solving the above ordinary differential equation, we get

$$\begin{aligned} a_0(t) &= a_1 e^t + a_2 e^{-t}, \quad a_1(t) = b_1 t + b_2, \quad b_1(t) = c_1 t + c_2, \\ a_n(t) &= d_1 \cos\sqrt{n^2-1}t + d_2 \sin\sqrt{n^2-1}t, \\ b_n(t) &= e_1 \cos\sqrt{n^2-1}t + e_2 \sin\sqrt{n^2-1}t, \end{aligned}$$

for $n = 2, 3, \cdots$, where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2$ are arbitrary constants.

Considering the initial condition $\int_{S^1} z(\theta,t)\mathrm{d}\theta = 0$, we have $a_0(t) = 0$. Therefore,

$$\begin{aligned} Z(\theta,t) =\ & (b_1 t + b_2)\cos\theta + (c_1 t + c_2)\sin\theta + \sum_{n=2}^{\infty}[(d_1 \cos\sqrt{n^2-1}t \\ & + d_2 \sin\sqrt{n^2-1}t)\sin n\theta + (e_1\cos\sqrt{n^2-1}t + e_2 \sin\sqrt{n^2-1}t)\cos n\theta], \end{aligned}$$

that is,

$$\begin{aligned} \frac{1}{k_1(\theta,t)} =\ & \frac{1}{k_2(\theta)} + (b_1 t + b_2)\cos\theta + (c_1 t + c_2)\sin\theta \\ & + \sum_{n=2}^{\infty}[(d_1 \sin n\theta + e_1 \cos n\theta)\cos\sqrt{n^2-1}t \\ & + (d_2 \sin n\theta + e_2 \cos n\theta)\sin\sqrt{n^2-1}t]. \end{aligned}$$

Frankly speaking, we have not derived any result from the expression of $\frac{1}{k_1(\theta,t)}$ for there are lots of arbitrary constant. But we conjecture that if we impose some conditions on the difference between $\frac{1}{k_1(\theta)}$ and $\frac{1}{k_2(\theta)}$ and make it controlled by the minimum width of $\gamma_1(\theta,t)$, where width means the sum of support function $u(\theta)+u(\theta+\pi)$, the flow will last forever without exploding.

The corresponding work will be done later.

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