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$*$-0-Distributive Almost Lattices

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Abstract. The concept of $*$-0-distributive almost lattice ($*$-0-DAL) is introduced. Some necessary and sufficient conditions for a 0-distributive almost lattice to become $*$-0-distributive almost lattice in topological and algebraic terms are proved.

Keywords: Almost lattice; 0-Distributive almost lattice; Boolean algebra; Dense elements; Ideal; Prime ideal; Minimal prime ideal; Filter; Prime filter; Maximal filter; Pseudo-complementation; Hull-kernel topology; Dual hull-kernel topology; $*$-0-Distributive almost lattice.

1. Introduction

The concept of almost lattice (AL) was introduced by G. Nanaji Rao and H.T. Alemu [3] as a common abstraction of almost all lattice theoretic generalization of Boolean algebra like distributive lattice, almost distributive lattice and lattices and established necessary and sufficient condition for an AL to become a lattice. The class of ALs with pseudo-complementation was introduced by G. Nanaji Rao and R. Venkata Aravinda Raju [6] and it was observed that an AL with 0 can have more than one pseudo-complementation unlike in the case of lattice. In fact, if there is a pseudo-complementation on an AL L then each maximal element of L corresponds to a pseudo-complementation on L and this correspondence is one-to-one. Also, it was proved that if $*$ is a pseudo-complementation on L then the set $L^* = \{a^* : a \in L\}$ is a Boolean algebra which is independent (upto isomorphism) of pseudo-complementation $*$. Later the concept of annihilator of a nonempty subset of an AL L with 0 was introduced by G. Nanaji Rao and R. Venkata Aravinda Raju [7] and they proved some basic properties of annihilators in L. Also, they introduced the concept of 0-distributive ALs and obtained necessary and sufficient conditions for an AL with 0 to become 0-distributive AL in terms of annihilators, ideals and pseudo-complementation. The concept of annihilator ideal in an AL with 0 was introduced by G. Nanaji Rao and R. Venkata Aravinda Raju [8] and they proved that the set AL of all annihilator ideals in an AL L is a complete Boolean algebra. Also, they introduced the concept of annihilator preserving homomorphism and established a sufficient condition for an AL homomorphism to become annihilator preserving homomorphism.

In this paper, we introduce the concept of $*$-0-distributive almost lattice. First we establish a necessary and sufficient condition for a prime ideal in a 0-distributive AL to become minimal prime ideal. We define a relation θ on an AL by $(x, y) \in \theta$ if and only if $[x]^* = [y]^*$ and prove that θ is a congruence relation on L. Also, we prove that if L is a 0-distributive AL then L is a $*$-0-distributive AL if and only if L/θ is a Boolean algebra. We also derive that L is a $*$-0-distributive AL if and only if $\mu = \{M_x : x \in L\}$ is a Boolean algebra. In other words, we characterise $*$-0-distributive almost lattice in algebraic terms. Also, some necessary and sufficient conditions for a 0-distributive AL L to become $*$-0-distributive AL in terms of the hull-kernel topology on the set of all (minimal) prime ideals of L.

2. Preliminaries

In this section we collect few important definitions and results which are already known and which will be used more frequently in the text.

Definition 2.1. *A partial order $\leq$ on a set P is called a total order, if for any $a,\ b \in R$, either $a \leq b$ or $b \leq a$ holds. In this case, the poset $(P, \leq)$ is called a totally ordered set or a chain.*

Definition 2.2. *(Zorn's Lemma) If every subchain of a nonempty partly ordered set P has an upper bound, then P contains a maximal element.*

Definition 2.3. *Let $(P, \leq)$ be a poset. Then P is said to be lattice ordered set if for every pair $x,\ y \in P$, $l.u.b\{x,\ y\}$ and $g.l.b\{x,\ y\}$ exists.*

Definition 2.4. *An algebra $(L, \vee, \wedge)$ of type $(2, 2)$ is called a lattice if it satisfies the following axioms. For any $x, y, z \in L$,*

(1) $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$ *(Commutative Law).*
(2) $(x \vee y) \vee z = x \vee (y \vee z)$ *and* $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ *(Associative Law).*
(3) $x \vee (x \wedge y) = x$ *and* $x \wedge (x \vee y) = x$ *(Absorption Law).*

It can be easily seen that in any lattice $(L, \vee, \wedge)$, $x \vee x = x$ and $x \wedge x = x$ (Idempotent Law).

Theorem 2.5. *Let* $(L, \leq)$ *be a lattice ordered set. If we define* $x \wedge y$ *is the g.l.b of* $\{x\ ,\ y\}$ *and* $x \vee y$ *is the l.u.b of* $\{x\ ,\ y\}$ $(x, y \in L)$*, then* $(L, \vee, \wedge)$ *is a lattice.*

Theorem 2.6. *Let* $(L, \vee, \wedge)$ *be a lattice. If we define a relation* $\leq$ *on* L*, by* $x \leq y$ *if and only if* $x = x \wedge y$*, or equivalently* $x \vee y = y$*. Then* $(L, \leq)$ *is a lattice ordered set.*

Important Note. Theorems 2.5 and 2.6 together imply that the concepts of lattice and lattice ordered set are the same. We refer to it as a lattice in future.

Definition 2.7. *Let* $(L, \vee, \wedge)$ *be a lattice. Then* L *is said to be a bounded lattice if* L *is bounded as a poset.*

Definition 2.8. *A bounded lattice* L *with bounds* 0 *and* 1 *is said to be complemented if to each* $x \in L$*, there exists* $y \in L$ *such that* $x \wedge y = 0$ *and* $x \vee y = 1$*.*

Theorem 2.9. *In any lattice* $(L, \vee, \wedge)$*, for any* $x,\ y,\ z \in L$*, the following statements are equivalent:*

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
(2) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.
(3) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
(4) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Definition 2.10. *A lattice* $(L, \vee, \wedge)$ *is called a distributive lattice if it satisfies any one of the four conditions in Theorem 2.9.*

Definition 2.11. *A complemented distributive lattice is called a Boolean algebra.*

Definition 2.12. *An algebra* $(L, \vee, \wedge, 0)$ *of type* $(2, 2, 0)$ *is called an almost lattice (AL) with* 0 *if, for any* $a, b, c \in L$*, it satisfies the following conditions:*

(1) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$,
(2) $(a \vee b) \wedge c = (b \vee a) \wedge c$,
(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,
(4) $(a \vee b) \vee c = a \vee (b \vee c)$,
(5) $a \wedge (a \vee b) = a$,
(6) $a \vee (a \wedge b) = a$,
(7) $(a \wedge b) \vee b = b$,
(8) $0 \wedge a = 0$.

It can be easily seen that $a \wedge b = a$ if and only if, $a \vee b = b$ in an AL.

Definition 2.13. *Let L be an AL and $a, b \in L$. Then we define a is less than or equal to b and write $a \leq b$ if and only if $a \wedge b = a$ or equivalently $a \vee b = b$.*

Theorem 2.14. *The relation $\leq$ is a partial ordering on an AL L and hence $(L, \leq)$ is a poset.*

Definition 2.15. *Let L be any nonempty set. Define, for any $x, y \in L$, $x \vee y = x = y \wedge x$. Then, clearly L is an AL and is called descrete AL.*

Theorem 2.16. *Let L be an AL and $m \in L$. Then the following statements are equivalent:*

(1) *m is maximal.*
(2) *$m \vee x = m$ for all $x \in L$.*
(3) *$m \wedge x = x$ for all $x \in L$.*

Theorem 2.17. *Let L be an AL. Then for any $m \in L$, the following statements are equivalent:*

(1) *m is minimal.*
(2) *$x \wedge m = m$ for all $x \in L$.*
(3) *$x \vee m = x$ for all $x \in L$.*

Definition 2.18. *Let L be an AL. Then a nonempty subset I of L is said to be an ideal of L if it satisfies the following conditions:*

(1) *If $x, y \in I$, then there exists $d \in I$ such that $d \wedge x = x$ and $d \wedge y = y$.*
(2) *If $x \in I$ and $a \in L$, then $x \wedge a \in I$.*

Lemma 2.19. *Let L be an AL and I an ideal of L. Then the following statements are equivalent:*

(1) *$x, y \in I$ implies $x \vee y \in I$.*
(2) *$x, y \in I$ implies there exists $d \in I$ such that $d \wedge x = x$ and $d \wedge y = y$.*

Theorem 2.20. *Let S be a nonempty subset of an AL L. Then $(S] = \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, \text{ for } 1 \leq i \leq n,\ x \in L \text{ and } n \in Z^+\}$ is the smallest ideal of L containing S.*

Corollary 2.21. *Let L be an AL and $a \in L$. Then $(a] = \{a \wedge x \mid x \in L\}$ is an ideal of L and is called principal ideal generated by a.*

Corollary 2.22. *Let L be an AL and $a,\ b \in L$. Then $a \in (b]$ if and only if $a = b \wedge a$.*

Corollary 2.23. *Let L be an AL and $a,\ b \in L$. Then $(a \wedge b] = (b \wedge a]$ and $(a \vee b] = (b \vee a]$.*

Theorem 2.24. *Let L be an AL. Then the set $\mathcal{I}(L)$ of all ideals of L form a lattice under set inclusion in which the glb and lub for any $I, J \in \mathcal{I}(L)$ are respectively $I \wedge J = I \cap J$ and $I \vee J = \{x \in L : (a \vee b) \wedge x = x$ for some $a \in I$ and $b \in J\}$.*

Theorem 2.25. *Let L be an AL. Then the set $P\mathcal{I}(L)$ of all principal ideals of L is a sublattice of the lattice $\mathcal{I}(L)$ of all ideals of L.*

Definition 2.26. *Let L be an AL. Then a proper ideal P of L is said to be prime if for any $x,\ y \in L$, $x \wedge y \in P$, then either $x \in P$ or $y \in P$.*

Definition 2.27. *Let L be an AL. Then a nonempty subset F of L is said to be a filter if it satisfies the following conditions:*
(1) *$x,\ y \in F$, implies $x \wedge y \in F$.*
(2) *$x \in F$ and $a \in L$, implies $a \vee x \in F$.*

Theorem 2.28. *Let L be an AL with a minimal element (say) m. If F is a filter in L such that $m \in F$, then $F = L$.*

Theorem 2.29. *Let L be an AL and S a nonempty subset of L. Then $[S) = \{x \vee (\bigwedge_{i=1}^{n} s_i)|\ x \in L, s_i \in S$ for $1 \leq i \leq n$ and $n \in Z^+\}$ is the smallest filter of L containing S.*

Corollary 2.30. *Let L be an AL and $a \in L$. Then $[a) = \{x \vee a | x \in L\}$ is the smallest filter of L containing a and is called a principal filter generated by a.*

Corollary 2.31. *Let L be an AL. Then for any $a,\ b \in L$, $a \in [b)$ if and only if $a = a \vee b$.*

Corollary 2.32. *Let L be an AL and $a,\ b \in L$. Then $a \in [b)$ if and only if $[a) \subseteq [b)$.*

Corollary 2.33. *Let L be an AL. Then for any $x,\ y \in L$, $[x \vee y) = [y \vee x)$.*

Theorem 2.34. *Let L be an AL. Then the set $\mathcal{F}(L)$ of all filters of L is a lattice under set inclusion, in which the glb and lub of any F and G in $\mathcal{F}(L)$ are given respectively by, $F \wedge G = F \cap G$, $F \vee G = \{x \in L | x \vee (a \wedge b) = x$ for some $a \in F$ and $b \in G\}$*

Theorem 2.35. *Let L be a an AL. Then a subset P of L is a prime ideal of L if and only if $L - P$ is a prime filter.*

Definition 2.36. *Let L be an AL with 0. Then a unary operation $a \mapsto a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$, it satisfies the following conditions:*
(P_1) *$a \wedge b = 0 \Rightarrow a^* \wedge b = b$.*

(P_2) $a \wedge a^* = 0$.
(P_3) $(a \vee b)^* = a^* \wedge b^*$.

Definition 2.37. *Let L be an AL with 0. Then for any nonempty subset A of L, define $A^* = \{x \in L : x \wedge a = 0$ for all $a \in A\}$. Here A^* is called the annihilator of A in L.*

Note that if $A = \{a\}$ then we write $[a]^*$ instead of A^*. In the following we prove some basic properties of annihilators.

Definition 2.38. *Let L be an AL with 0. Then,*

(1) *An element a of an AL L is called dense element if $[a]^* = \{0\}$.*
(2) *L is said to be dense AL if every nonzero element in L is dense.*

Note that the set of all dense elements in an AL with 0 is denoted by D.

Theorem 2.39. *Let L be an AL with 0. Then for any $x, y \in L$, the following statements hold:*

(1) $(x] \cap [x]^* = (0]$,
(2) $[x]^* \cap [x]^{**} = (0]$,
(3) $(x]^* = [x]^*$,
(4) $(x]^* \cap [x]^{**} = (0]$,
(5) $x \leq y \Rightarrow [y]^* \subseteq [x]^*$,
(6) $[x \wedge y]^* = [y \wedge x]^*$,
(7) $[x \vee y]^* = [y \vee x]^*$,
(8) $(x] \subseteq [x]^{**}$,
(9) $[x]^{***} = [x]^*$,
(10) $[x]^* \subseteq [y]^* \Leftrightarrow [y]^{**} \subseteq [x]^{**}$,
(11) $[x \wedge y]^{**} = [x]^{**} \cap [y]^{**}$.

Definition 2.40. *Let L be an AL with 0. Then L is said to be 0-distributive if for any $a, b, c \in L$, $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$.*

Theorem 2.41. *Let L be an AL with 0. Then L is $0-$distributive if and only if for any nonempty subset A of L, A^* is an ideal of L.*

Theorem 2.42. *Every pseudo-complemented AL is a $0-$distributive AL.*

Corollary 2.43. *Let L be 0-distributive AL. Then for any $x, y \in L$, $[x \vee y]^* = [x]^* \cap [y]^*$.*

Lemma 2.44. *Let L be a 0-distributive AL. For any subset A of L, $A \cap A^* = \{0\}$*

3. Notations and Theorems on 0-Distributive ALs

Let L be a 0-distributive AL and $Y(M)$ the set of all prime(minimal) ideals in L. Let $\Im_h^Y$ be the hull-kernel topology on Y. That is, the topology on Y for which $\{Y_x : x \in L\}$ is a basis, where $Y_x = \{P \in Y : x \notin P\}$ for any $x \in L$. We write $\Im_h^M$ for the topology on M induced from $\Im_h^Y$. In this topology, for any $x \in L$ the corresponding basic open set $M \cap Y_x$ is denoted by M_x. The dual hull-kernel topology on $Y(M)$ is denoted by $\Im_d^Y(\Im_d^M)$. That is, the topology, for which $\{h_Y(x) : x \in L\}$ $(\{h_M(x) : x \in L\})$ is a basis, where $h_Y(x) = \{P \in Y : x \in P\}$ $(h_M(x) = \{P \in M : x \in P\})$ for any $x \in L$, .

In the hull-kernel topology on Y, open (closed) sets are of the form $Y_I(h_Y(I))$ where $Y_I = \{P \in Y : I \not\subset P\}$ and $h_Y(I) = Y - Y_I$. Similarly, in the hull-kernel topology on M, the open (closed) sets are of the form $M_I(h_M(I))$ where $M_I = \{P \in M : I \not\subset P\}$ and $h_M(I) = M - M_I$. Also, for any subset F of Y, the closure $\bar{F}$ of F in the hull-kernel topology on Y is given by $\bar{F} = \{Q \in Y : \bigcap_{P \in F} P \subseteq Q\}$.

We first prove some important results on prime ideals, prime filters, minimal prime ideals and maximal filters in 0-distributive ALs. We begin this section with the following theorem.

Theorem 3.1. *Let L be 0-distributive AL. Then every maximal filter in L is a prime filter.*

Proof. Suppose F is a maximal filter in L. Now, we shall prove F is a prime filter. Let $x, y \in L$ such that $x \notin F$ and $y \notin F$. Then $[x) \vee F$ and $[y) \vee F$ are filters of L which is properly contains F. Since F is maximal, $[x) \vee F = L$ and $[y) \vee F = L$. Now, since $0 \in L = [x) \vee F$, $0 \vee (a_1 \wedge b_1) = 0$ where $a_1 \in [x)$ and $b_1 \in F$. It follows that $a_1 \wedge b_1 = 0$ and hence $(a_1 \vee x) \wedge b_1 = 0$. Similarly, we get $(a_2 \vee y) \wedge b_2 = 0$ where $a_2 \in [y)$ and $b_2 \in F$. It follows that $x \wedge b_1 = 0$ and $y \wedge b_2 = 0$. This implies $(b_1 \wedge b_2) \wedge x = 0$ and $(b_1 \wedge b_2) \wedge y = 0$. Since L is 0-distributive, $(b_1 \wedge b_2) \wedge (x \vee y) = 0$. Now, if $x \vee y \in F$, then $0 = (b_1 \wedge b_2) \wedge (x \vee y) \in F$. It follows that $0 \in F$. Hence $F = L$, a contradiction to F is proper. Thus $x \vee y \notin F$. Therefore F is a prime filter. ∎

In the following we prove that every proper filter is contained in a maximal filter.

Theorem 3.2. *Let L be an AL with 0. Then every proper filter is contained in a maximal filter.*

Proof. Suppose F is a proper filter in L. Now, put $\mathcal{S} = \{H : H$ is a proper filter in L and $F \subseteq H\}$. Then clearly, $\mathcal{S}$ is nonempty, since $F \in \mathcal{S}$ and also, clearly $\mathcal{S}$ is a poset w.r.to set inclusion and satisfies the hypothesis of Zorn's lemma. Therefore by Zorn's lemma, $\mathcal{S}$ has a maximal element (say) M. Then M is a proper filter in L and $F \subseteq M$. Clearly M is a maximal filter containing F. ∎

It can be easily seen that if a filter F in an AL L contains the zero element then $F = L$. Now, we have the following corollary.

Corollary 3.3. *Let L be an AL with 0. Then every non-zero element in L is contained in a maximal filter.*

Proof. Suppose a is a non-zero element in L. Then clearly $[a)$ is a proper filter. Hence by Theorem 3.2, there exists a maximal filter (say) F in L such that $[a) \subseteq F$. It follows that $a \in F$. ■

Next, we prove that maximal filters and maximal prime filters are equivalent.

Theorem 3.4. *Let L be a 0-distributive AL. Then a filter F of L is maximal if and only if F is a maximal prime filter.*

Proof. Suppose F is a maximal filter in L. Then, by Theorem 3.1, F is a prime filter. Suppose H is a prime filter in L such that $F \subseteq H$. Now, if $F \neq H$ then $F \subsetneqq H$, a contradiction to F is maximal, since $H \neq L$. Therefore $F = H$. Thus F is a maximal prime filter. The converse follows by Theorems 3.1 and 3.2. ■

In the following we prove that the complement of a minimal prime ideal is a maximal filter and vice versa.

Theorem 3.5. *Let L be an AL and P a prime ideal of L. Then P is a minimal prime ideal if and only if $L - P$ is a maximal prime filter.*

Proof. Suppose P is a minimal prime ideal. Then P is nonempty proper subset of L and hence $L - P$ is nonempty proper subset of L. Also, by Theorem 2.35, $L - P$ is a prime filter. Suppose G is a prime filter of L such that $L - P \subseteq G$. Then $L - G \subseteq P$ and $L - G$ is a prime ideal. Therefore $L - G = P$. Hence $G = L - P$. Therefore $L - P$ is a maximal prime filter. Conversely, suppose $L - P$ is a maximal prime filter. Then we have P is a prime ideal. Suppose Q is a prime ideal of L such that $Q \subseteq P$. Then $L - P \subseteq L - Q$ and $L - Q$ is a prime filter. It follows that $L - P = L - Q$. Hence $P = Q$. Therefore P is a minimal prime ideal. ■

Corollary 3.6. *Let L be a 0-distributive AL. Then P is a minimal prime ideal if and only if $L - P$ is a maximal filter.*

Corollary 3.7. *Let L be a 0-distributive AL. If $a(\neq 0) \in L$, then there exists a minimal prime ideal of L not containing a.*

In the following, we derive a necessary and sufficient condition for a prime ideal of a 0-distributive AL to become a minimal prime ideal.

Theorem 3.8. *Let L be a 0-distributive AL and P a prime ideal of L. Then P is a minimal prime ideal if and only if $[x]^* - P \neq \emptyset$ for every $x \in P$.*

Proof. Suppose P is a minimal prime ideal of L. Then we have $L-P$ is a maximal filter of L. Let $x \in P$. Then $x \notin L-P$. It follows that $[x) \vee (L-P) = L$. Now, since $0 \in L = [x) \vee (L-P)$, $0 = 0 \vee (a \wedge b)$ where $a \in [x)$ and $b \in L-P$. It follows that $(a \vee x) \wedge b = 0$, $b \in L-P$. Hence we get $x \wedge b = 0$ and $b \in L-P$. Thus $b \in [x]^*$ and $b \notin P$. Therefore $[x]^* - P \neq \emptyset$. Conversely, assume the condition. Now, we shall prove P is a minimal prime ideal. Suppose Q is a prime ideal of L such that $Q \subsetneqq P$. Then there exists $x \in P$ such that $x \notin Q$. Therefore by assumption, $[x]^* - P \neq \emptyset$. Hence choose $t \in [x]^*$ such that $t \notin P$. This implies $t \wedge x = 0 \in Q$ and $x \notin Q$. Hence $t \in Q$ and $t \notin P$, a contradiction to $Q \subsetneqq P$. Therefore $Q = P$ and hence P is a minimal prime ideal. ∎

Corollary 3.9. *Let P be a minimal prime ideal in a 0-distributive AL L and let $x \in L$. Then $x \notin P$ if and only if $[x]^* \subseteq P$.*

Corollary 3.10. *Let P be a minimal prime ideal in a 0-distributive AL L and let $x \in L$. Then $[x]^{**} \nsubseteq P$ if and only if $[x]^* \subseteq P$.*

Lemma 3.11. *Let L be a 0-distributive AL. Then for any $x \in L$, $\bigcap_{P \in Y_x} P = [x]^*$.*

Proof. Suppose $a \notin [x]^*$. Then $a \wedge x \neq 0$. Hence there exists $P \in Y$ such that $a \wedge x \notin P$. It follows that $a \notin P$ and $x \notin P$. This implies $a \notin P$ and $P \in Y_x$. Thus $a \notin \bigcap_{P \in Y_x} P$. Therefore $\bigcap_{P \in Y_x} P \subseteq [x]^*$. Conversely, suppose $a \in [x]^*$ and $P \in Y_x$. Then $a \wedge x = 0 \in P$. Hence $a \in P$. Therefore $a \in \bigcap_{P \in Y_x} P$. Thus $[x]^* \subseteq \bigcap_{P \in Y_x} P$. Therefore $\bigcap_{P \in Y_x} P = [x]^*$. ∎

Lemma 3.12. *Let L be a 0-distributive AL. Then $\bigcap_{P \in Y} P = \{0\}$.*

Proof. The proof follows immediately from Corollary 3.7. ∎

Lemma 3.13. *Let L be a 0-distributive AL and I an ideal of L. Then $h_Y(I) = Y$ if and only if $I = \{0\}$.*

Proof. Suppose $h_Y(I) = Y$. Then $I \subseteq P$ for all $P \in Y$. Hence $I \subseteq \bigcap_{P \in Y} P = \{0\}$, we get $I = \{0\}$. Conversely, suppose $I = \{0\}$. Since $0 \in P$ for all $P \in Y$, $I = \{0\} \subseteq P$ for all $P \in Y$. Therefore $I \subseteq P$ for all $P \in Y$. Hence $P \in h_Y(I)$ for all $P \in Y$. Therefore $h_Y(I) = Y$. ∎

Now we improve some important relations between the ideals of 0-distributive AL and the corresponding open sets in hull-kernel topology on M which we will use in section 4 to characterize ∗-0-distributive ALs topologically.

Lemma 3.14. *Let L be a 0-distributive AL. Then for any $I, J \in \mathcal{I}(L)$, we have the following statements:*

(1) $I \subseteq J \Rightarrow M_I \subseteq M_J$.
(2) $I \subseteq J \Rightarrow h_M(J) \subseteq h_M(I)$.
(3) $M_I \cup M_J = M_{I \vee J}$.
(4) $M_I \cap M_J = M_{I \cap J}$.
(5) $h_M(I) \cup h_M(J) = h_M(I \cap J)$.
(6) $h_M(I) \cap h_M(J) = h_M(I \vee J)$.

Proof. (1) Suppose $I \subseteq J$ and suppose $P \in M_I$. Then $I \not\subseteq P$. So that $J \not\subseteq P$. Hence $P \in M_J$. Therefore $M_I \subseteq M_J$.

(2) Suppose $I \subseteq J$ and suppose $P \in h_M(J)$. Then $J \subseteq P$ and hence $I \subseteq P$. Thus $P \in h_M(I)$. Therefore $h_M(J) \subseteq h_M(I)$.

(3) We have $I, J \subseteq I \vee J$. Hence by condition (1) $M_I, M_J \subseteq M_{I \vee J}$. Therefore $M_I \cup M_J \subseteq M_{I \vee J}$. Conversely, suppose $P \notin M_I \cup M_J$. Then $P \notin M_I$ and $P \notin M_J$. This implies $I \subseteq P$ and $J \subseteq P$. It follows that $I \vee J \subseteq P$. Hence $P \notin M_{I \vee J}$. Thus $M_{I \vee J} \subseteq M_I \cup M_J$. Therefore $M_I \cup M_J = M_{I \vee J}$.

(4) We have $I \cap J \subseteq I, J$. Hence by condition (1) $M_{I \cap J} \subseteq M_I, M_J$. Therefore $M_{I \cap J} \subseteq M_I \cap M_J$. Conversely, suppose $P \in M_I \cap M_J$. Then $P \in M_I$ and $P \in M_J$. Then $I \not\subseteq P$ and $J \not\subseteq P$. This implies $I \cap J \not\subseteq P$, since P is a prime ideal. Hence $P \in M_{I \cap J}$. Thus $M_I \cap M_J \subseteq M_{I \cap J}$. Therefore $M_I \cap M_J = M_{I \cap J}$

(5) $h_M(I) \cup h_M(J) = M_I^C \cup M_J^C = (M_I \cap M_J)^C = (M_{I \cap J})^C = h_M(I \cap J)$.

(6) $h_M(I) \cap h_M(J) = M_I^C \cap M_J^C = (M_I \cup M_J)^C = (M_{I \vee J})^C = h_M(I \vee J)$ ■

Corollary 3.15. *Let L be a 0-distributive AL. Then for any $x, y \in L$, we have the following statements:*

(1) $x \leq y \Rightarrow M_x \subseteq M_y$.
(2) $x \leq y \Rightarrow h_M(y) \subseteq h_M(x)$.
(3) $M_x \cup M_y = M_{x \vee y}$.
(4) $M_x \cap M_y = M_{x \wedge y}$.
(5) $h_M(x) \cup h_M(y) = h_M(x \wedge y)$.
(6) $h_M(x) \cap h_M(y) = h_M(x \vee y)$.

The following lemma exhibits the relation between the annihilator ideals of L, the basic open sets and the basic closed sets of M in the hull-kernel topology.

Lemma 3.16. *Let L be a 0-distributive AL. Then for any $x, y, z \in L$, we have the following statements:*

(1) $M_x = h_M([x]^*)$.
(2) $h_M(x) = h_M([x]^{**})$.
(3) $[x]^* \subseteq [y]^* \Leftrightarrow h_M(x) \subseteq h_M(y)$.
(4) $[x]^* \subseteq [y]^* \Leftrightarrow M_y \subseteq M_x$.
(5) $[z]^* = [x]^* \cap [y]^* \Leftrightarrow h_M(z) = h_M(x) \cap h_M(y)$.
(6) $[x]^{**} = [y]^* \Leftrightarrow h_M(x) = h_M([y]^*)$.

Proof. (1) We have $P \in M_x \Leftrightarrow x \notin P \Leftrightarrow [x]^* \subseteq P$ (by Corollary 3.9) $\Leftrightarrow P \in h_M([x]^*)$.

(2) Suppose $P \in h_M(x)$. Then $x \in P$. This implies $[x]^* \nsubseteq P$. Hence $[x]^{**} \subseteq P$, since $[x]^* \cap [x]^{**} = \{0\} \subseteq P$. It follows that $P \in h_M([x]^{**})$. Thus $h_M(x) \subseteq h_M([x]^{**})$. Conversely, suppose $P \in h_M([x]^{**})$. Then $[x]^{**} \subseteq P$. This implies $x \in P$. Hence $P \in h_M(x)$. Thus $h_M([x]^{**}) \subseteq h_M(x)$. Therefore $h_M(x) = h_M([x]^{**})$.

(3) Suppose $[x]^* \subseteq [y]^*$ and suppose $P \in h_M(x)$. Then $x \in P$. Hence by Corollary 3.9, we get $[x]^* \nsubseteq P$. Therefore $[y]^* \nsubseteq P$. Hence $y \in P$, we get $P \in h_M(y)$. Thus $h_M(x) \subseteq h_M(y)$. Conversely, suppose $h_M(x) \subseteq h_M(y)$. Let $a \notin [y]^*$. Then $a \wedge y \neq 0$. Therefore by Corollary 3.7, there exists a minimal prime ideal (say) P of L such that $a \wedge y \notin P$. Therefore $a \notin P$ and $y \notin P$. Hence we get $a \notin P$ and $P \notin h_M(y)$. Thus $a \notin P$ and $P \notin h_M(x)$, we get $a \notin P$ and $x \notin P$. Therefore $a \wedge x \notin P$, since P is a prime ideal. Hence $a \wedge x \neq 0$, we get $a \notin [x]^*$. Thus $[x]^* \subseteq [y]^*$.

(4) We have $[x]^* \subseteq [y]^* \Leftrightarrow h_M(x) \subseteq h_M(y) \Leftrightarrow M - h_M(y) \subseteq M - h_M(x) \Leftrightarrow M_y \subseteq M_x$. Therefore $[x]^* \subseteq [y]^* \Leftrightarrow M_y \subseteq M_x$.

(5) Suppose $[z]^* = [x]^* \cap [y]^*$. Hence $[z]^* = [x \vee y]^*$. Therefore by condition (3), we get $h_M(z) = h_M(x \vee y) = h_M(x) \cap h_M(y)$. Thus $h_M(z) = h_M(x) \cap h_M(y)$. Conversely, suppose $h_M(z) = h_M(x) \cap h_M(y)$. Then $h_M(z) = h_M(x \vee y)$. Hence by condition (3), we get $[z]^* = [x \vee y]^* = [x]^* \cap [y]^*$. Therefore $[z]^* = [x]^* \cap [y]^*$.

(6) Suppose $[x]^{**} = [y]^*$. Then $h_M([x]^{**}) = h_M([y]^*)$. Hence by condition (2), we get $h_M(x) = h_M([y]^*)$. Conversely, suppose $a \notin [x]^{**}$. Then there exists $t \in [x]^*$ such that $a \wedge t \neq 0$. Therefore there exists a minimal prime ideal (say) P of L such that $a \wedge t \notin P$ and hence $a \notin P$ and $t \notin P$. Since $t \wedge x = 0 \in P$, we have $x \in P$. Therefore $P \in h_M(x) = h_M([y]^*)$. Hence $[y]^* \subseteq P$, we get $a \notin [y]^*$. Thus $[y]^* \subseteq [x]^{**}$. Similarly, we get $[x]^{**} \subseteq [y]^*$. Hence $[x]^{**} = [y]^*$. ■

4. ∗-0-Distributive Almost Lattices

It can be easily seen that, if L is a pseudo-complemented AL then for any $x \in L$, $[x]^* = (x]^* = (x^*]$ and hence $[x]^{**} = (x^*]^* = [x^*]^*$. This motivate us to introduce a new class of 0-distributive ALs which are called ∗-0-distributive ALs in this section and observe that this class contains pseudo-complemented ALs. We derive a set of identities for a 0-distributive AL L to become ∗-0-distributive ALs in topological terms. Next, we define a congruence relation on an AL L, and prove that if ψ is a congruence relation on L then the congruence of class 0 is an ideal of L. Later, we prove that a relation θ on an AL L defined by $(x, y) \in \theta$ if and only if $[x]^* = [y]^*$ is a congruence relation if L is 0-distributive. We derive a necessary and sufficient condition for an AL to become 0-distributive AL and establish necessary and sufficient conditions for a 0-distributive AL to become ∗-0-distributive AL. Finally, we derive a set of identities for a 0-distributive AL to become ∗-0-distributive AL in topological and algebraic terms.

Definition 4.1. *Let L be a 0-distributive AL. Then L is said to be a $*$-0-distributive AL if, to each $x \in L$, there exists $x' \in L$ such that $[x]^{**} = [x']^*$.*

Note that, here onwards, we denote $*$-0-distributive AL by $*$-0-DAL.

Example 4.2. Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ALs. Now, put $L = A \times B = \{(0, \ 0), (0, \ b_1), (0, \ b_2), (a, \ 0), (a, \ b_1), (a, \ b_2)\}$ and define operations $\vee$ and $\wedge$ on L as follows.

$\vee$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,0)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,b_1)$	$(0,b_1)$	$(0,b_1)$	$(0,b_1)$	(a,b_1)	(a,b_1)	(a,b_1)
$(0,b_2)$	$(0,b_2)$	$(0,b_2)$	$(0,b_2)$	(a,b_2)	(a,b_2)	(a,b_2)
$(a,0)$	$(a,0)$	(a,b_1)	(a,b_2)	$(a,0)$	(a,b_1)	(a,b_2)
(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)
(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)

$\wedge$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(0,b_1)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(a,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(a,0)$	$(a,0)$	$(a,0)$
(a,b_1)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
(a,b_2)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)

Then clearly, L is a $0-$distributive AL with $(0, 0)$ as its zero element. Now, it can be observed that $[(0, 0)]^{**} = [(a, b_1)]^*$, $[(0, b_1)]^{**} = [(a, 0)]^*$,$[(0, b_2)]^{**} = [(a, 0)]^*$, $[(a, 0)]^{**} = [(0, b_1)]^*$,$[(a, b_1)]^{**} = [(0, 0)]^*$ and $[(a, b_2)]^{**} = [(0, 0)]^*$. Hence L is a $*$-0-DAL.

Theorem 4.3. *Every pseudo-complemented AL is a $*$-0-DAL.*

Proof. Suppose L is a pseudo-complemented AL. Then by Theorem 2.42, L is a 0-distributive AL. Clearly, L is a $*$-0-DAL, since for any $x \in L$, $[x]^{**} = [x^*]^*$. ■

In the following we derive a set of identities for a 0-distributive AL to become $*$-0-DAL in topological terms. For this, first we need the following lemma.

Lemma 4.4. *Let L be a $*$-0-DAL and let $x \in L$ such that $[x]^{**} = [x']^*$ for some $x' \in L$. Then we have the following statements:*

(1) $[x]^* \cap [x']^* = \{0\}$.
(2) $M - M_x = M_{x'}$.

Proof. (1) Suppose $x \in L$. Then $[x]^* \cap [x']^* = [x]^* \cap [x]^{**} = \{0\}$. Therefore $[x]^* \cap [x']^* = \{0\}$.

(2) We have $M - M_x = h_M(x) = h_M([x]^{**}) = h_M([x']^*) = M_{x'}$ (by Lemma 3.16). ■

Lemma 4.5. *Let L be a 0-distributive AL and let $x \in L$. Then $M_x = \emptyset$ if and only if $x = 0$.*

Proof. Suppose $M_x \neq \emptyset$. Then we can choose a minimal prime ideal $P \in M$ such that $P \in M_x$. Hence $x \notin P$, we get $x \neq 0$. Conversely, suppose $x \neq 0$. Then, by Corollary 3.7, there exists $P \in M$ such that $x \notin P$. Hence $P \in M_x$, we get $M_x \neq \emptyset$. Therefore if $M_x = \emptyset$, then $x = 0$. ■

Lemma 4.6. *Let L be a 0-distributive AL and A a subset of L. Then $h_M(A) = \bigcap_{a\in A} h_M(a)$.*

Proof. Suppose A is a subset of L. Then $P \in h_M(A) \Leftrightarrow A \subseteq P \Leftrightarrow a \in P$ for all $a \in A \Leftrightarrow P \in h_M(a)$ for all $a \in A \Leftrightarrow P \in \bigcap_{a\in A} h_M(a)$. Therefore $h_M(A) = \bigcap_{a\in A} h_M(a)$. ■

Theorem 4.7. *Let L be a 0-distributive AL. Then the following statements are equivalent:*

(1) *L is a $*$-0-DAL.*
(2) $\Im_h^M = \Im_d^M$.
(3) *M is compact in the hull-kernel topology.*

Proof. (1)$\Rightarrow$(2). Suppose L is a $*$-0-DAL. Let $M_x \in \Im_h^M$. Since L is a $*$-0-DAL, there exists $x' \in L$ such that $[x]^{**} = [x']^*$. Now $M_x = h_M([x]^*) = h_M([x']^{**}) = h_M(x') \in \Im_d^M$. Hence $\Im_h^M \subseteq \Im_d^M$. Similarly, we get $\Im_d^M \subseteq \Im_h^M$. Therefore $\Im_h^M = \Im_d^M$.

(2)$\Rightarrow$(3). Suppose $\Im_h^M = \Im_d^M$. Let $\{M_x : x \in \Delta\}$ be a family of closed sets in M with finite intersection property. Now, put $F = [\Delta)$. Suppose $F = L$. Then we have $0 \in L = [\Delta)$. Therefore $0 = x \vee (\bigwedge_{i=1}^n y_i)$, where $y_i \in \Delta$ for all i and $x \in L$. It follows that $x = 0$ and $\bigwedge_{i=1}^n y_i = 0$. Now, consider $\bigcap_{i=1}^n M_{y_i} = M_{\bigwedge_{i=1}^n y_i} = M_0 = \emptyset$, a contradiction. Thus $F \neq L$. Hence F is a proper filter. Therefore there exists a maximal filter (say) G of L such that $F \subseteq G$. Therefore $L - G$ is a minimal prime ideal. Clearly, $\Delta \subseteq G$. Now, let $x \in \Delta$. Then $x \in G$. This implies $x \notin L - G$. Hence $L - G \in M_x$. It follows that $L - G \in \bigcap_{x\in\Delta} M_x$. Therefore $\bigcap_{x\in\Delta} M_x$ is nonempty. Hence M is compact in the hull-kernel topology.

(3)$\Rightarrow$ (1). Suppose M is compact in the hull-kernel topology. We shall prove L is a $*$-0-DAL. Let $x \in L$. Then $M_x \in \Im_h^M$ and hence $h_M(x) = M - M_x$ is closed. Hence $h_M(x)$ is a closed subset of a compact space M. Thus $h_M(x)$is compact. Now, we have $h_M(x) \cap h_M([x]^*) = \emptyset$. This implies $h_M(x) \cap \bigcap_{t\in[x]^*} h_M(t) = \emptyset$ and $\{h_M(x) \cap h_M(t) : t \in [x]^*\}$ is class of closed sets in the compact space $h_M(x)$. Hence there exists $t_1, t_2, \dots, t_n \in [x]^*$ such that $h_M(x) \cap (h_M(t_1) \cap \dots. \cap h_M(t_n)) = \emptyset$. That is, $h_M(x) \cap h_M(\bigvee_{i=i}^n t_i) = \emptyset$. Now, put $x' = \bigvee_{i=i}^n t_i$. Then

$h_M(x) \cap h_M(x') = \emptyset$. It follows that $M_x \cup M_{x'} = M$ and $M_x \cap M_{x'} = M_{x \wedge x'} = M_{x \wedge (\bigvee_{i=i}^{n} t_i)} = M_0 = \emptyset$, since L is 0-distributive AL. Thus $M_{x'} = h_M(x)$ and $M_x = h_M(x')$. Hence, we get $h_M([x]^{**}) = h_M(x) = M_{x'} = h_M([x']^*)$. Therefore $h_M([x]^{**}) = h_M([x']^*)$. Thus, we get $[x]^{**} = [x']^*$. Therefore L is a $*$-0-DAL. ∎

Next, we define a congruence relation on an AL and prove that the congruence class of 0 is an ideal.

Definition 4.8. *Let L be an AL. Then an equivalence relation ψ on L is said to be a congruence relation on L if for any $(a, b), (c, d) \in \psi$, $(a \wedge c, b \wedge d), (a \vee c, b \vee d) \in \psi$.*

It can be easily seen that if ψ is an equivalence relation on an AL L then ψ is a congruence relation if and only if for any $(a, b) \in \psi$ and $x \in L$, $(a \vee x, b \vee x), (a \wedge x, b \wedge x) \in \psi$. Note that if $x \in L$, then the congruence class of x with respect to congruence relation ψ is denoted by x/ψ. Also, note that the set of all congruence classes with respect to congruence relation ψ is denoted by L/ψ. Therefore $L/\psi = \{x/\psi : x \in L\}$.

Lemma 4.9. *Let L be a 0-distributive AL. Then $0/\psi$ is an ideal of L.*

Proof. We have $0/\psi = \{x \in L : (0, x) \in \psi\}$. Then clearly, $0 \in 0/\psi$ and hence $0/\psi$ is nonempty subset of L. Now, let $x, y \in 0/\psi$. Then $(0, x), (0, y) \in \psi$. This implies $(0 \vee 0, x \vee y) \in \psi$. Hence $(0, x \vee y) \in \psi$. Therefore $x \vee y \in 0/\psi$. Again, let $x \in 0/\psi$ and $a \in L$. Then $(0, x) \in \psi$ and $a \in L$. Hence $(0 \wedge a, x \wedge a) \in \psi$. Thus $(0, x \wedge a) \in \psi$. Therefore $x \wedge a \in 0/\psi$. Hence $0/\psi$ is an ideal. ∎

In the following we define a relation θ on an AL L and prove that if L is a 0-distributive AL then θ is a congruence relation on L.

Theorem 4.10. *Let L be a 0-distributive AL. Define a relation θ on L by $(x, y) \in \theta$ if and only if $[x]^* = [y]^*$. Then θ is a congruence relation on L.*

Proof. Clearly, θ is an equivalence relation on L. Let $(a, b), (c, d) \in \theta$. Then $[a]^* = [b]^*$ and $[c]^* = [d]^*$. Now, we shall prove $(a \vee c, b \vee d), (a \wedge c, b \wedge d) \in \theta$. Now, consider, $[a \vee c]^* = [a]^* \cap [c]^* = [b]^* \cap [d]^* = [b \vee d]^*$. Therefore $(a \vee c, b \vee d) \in \theta$. Again, consider $[a \wedge c]^{**} = [a]^{**} \cap [c]^{**} = [b]^{**} \cap [d]^{**} = [b \wedge d]^{**}$. It follows that $[a \wedge c]^* = [b \wedge d]^*$. Therefore $(a \wedge c, b \wedge d) \in \theta$. Thus θ is a congruence relation. ∎

If θ is a congruence relation on L and $x \in L$ then $x/\theta = \{y \in L : (x, y) \in \theta\}$ is called the congruence class of x with respect to θ.

Corollary 4.11. *Let L be a 0-distributive AL and let m_1 and m_2 be two maximal elements in L. Then $m_1/\theta = m_2/\theta$.*

Proof. Since m_1 and m_2 are maximal, $[m_1]^* = \{0\} = [m_2]^*$. Therefore $[m_1]^* = [m_2]^*$. Hence $(m_1, m_2) \in \theta$. Therefore $m_1/\theta = m_2/\theta$. ∎

Corollary 4.12. *Let L be a 0-distributive AL. Then for any maximal element m in L, m/θ is a filter.*

Proof. We have $m/\theta = \{x \in L : (m, x) \in \theta\}$. Clearly m/θ is nonempty, since $m \in \theta$. Let $x, y \in m/\theta$. Then $(m, x), (m, y) \in \theta$. This implies $(m \wedge m, x \wedge y) \in \theta$. It follows that $(m, x \wedge y) \in \theta$. Thus $x \wedge y \in m/\theta$. Again, let $x \in m/\theta$ and $t \in L$. Then $(m, x) \in \theta$ and $t \in L$. This implies $(t \vee m, t \vee x) \in \theta$. Thus $t \vee x \in (t \vee m)/\theta = m/\theta$, since both $t \vee m$ and m are maximal. Hence $t \vee x \in m/\theta$. Therefore m/θ is a filter. ∎

Next, we give a necessary and sufficient condition for an AL with 0 to become 0-distributive AL. For this, first we need the following lemma.

Lemma 4.13. *Let L be a $0-$distributive AL. Then $L/\theta = \{x/\theta : x \in L\}$ is a lattice under the operations $\vee$ and $\wedge$ defined on L/θ by $x/\theta \vee y/\theta = (x \vee y)/\theta$, $x/\theta \wedge y/\theta = (x \wedge y)/\theta$.*

Proof. Suppose L is 0-distributive AL. First we shall prove that operations $\vee$ and $\wedge$ on L are well defined. Suppose $x/\theta = a/\theta$ and $y/\theta = b/\theta$. Then $(x, a) \in \theta$ and $(y, b) \in \theta$. It follows that $[x]^* = [a]^*$ and $[y]^* = [b]^*$. Now, consider $[x \vee y]^* = [x]^* \cap [y]^* = [a]^* \cap [b]^* = [a \vee b]^*$. Therefore $(x \vee y, a \vee b) \in \theta$. Thus $(x \vee y)/\theta = (a \vee b)/\theta$. Therefore $x/\theta \vee y/\theta = a/\theta \vee b/\theta$. Again, let $t \in [x \wedge y]^*$. Then $t \wedge (x \wedge y) = 0$. This implies $(t \wedge x) \wedge y = 0$. Thus $t \wedge x \in [y]^* = [b]^*$. It follows that $(t \wedge x) \wedge b = 0$. This implies $(t \wedge b) \wedge x = 0$. Thus $t \wedge b \in [x]^* = [a]^*$. This implies $(t \wedge b) \wedge a = 0$. Hence $t \wedge (a \wedge b) = 0$. Therefore $t \in [a \wedge b]^*$. Thus $[x \wedge y]^* \subseteq [a \wedge b]^*$. Similarly, we can prove that $[a \wedge b]^* \subseteq [x \wedge y]^*$. Thus $[x \wedge y]^* = [a \wedge b]^*$. Hence $(x \wedge y, a \wedge b) \in \theta$. It follows that $(x \wedge y)/\theta = (a \wedge b)/\theta$. Thus $x/\theta \wedge y/\theta = a/\theta \wedge b/\theta$. Now, we shall prove that $(L/\theta, \wedge, \vee)$ is a lattice. We have $[x \vee y]^* = [y \vee x]^*$ and $[x \wedge y]^* = [y \wedge x]^*$. Therefore $x/\theta \vee y/\theta = y/\theta \vee x/\theta$ and $x/\theta \wedge y/\theta = y/\theta \wedge x/\theta$. It follows that L/θ is a lattice. ∎

Theorem 4.14. *Let L be an AL with 0. Then L is a 0-distributive if and only if L/θ is a distributive lattice.*

Proof. Suppose L is 0-distributive AL. Then by Lemma 4.13, we get L/θ is a lattice. First we shall prove that for any $x, y, z \in L, [(x \vee y) \wedge z]^* = [(x \wedge z) \vee (y \wedge z)]^*$. Let $t \in [(x \vee y) \wedge z]^*$. Then $t \wedge ((x \vee y) \wedge z) = 0$. This implies $(t \wedge ((x \vee y) \wedge z)) \wedge x = 0$. It follows that $t \wedge (x \wedge z) = 0$. Similarly, we get $t \wedge (y \wedge z) = 0$. Since L is 0-distributive AL, $t \wedge ((x \wedge z) \vee (y \wedge z)) = 0$. Thus $t \in [(x \wedge z) \vee (y \wedge z)]^*$. Therefore $[(x \vee y) \wedge z]^* \subseteq [(x \wedge z) \vee (y \wedge z)]^*$. Conversely, suppose $t \in [(x \wedge z) \vee (y \wedge z)]^*$. Then $t \wedge ((x \wedge z) \vee (y \wedge z)) = 0$. This implies $(t \wedge ((x \wedge z) \vee (y \wedge z))) \wedge (x \wedge z) = 0$. It follows that $t \wedge (x \wedge z) = 0$. Similarly, we get $t \wedge (y \wedge z) = 0$. Hence $(t \wedge x) \wedge z = 0$ and $(t \wedge y) \wedge z = 0$. Since L is 0-distributive AL, we get $z \wedge ((t \wedge x) \vee (t \wedge y)) = 0$. It follows that $t \wedge (z \wedge ((t \wedge x) \vee (t \wedge y))) = 0$. This implies $((t \wedge z) \wedge ((t \wedge x) \vee (t \wedge y))) \wedge (t \wedge x) = 0$. It follows that $(t \wedge z) \wedge (t \wedge x) = 0$. Similarly, we get $(t \wedge z) \wedge (t \wedge y) = 0$. Therefore $(t \wedge z) \wedge x = 0$ and $(t \wedge z) \wedge y = 0$.

Hence $(t \wedge z) \wedge (x \vee y) = 0$. Thus $t \wedge (z \wedge (x \vee y)) = 0$. It follows that $t \wedge ((x \vee y) \wedge z) = 0$. Hence $t \in [(x \vee y) \wedge z]^*$. Thus $[(x \wedge z) \vee (y \wedge z)]^* \subseteq [(x \vee y) \wedge z]^*$. Therefore $[(x \vee y) \wedge z]^* = [(x \wedge z) \vee (y \wedge z)]^*$. Therefore $((x \vee y) \wedge z, (x \wedge z) \vee (y \wedge z)) \in \theta$. Hence $(x/\theta \vee y/\theta) \wedge z/\theta = ((x \vee y) \wedge z)/\theta = ((x \wedge z) \vee (y \wedge z))/\theta = (x/\theta \wedge z/\theta) \vee (y/\theta \wedge z/\theta)$. Thus L/θ is a distributive lattice.

Conversely, suppose L/θ is a distributive lattice. Now, we shall prove L is 0-distributive AL. Let $a, b, c \in L$ such that $a \wedge b = 0$ and $a \wedge c = 0$. This implies $(a \wedge b) \vee (a \wedge c) = 0$. It follows that $((a \wedge b) \vee (a \wedge c))/\theta = 0/\theta$. Thus $(a/\theta \wedge b/\theta) \vee (a/\theta \wedge c/\theta) = 0/\theta$. Since L/θ is distributive, $a/\theta \wedge (b/\theta \vee c/\theta) = 0/\theta$. It follows that $(a \wedge (b \vee c))/\theta = 0/\theta$. Therefore $[a \wedge (b \vee c)]^* = [0]^* = L$. It follows that $a \wedge (b \vee c) = 0$. Hence L is 0-distributive AL. ∎

Next, we establish a set of identities for a 0-distributive AL to become a $*$-0-DAL in topological and algebraic terms. For this first we need the following theorem.

Theorem 4.15. *Let L be a 0-distributive AL. Then L is a $*$-0-DAL if and only if $\mu = \{M_x : x \in L\}$ is a Boolean algebra under the operations $\cup$ and $\cap$.*

Proof. Suppose L is a $*$-0-DAL. Now, we shall prove μ is a Boolean algebra. Let $M_x, M_y \in \mu$. Then by Corollary 3.15, we have $M_x \cup M_y = M_{x \vee y}$ and $M_x \cap M_y = M_{x \wedge y}$. Therefore μ is closed closed under $\cup$ and $\cap$. It can be easily seen that $(\mu, \cup, \cap)$ is a lattice. Since $0 \in L$, $\emptyset = M_0 \in \mu$. Clearly, M_0 is the least element in μ. Also, since $0 \in L$ and L is a $*$-0-DAL, there exists $0' \in L$ such that $[0]^{**} = [0']^*$. Now, $M = h_M(0) = h_M([0]^{**}) = h_M([0']^*) = M_{0'}$. Therefore $M = M_{0'} \in \mu$ and clearly, $M_{0'}$ is the greatest element in the lattice μ. Thus μ is a bounded lattice. Now, we shall prove that μ is complemented. Let $M_x \in \mu$. Then $x \in L$. Since L is $*$-0-DAL, there exists $x' \in L$ such that $[x]^{**} = [x']^*$. Now, consider $M_x \cap M_{x'} = M_{x \wedge x'} = M_0 = \emptyset$ and $M_x \cup M_{x'} = M_{x \vee x'} = M_{0'}$, since $[x \vee x']^* = [x]^* \cap [x']^* = [x]^* \cap [x]^{**} = \{0\} = [0]^{**} = [0']^*$. Let $x \in L$. There exists $x' \in L$ such that $[0]^{**} = [0']^*$. Hence by Lemma 4.4, $M - M_x = M_{x'}$, we get every element in μ is complemented. Therefore μ is complemented lattice. Finally, we shall prove μ is a distributive lattice. Now, let $M_x, M_y, M_z \in \mu$. Then by Theorem 4.14, we get $(M_x \cup M_y) \cap M_z = (M_x \cap M_z) \cup (M_y \cap M_z)$. Therefore $(\mu, \cup, \cap)$ is a distributive lattice. Thus $(\mu, \cup, \cap, M_0, M_{0'})$ is a Boolean algebra.

Conversely, suppose $\mu = \{M_x : x \in L\}$ is a Boolean algebra. Let $x \in L$. Then $M_x \in \mu$. Hence there exists $M_{x'} \in L$ such that $M_x \cap M_{x'} = \emptyset = M_0$ and $M_x \cup M_{x'} = M_a$, where M_0 and M_a are least and greatest elements in μ respectively. Then $M_{x \wedge x'} = M_0$ and $M_{x \vee x'} = M_a$. This implies $x \wedge x' = 0$ and $[x \vee x']^* = [a]^*$. Now, let $t \in [a]^*$. It follows that $t \in [t \vee t']^* = [t]^* \cap [t']^*$. Hence $t \in [t]^*$. Therefore $t \wedge t = 0$. Hence $t = 0$. Thus $[x \vee x']^* = \{0\}$. Therefore $x \wedge x' = 0$ and $x \vee x' \in D$. This implies $x \in [x']^*$. Hence $(x] \subseteq [x']^*$. It follows that $[x]^{**} \subseteq [x']^*$. Now, let $t \in [x']^*$ and $s \in [x]^*$. Then $t \wedge s \in [x]^*$ and $t \wedge s \in [x']^*$. Therefore $t \wedge s \in [x]^* \cap [x']^* = [x \vee x']^* = \{0\}$, since $x \vee x' \in D$.

Hence $t \wedge s = 0$. Thus $t \in [x]^{**}$. Therefore $[x']^* \subseteq [x]^{**}$. Thus $[x]^{**} = [x']^*$. Therefore L is a $*$-0-DAL. ■

Theorem 4.16. *Let L be a 0-distributive AL. Then L is a $*$-0-DAL if and only if L/θ is a Boolean algebra.*

Proof. Suppose L is a $*$-0-DAL. Then By Theorem 4.14, we get L/θ is a distributive lattice. Since $0 \in L$, $0/\theta \in L/\theta$. Now, for any $x/\theta \in L/\theta$, $0/\theta \wedge x/\theta = (0 \wedge x)/\theta = 0/\theta$. Hence $0/\theta \leq x/\theta$. Therefore $0/\theta$ is the least element in L/θ. Again, since $0 \in L$ and L is $*$-0-DAL, there exists $0' \in L$ such that $[0]^{**} = [0']^*$. Let $x/\theta \in L/\theta$. Now, consider, $[x \vee 0']^* = [x]^* \cap [0']^* = [x]^* \cap [0]^{**} = [x]^* \cap \{0\} = \{0\} = [0]^{**} = [0']^*$. Hence $(x \vee 0')/\theta = 0'/\theta$. This implies $x/\theta \vee 0'/\theta = 0'/\theta$. Thus $x/\theta \leq 0'/\theta$. Therefore $0'/\theta$ is the greatest element in L/θ. Thus L/θ is a bounded lattice. Let $x/\theta \in L/\theta$. Then $x \in L$. Since L is $*$-0-DAL, there exists $x' \in L$ such that $[x]^{**} = [x']^*$. Now, consider $x/\theta \wedge x'/\theta = (x \wedge x')/\theta = 0/\theta$, since $x \in [x]^{**} = [x']^*$. Again, consider $[x \vee x']^* = [x]^* \cap [x']^* = [x]^* \cap [x]^{**} = \{0\} = [0]^{**} = [0']^*$. Therefore $x/\theta \vee x'/\theta = (x \vee x')/\theta = 0'/\theta$. Thus L/θ is complemented and hence is a Boolean algebra.

Conversely, suppose L/θ is a Boolean algebra. Now, we shall prove L is a $*$-0-DAL. Let $x \in L$. Then $x/\theta \in L/\theta$. Therefore there exists $x'/\theta \in L/\theta$ such that $x/\theta \vee x'/\theta = (x \vee x')/\theta = a/\theta$ and $x/\theta \wedge x'/\theta = (x \wedge x')/\theta = 0/\theta$, where $0/\theta$ and a/θ are least and greatest elements in L/θ respectively. Now, since $x/\theta \vee x'/\theta = a/\theta$, $(x \vee x')/\theta = a/\theta$. Thus $(x \vee x', a) \in \theta$. Therefore $[x \vee x']^* = [a]^*$, this is true for all $x \in L$. Now, let $t \in [a]^*$. Then $t \in [t \vee t']^* = [t]^* \cap [t']^*$. It follows that $t = 0$. Therefore $[a]^* = \{0\}$. Thus $[x \vee x']^* = \{0\}$. Therefore $x \vee x' \in D$. This implies $[x]^* \cap [x']^* = \{0\}$. Again, since $x/\theta \wedge x'/\theta = 0/\theta$, $[x \wedge x']^* = [0]^* = L$. Now, we have $x \wedge x' \in L = [x \wedge x']^*$. Therefore $x \wedge x' = 0$. This implies $x \in [x']^*$. Hence $(x] \subseteq [x']^*$. It follows that $[x]^{**} \subseteq [x']^*$. Now, let $t \in [x']^*$ and $a \in [x]^*$. Then $t \wedge a \in [x]^*$ and $t \wedge a \in [x']^*$. Therefore $t \wedge a \in [x]^* \cap [x']^* = [x \vee x']^* = \{0\}$, since $x \vee x' \in D$. Hence $t \wedge a = 0$. Thus $t \in [x]^{**}$. Therefore $[x']^* \subseteq [x]^{**}$. Thus $[x]^{**} = [x']^*$. Therefore L is a $*$-0-DAL. ■

Theorem 4.17. *Let L be a 0-distributive AL. Then L is a $*$-0-DAL if and only if for any $x \in L$, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in D$.*

Proof. Suppose L is a $*$-0-DAL and suppose $x \in L$. Then there exists $x' \in L$ such that $[x]^{**} = [x']^*$. Since $x \in [x]^{**} = [x']^*$, $x \wedge x' = 0$. Again, consider $[x \vee x']^* = [x]^* \cap [x']^* = [x]^* \cap [x]^{**} = \{0\}$. Thus $[x \vee x']^* = \{0\}$. Therefore $[x \vee x']^* \in D$. Conversely, assume the condition. Now, let $x \in L$. Then by assumption, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in D$. This implies $x \in [x']^*$. Hence $(x] \subseteq [x']^*$. It follows that $[x]^{**} \subseteq [x']^*$. Now, let $t \in [x']^*$ and $a \in [x]^*$. Then $t \wedge a \in [x]^*$ and $t \wedge a \in [x']^*$. Therefore $t \wedge a \in [x]^* \cap [x']^* = [x \vee x']^* = \{0\}$, since $x \vee x' \in D$. Hence $t \wedge a = 0$. Thus

$t \in [x]^{**}$. Hence, $[x']^{*} \subseteq [x]^{**}$ and so $[x]^{**} = [x']^{*}$. Therefore L is a $*$-0-DAL. ■

It can be easily seen that if L is a $*$-0-DAL then $M_{0'} = M$ and $h_M(0') = \emptyset$ where $[0]^{**} = [0']^{*}$. Finally, we prove the following theorem.

Theorem 4.18. *Let L be an 0-distributive AL. Then the following conditions are equivalent:*

(1) *L is a $*$-0-DAL.*
(2) *$\mu = \{M_x : x \in L\}$ is a Boolean algebra.*
(3) *L/θ is a Boolean algebra.*
(4) *For any $x \in L$, there is $x' \in L$ such that $x \wedge x' = 0, x \vee x' \in D$.*
(5) *$\Im_h^M = \Im_d^M$.*
(6) *M is compact in the hull-kernel topology.*
(7) *$\{h(x) : x \in L\}$ is a subbasis for open sets of $(M, \Im_h^M)$.*
(8) *$\{M(x) : x \in L\}$ is a subbasis for open sets of $(M, \Im_d^M)$.*

Proof. The equivalence of (1), (2), (3) and (4) follows by Theorems 4.15, 4.16 and 4.17 and equivalence of (1), (5) and (6) follows by Theorem 4.7. Finally, the equivalence of (5), (7) and (8) is trivial, since the topologies are completely determined by any of their subbasis. ■

References

[1] G. Birkhoff, Lattice theory, *Amer. Math. Soc. Colloq.* **XXV**, Providence, USA, 1967.

[2] S. Burris, H.P. Sankappanavar, *A Course in Universal Algebra*, Springer Verlag, 1981.

[3] G. Nanaji Rao, H.T. Alemu, Almost lattices, *J. Int. Mat. Var. Inst.* **9** (2019) 155–171.

[4] G. Nanaji Rao, H.T. Alemu, Ideals in almost lattices, *Bull. Int. Mat. Virtual Inst.* **10** (1) (2019) 37–50.

[5] G. Nanaji Rao, H.T. Alemu, Filters in almost lattices, *Bull. Int. Mat. Virtual Inst.* **10** (1) (2020) 69–82.

[6] G. Nanaji Rao, R. Venkata Aravinda Raju, Pseudo-complementation on almost lattices, *Annals of Pure and Applied Mathematics* **19** (1) (2019) 37–51.

[7] G. Nanaji Rao, R. Venkata Aravinda Raju, 0-Distributive almost lattices, *Bull. Int. Mat. Virtual Inst.* **10** (2) (2020) 239–248.

[8] G. Nanaji Rao, R. Venkata Aravinda Raju, Annihilator ideals in 0-distributive almost lattices, *Bull. Int. Mat. Virtual Inst.* **11** (1) (2021) 1–13.

[9] Y.S. Pawar, N.K. Thakare, The space of minimal prime ideals in a 0-distributive semilattices, *Periodica Mathematica Hungarica* **13** (4) (1982) 309–319.

[10] G.C. Rao, *Almost Distributive Lattices*, Doctoral Thesis, Andhra University, Waltair, 1980.

[11] T.P. Speed, On Stone lattices, *J. Austral. Math. Soc.* **9** (1969) 297–307.

[12] U.M. Swamy, G.C. Rao, Almost distributive lattices, *J. Austral Math. Soc.(Series)* **31** (1981) 77–91.

[13] U.M. Swamy, G.C. Rao, G. Nanaji Rao, Stone almost distributive lattices, *Southeast Asian Bull. Math.* **27** (2003) 513–526.