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Hopf Algebra Action on Semiring of Quotients

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Abstract. In this paper, we introduce the semiring of quotients of a $H-$ semimodule semialgebra A with respect to the filter $\mathfrak{F}$ of $H-$stable ideals I of A with right and left annihilators of I are zero. We establish a connection between semiring of quotients of a semialgebra A and the ring of quotients of the algebra of differences $A^{\triangle}$. We also introduce the Hopf algebra action of the semiring of quotients and study the connection between the smash product of semialgebra of quotients and the smash product of algebra of quotients.

Keywords: Semiring of quotients; Semialgebra; Hopf algebra; Smashproduct.

1. Introduction

The theory of ring of quotients began with the work of Ore in 1930. In fact, Ore established a criterion for a ring R to have a classical ring of quotients. In [9], Martindale introduced the ring of quotients for a prime rings. In 1972, S.A. Amitsur [1] generalised the constructions of Martindale ring of quotients to semiprime rings. The study of ring of quotients has been found very useful in the study of Galois theory for non-commutative rings. In [2], M. Cohen extended the Hopf algebra actions to the ring of quotients of an Hopf module algebra. In [8], C. Lomp has given a sufficient condition to extend Hopf actions to the algebra of quotients of an $H-$module algebra.

In this paper, we consider the action of a Hopf algebra(H) on the semiring of quotients $Q_{\mathfrak{F}}(A)$ of an $H-$semimodule semialgbra A with respect to the filter

$\mathfrak{F}$ of $H-$stable ideals I of A with right and left annihilator of I in A are zero. This paper organised as follows:

The second section contains some basic definitions and results on semirings and Hopf algebra that are needed for latter sections. In the third section, we introduced the action of Hopf algebra over semialgebra A and define the smash product semialgbra $A\#H$. Also, we have introduced that the semiring of quotients of A with respect to the filter $\mathfrak{F}$ of $H-$stable ideals I of A with right and left annihilator of I in A are zero. In this section we have proved that there is a one to one homomorphism between the semiring of quotients of the semialgebra A and the ring of quotients of the algebra of their differences $A^{\triangle}$. Also, we have proved that there is a one to one homomorphism between $(Q_{\mathfrak{F}}(A))^{\triangle}$ and $(Q_{\mathfrak{F}^{\triangle}}(A^{\triangle}))$.

In the fourth section, we introduced the Hopf algebra action on the semiring of quotients of an $H-$semimodule semialgebra A and proved that, there is a one to one homomorphism between the smash products $(Q_{\mathfrak{F}}(A))^{\triangle}\#H$ and $Q_{\mathfrak{F}^{\triangle}}(A^{\triangle})\#H$.

2. Preliminaries

For basic definitions and results in semiring theory we refer to J.S. Golan [6]:

Definition 2.1. *A semiring is a nonempty set R equipped with two binary operations $'+'$ and $'\cdot'$ called addition and multiplication such that, for $a, b, c \in R$,*

(i) $(R, +)$ *is a commutative monoid with identity element 0.*
(ii) $(R, \cdot)$ *is a monoid with identity element 1.*
(ii) *Multiplication distributes over addition from either side.*
  (a) $a \cdot (b + c) = a \cdot b + a \cdot c$
  (b) $(a + b) \cdot c = a \cdot c + b \cdot c$
(iii) $a \cdot 0 = 0 \cdot a = 0$, *for all* $a \in R$.
(iv) $1 \neq 0$.

Definition 2.2. *A semiring R is zerosumfree if and only if $r + r' = 0$ implies that* $r = r' = 0$.

Definition 2.3. *A semiring R is yoked if and only if for each $a, b \in R$ there exists $r \in R$ such that either $a + r = b$ or $b + r = a$.*

Definition 2.4. *An ideal I of a semiring R is a nonempty subset of R satisfying the following conditions:*

(i) *If $a, b \in I$, then $a + b \in I$;*
(ii) *If $a \in I$ and $r \in R$, then ar and $ra \in I$;*

Definition 2.5. *A nonempty subset(ideal) I of a semiring R is subtractive if and only if $a \in I$ and $a + b \in I$ implies that $b \in I$.*

Definition 2.6. [6, 11] *If R is an additively cancellative semiring, then the ring of differences of R, denoted by R^{Δ}, is given by*

$$R^{\Delta} = \{a - b | a, b \in R\}.$$

In R^{Δ}, we have $a - b = c - d$ if and only if there exist elements $r, r' \in R$ such that $a + r = c + r'$ and $b + r = d + r'$. Moreover R^{Δ} becomes a ring under addition and multiplication is given by

$$\begin{aligned}(a - b) + (c - d) &= (a + c) - (b + d) \\ (a - b) \cdot (c - d) &= (ac + bd) - (ad + bc).\end{aligned}$$

The zero element of R^{Δ} is $a - a$ and the multiplicative identity element of R^{Δ} is $1 = 1 - 0$. Also the map $a \mapsto a - 0$ is the natural embedding of R into R^{Δ}.

Lemma 2.7. [11] *Let R be additively cancellative semiring and $R^{\triangle}$ is its ring of differences. Let A, B be two ideals of R and I, J two ideals of $R^{\triangle}$. Then:*

(i) $A^{\triangle}B^{\triangle} = (AB)^{\triangle}$;

(ii) $(I \cap R)(J \cap R) \subseteq (IJ) \cap R$;

(iii) $(I \cap R)^{\triangle} \subseteq I$. *Equality holds if R is a yoked semiring;*

(iv) *For any two subsets A, B of R, $(A \cap B)^{\triangle} \subseteq A^{\triangle} \cap B^{\triangle}$. Equality holds if A and B are subtractive subsets of R and R is yoked;*

(v) *$I \cap R$ is subtractive, for every ideal I of $R^{\triangle}$.*

We denote the binary operation on an arbitrary monoid M as $'+'$.

Definition 2.8. [7] (Tensor product of semimodules) *Let K be a commutative semiring, $F \in Smod - K$ and $G \in K - Smod$. Then the tensor product $F \otimes_K G$, is defined by Sharma et al. in* [10] *as the factor monoid $(F \otimes G)/\sigma$, where σ is a congruence on $(F \otimes G)$ generated by the pairs $< (ak \otimes b), (a \otimes kb) >, \forall a \in F, b \in G$ and $k \in K$, such that for any balanced product (C, f) of F and G, there exists a unique morphism of monoids $\phi : F \otimes_K G \to C$, satisfying $f = \phi \circ g$, where $g : F \times G \to F \otimes_K G$, is given by $(m, n) \mapsto m \otimes n$.*

If K is a commutative semiring, then every left $K-$semimodule is a right $K-$semimodule and vice-versa. Also, if $F, G \in K - Smod$, then $F \otimes_K G$ is a commutative monoid and it becomes a $K-$semimodules by defining $\alpha(a \otimes b) = \alpha a \otimes b = a \otimes \alpha b$, for $a \in F, b \in G$ and $\alpha \in K$.

Theorem 2.9. [10] *Let K be commutative semiring. Then $(Smod - K, \otimes_K, K)$ is a monoidal category.*

Definition 2.10. [10] *The monoids in the monoidal category $(Smod - K, \otimes_K, K)$ are called $K-$semialgebras.*

Therefore, a $K-$semialgebra can be defined as a triple (A, M, u) with A a $K-$semimodule, $M : A \otimes A \to A$, a map called multiplication, $u : K \to A$, a map called the unit map, and such that the following diagrams are commutative,

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "I\otimes M"] \arrow[d, "M\otimes I"'] & A \otimes A \arrow[d, "M"] \\
A \otimes A \arrow[r, "M"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \otimes A \arrow[dd, "M"] & \\
K \otimes A \arrow[ur, "u\otimes I"] \arrow[dr] & & A \otimes K \arrow[ul, "I\otimes u"'] \arrow[dl] \\
& A &
\end{tikzcd}
$$

Note that $u(1_k)$ is the unital element of semialgebra A.

Now we recall the definitions of a Hopf algebra from Sweedler [12].

Definition 2.11. *A system $(H, M, u, \triangle, \epsilon)$, where H has algebra structure over a commutative ring k with multiplication M and unit u and H has coalgebra structure over k with co-multiplication $\triangle$ and co-unit ϵ satisfying:*

(i) *M, u are co-algebra maps.*
(ii) *$\triangle, \epsilon$ are algebra maps,*

is called a bialgebra.

Definition 2.12. *Let H be a bialgebra. The map $S : H \to H$ satisfying*

$$\sum_{(h)} S(h_1)h_2 = \epsilon(h)1_H = \sum_{(h)} h_1 S(h_2)$$

where $\Delta(h) = \sum_{(h)} h_1 \otimes h_2$, is called an antipode for H.

Definition 2.13. *A bialgebra with an antipode is a Hopf algebra.*

3. Semiring of Quotients of H-Semimodule Semialgebra

Throughout this paper A denotes semialgebra over a commutative semiring K and H denotes a Hopf algebra over the ring of differences $K^{\triangle}$ of K.

Definition 3.1. [5] *Let A be a $K-$semialgebra with identity 1_A. We say A is called an $H-$semimodule semialgebra if:*

(i) *A is an $H-$semimodule, where we denote the action of H on A by $h \cdot a$.*
(ii) *$h \cdot (ab) = \sum_{(h)}(h_1 \cdot a)(h_2 \cdot b)$, where $a, b \in A, h \in H$, and $\Delta(h) = \sum_{(h)}(h_1 \otimes h_2)$.*

(iii) $h \cdot 1_A = \epsilon(h)1_A$, *for all* $h \in H$.

Remark 3.2. Defining $h \cdot (a - b) = h \cdot a - h \cdot b, \forall h \in H, a, b \in A$, it can be easily seen that the algebra of differences $A^{\triangle}$ is an $H-$module algebra [4].

Definition 3.3. [5] *Let A be a left $H-$semimodule semialgebra. Then the smash product semialgebra $A\#H$ is defined as follows, for all $a, b \in A, h, k \in H$:*

(i) *As $K-$semimodule, $A\#H = A \otimes_K H$. We write $a\#h$ for the element $a \otimes h$.*
(ii) *Multiplication is given by*

$$(a\#h)(b\#g) = \sum_{(h)} a(h_1 \cdot b)\#h_2 g.$$

It is clear that A and H are embedded in $A\#H$ via the maps $a \mapsto a\#1_H$ and $h \mapsto 1_A\#h$ respectively. Also we write ah in place of $a\#h$.

Definition 3.4. [5] *An ideal I of a $H-$semimodule semialgebra A is said to be $H-$stable if $h \cdot a \in I, \forall h \in H, a \in I$.*

Definition 3.5. *An $H-$semimodule semialgebra A is called $H-$prime (respectively $H-$semiprime) if for any $H-$stable ideals I, J of A, then $IJ = 0$(resp. $I^2 = 0$) implies $I = 0$ or $J = 0$ (resp. I=0).*

Proposition 3.6. *Let A be $H-$semimodule semialgebra and I be $H-$stable ideal in A. Then the left annihilator of I in A is $H-$stable. Further if, S^{-1} exists, then the right annihilator of I in A is also $H-$stable.*

Proof. The proof is similar to the proof given in [3, Corollary 2]. ∎

Now we introduce the semiring of quotients of A with respect to a filter $\mathfrak{F}$ of ideals of A.

Definition 3.7. *Let A be $H-$semimodule semialgebra and $\mathfrak{F}$ be the family of all $H-$stable ideals I of A whose right and left annihilators are zero. If $f : I \to A$ and $g : J \to A$ are left $A-$semimodule homomorphisms, with $I, J \in \mathfrak{F}$, then f is said to equivalent to g if they agree on their common domain. Let $[f]$ denote the equivalence class of f, and let $Q_{\mathfrak{F}}(A)$ be the set of all such equivalence classes $q = [f]$. Addition and multiplication in $Q_{\mathfrak{F}}(A)$ are given as follows: If $q_1 = [f], q_2 = [g]$ are in $Q_{\mathfrak{F}}(A)$, then define $q_1 + q_2$ as the class of $f + g : I \cap J \to A$ and define $q_1 q_2$ as the class of composite map $fg : JI \to A$. Under the above operations $Q_{\mathfrak{F}}(A)$ becomes a semiring and is called the semiring of quotients of A with respect to $\mathfrak{F}$. If $\mathfrak{F}_1$ is the family of all ideals I of A whose right and left annihilators are zero then $Q_{\mathfrak{F}_1}(A)$ is called right Martindale semiring of quotients.*

Lemma 3.8. *If A is $H-$prime, $H-$semimodule semialgebra and the algebra of differences $A^{\triangle}$ of A is $H-$prime, $H-$module algebra.*

Proof. Let I and J be two $H-$stable ideals in A, such that $IJ = 0$. Then $IJ \cap A = 0$, by Lemma 2.7(ii), $(I \cap A)(J \cap A) = 0$. Since A is $H-$prime, yoked and by Lemma 2.7(iii), they give $I = 0$ or $J = 0$. ∎

Let A be additively cancellative and yoked semiring. We denote

$$\begin{aligned}
\mathfrak{F} &= \{I \subseteq A | I \,\text{is}\, H-\text{stable ideal of}\, A \,\text{and}\, l_A(I) = 0 = r_A(I)\},\\
\mathfrak{F}' &= \{I \subseteq A^{\triangle} | I \,\text{is}\, H-\text{stable ideal of}\, A^{\triangle} \,\text{and}\, l_{A^{\triangle}}(I) = 0 = r_{A^{\triangle}}(I)\}\\
\mathfrak{F}^{\triangle} &= \{I \subseteq A^{\triangle} | I \,\text{is}\, H-\text{stable ideal of}\, A^{\triangle} \,\text{such that}\, I = J^{\triangle} \,\text{for some}\, J \in \mathfrak{F}\},
\end{aligned}$$

where $l_A(I), r_A(I)$ are left and right annihilators of I in A respectively.

With the notation as above, we have the following:

Proposition 3.9. *If A is $H-$prime semimodule semialgebra, then $\mathfrak{F}' = \mathfrak{F}^{\triangle}$.*

Proof. First we prove that $\mathfrak{F}^{\triangle} \subseteq \mathfrak{F}'$. Let $I \subseteq \mathfrak{F}^{\triangle}$. Then $0 \neq I$ is $H-$ stable ideal of $A^{\triangle}$ such that $I = J^{\triangle}$ for some $J \in \mathfrak{F}$. Since A is $H-$prime and by Lemma 3.8, $A^{\triangle}$ is $H-$prime. Now, by $H-$primeness of $A^{\triangle}$ it follows that $l_{A^{\triangle}}(J^{\triangle}) = 0 = r_{A^{\triangle}}(J^{\triangle})$. Hence $I \subseteq \mathfrak{F}'$. Thus we have proved that $\mathfrak{F}^{\triangle} \subseteq \mathfrak{F}'$. To prove the reverse side inclusion $\mathfrak{F}' \subseteq \mathfrak{F}^{\triangle}$, let $J \in \mathfrak{F}'$. Then J is a nonzero $H-$stable ideal of $A^{\triangle}$, with left and right annihilators of J in $A^{\triangle}$ are 0. It is required to prove that $J = I^{\triangle}$ for some $I \in \mathfrak{F}$. Since A is yoked and by Lemma 2.7(iii), $J = (J \cap A)^{\triangle}$. So, it is remains to prove $(J \cap A) \in \mathfrak{F}$. Since $J \neq 0$, it follows from Lemma 2.7(iii) that $(J \cap A) \neq 0$. Also, it is clear that $J \cap A$ is $H-$stable ideal in A. Next, we proceed to show that left annihilator of $J \cap A$ in A is zero. For, suppose there exists $r \in A$ such that $r(J \cap A) = 0$. Then

$$\begin{aligned}
&rj = 0, \forall j \in J \cap A\\
&\Rightarrow rj_1 - rj_2 = 0, \forall j_1, j_2 \in J \cap A\\
&\Rightarrow r(j_1 - j_2) = 0, \forall j_1 - j_2 \in (J \cap A)^{\triangle} = J \text{ (by Lemma 2.7(iii))}\\
&\Rightarrow r \in l_{A^{\triangle}}(J) = 0
\end{aligned}$$

Hence, $l_A(J \cap A) = 0$. Similarly we can show right annihilator of $J \cap A$ in A is zero. This proves $J \cap A \in \mathfrak{F}$. Hence $\mathfrak{F}' = \mathfrak{F}^{\triangle}$. ∎

The following Proposition establishes an injective homomorphism from the semiring of quotients of A to the ring of quotients of $A^{\triangle}$ of A.

Proposition 3.10. *Let A be $H-$semimodule semialgebra and $\psi : Q_{\mathfrak{F}}(A) \to Q_{\mathfrak{F}^{\triangle}}(A^{\triangle})$ be the map defined by $\psi(q) = \tilde{q}$, where $q = [f], f : I \to A$ and $\tilde{q} = [\tilde{f}], \tilde{f} : I^{\triangle} \to A^{\triangle}$, given by $(a - b)\tilde{f} = af - bf, \forall a - b \in I^{\triangle}$. Then ψ is injective homomorphism.*

Proof. Let $\psi : Q_{\mathfrak{F}}(A) \to Q_{\mathfrak{F}^\triangle}(A^\triangle)$ defined by $\psi(q) = \tilde{q}$, where $q = [f]$, where $f : I \to A$ and $\tilde{q} = [\tilde{f}], \tilde{f} : I^\triangle \to A^\triangle$ given by $(a-b)\tilde{f} = af - bf, \forall a-b \in I^\triangle$. Let $q_1, q_2 \in Q_{\mathfrak{F}}(A)$. Then $q_1 = [f_1]$ and $q_2 = [f_2]$ where $f_1 : I_1 \to A$ and $f_2 : I_2 \to A$ are left $A-$homomorphism.

First we prove that to preserve addition, we prove that $\psi(q_1 + q_2) = \psi(q_1) + \psi(q_2)$ i.e., $(\widetilde{q_1 + q_2}) = \tilde{q_1} + \tilde{q_2}$. Let $q_1 + q_2 = [f_1] + [f_2]$, where $f_1 + f_2 : I_1 \cap I_2 \to A$ defined by $(a)(f_1 + f_2) = af_1 + af_2, \forall a \in I_1 \cap I_2$. Then $(\widetilde{q_1 + q_2}) = [\widetilde{f_1 + f_2}]$ where $\widetilde{f_1 + f_2} : (I_1 \cap I_2)^\triangle \to A^\triangle$ defined by $(a-b)(\widetilde{f_1 + f_2}) = a(f_1 + f_2) - b(f_1 + f_2), \forall a, b \in I_1 \cap I_2$. But $\tilde{q_1} + \tilde{q_2} = [\tilde{f_1}] + [\tilde{f_2}]$ where $\tilde{f_1} + \tilde{f_2} : (I_1^\triangle \cap I_2^\triangle) \to A^\triangle$, is defined by $(a-b)(\tilde{f_1} + \tilde{f_2}) = (a-b)\tilde{f_1} + (a-b)\tilde{f_2}, \forall a - b \in (I_1^\triangle \cap I_2^\triangle)$. Now since $(I_1 \cap I_2)^\triangle \subseteq (I_1^\triangle \cap I_2^\triangle)$, it follows that $(I_1 \cap I_2)^\triangle \cap (I_1^\triangle \cap I_2^\triangle) = (I_1 \cap I_2)^\triangle$. Hence, for any $a - b \in (I_1 \cap I_2)^\triangle$, we have $(a-b)(\tilde{f_1} + \tilde{f_2}) = (a-b)\tilde{f_1} + (a-b)\tilde{f_2}$. Then

$$\begin{aligned}
(a-b)(\tilde{f_1} + \tilde{f_2}) &= (a-b)\tilde{f_1} + (a-b)\tilde{f_2} \\
&= (af_1 - bf_1) + (af_2 - bf_2) \\
&= a(f_1 + f_2) - b(f_1 + f_2) \\
&= (a-b)(\widetilde{f_1 + f_2}), \ \forall a - b \in (I_1 \cap I_2)^\triangle.
\end{aligned}$$

Thus, $(\widetilde{f_1 + f_2}) = \tilde{f_1} + \tilde{f_2}$ on $(I_1 \cap I_2)^\triangle$ and hence $(\widetilde{q_1 + q_2}) = \tilde{q_1} + \tilde{q_2}$.

Next we prove that $\psi(q_1 q_2) = \psi(q_1)\psi(q_2)$. We have $q_1 q_2 = [f_1][f_2]$, where $f_1 f_2 : I_2 I_1 \to A$ defined by $(\sum_i a_i b_i)(f_1 f_2) = (\sum_i a_i(b_i f_1))f_2$. Also, $\widetilde{q_1 q_2} = [\widetilde{f_1 f_2}]$, where $\widetilde{f_1 f_2} : (I_2 I_1)^\triangle \to A^\triangle$ defined by $(\sum_i a_i b_i - \sum_i c_i d_i)(\widetilde{f_1 f_2}) = (\sum_i a_i b_i)(f_1 f_2) - (\sum_i c_i d_i)(f_1 f_2) = (\sum_i a_i(b_i f_1))f_2 - (\sum_i c_i(d_i f_1))f_2$. On the other hand, we have $\tilde{q_1}\tilde{q_2} = [\tilde{f_1}\tilde{f_2}]$, where $\tilde{f_1}\tilde{f_2} : (I_2 I_1)^\triangle \to A^\triangle$ defined by $(\sum_i a_i b_i - \sum_i c_i d_i)(\tilde{f_1}\tilde{f_2}) = (\sum_i a_i(b_i f_1) - \sum_i c_i(d_i f_1))\tilde{f_2} = (\sum_i a_i(b_i f_1))f_2 - (\sum_i c_i(d_i f_1))f_2$. Hence, $\psi(q_1 q_2) = \psi(q_1)\psi(q_2)$.

Finally we shall prove that ψ is one-to-one. For, let $\psi(q_1) = \psi(q_2)$, i.e., $\tilde{q_1} = \tilde{q_2}$, on $I_1^\triangle \cap I_2^\triangle$ where $\tilde{q_1} = [\tilde{f_1}], \tilde{f_1} : (I_1)^\triangle \to A^\triangle, \tilde{q_2} = [\tilde{f_2}], \tilde{f_2} : (I_2)^\triangle \to A^\triangle$. So, in particular $\tilde{q_1} = \tilde{q_2}$, on $(I_1 \cap I_2)^\triangle \subseteq I_1^\triangle \cap I_2^\triangle$, that is,

$$\begin{aligned}
& [\tilde{f_1}] = [\tilde{f_2}] \\
\Leftrightarrow\ & (a-b)\tilde{f_1} = (a-b)\tilde{f_2}, \ \ \forall a - b \in (I_1 \cap I_2)^\triangle \\
\Rightarrow\ & af_1 - bf_1 = af_2 - bf_2, \ \ \forall a - b \in (I_1 \cap I_2)^\triangle
\end{aligned}$$

In particular,

$$\begin{aligned}
& af_1 - 0f_1 = af_2 - 0f_2, \quad \text{take } b = 0 \\
\Rightarrow\ & af_1 = af_2, \ \forall a \in I_1 \cap I_2 \\
\Rightarrow\ & q_1 = q_2.
\end{aligned}$$

Thus ψ is one to one. ■

Proposition 3.11. *The $H-$semimodule semialgebra A is embedded in $Q_{\mathfrak{F}}(A)$ via $a \mapsto \phi_a$, where ϕ_a denote right multiplication on A by a.*

Proof. Follows from the fact that A is a semialgebra with identity 1_A ■

Proposition 3.12. *Let A be zerosumfree semiring. Then $Q_{\mathfrak{F}}(A)$ is a semiring but not a ring.*

Proof. By assumption, there exists $0 \neq a \in A$ such that $a + a' \neq 0$, $\forall a' \in A$. Let $\phi_a : A \to A$ defined by $r\phi_a = ra$. Let $q = [\phi_a] \in Q_{\mathfrak{F}}(A)$. Suppose that $q' = [f] \in Q_{\mathfrak{F}}(A), f : I \to A$ where $I \in \mathfrak{F}$, is additive inverse of q. Then $q + q' = 0$, where $q + q' = [\phi_a] + [f] = 0$. That is, $0 = b(\phi_a + f) = ba + bf$, $\forall b \in I$. Since A is zerosumfree, $ba = 0, \forall b \in I$ and $f \equiv 0$. This implies that $\phi_a \equiv 0$. But, since A is a semialgebra with 1_A, it follows that $a = 0$. This contradicts to our assumption that $a \neq 0$. Hence $Q_{\mathfrak{F}}(A)$ is semiring but not a ring. ■

Note 3.13. The map $\psi : Q_{\mathfrak{F}}(A) \to Q_{\mathfrak{F}^\triangle}(A^\triangle)$ is not onto: Suppose ψ is onto. Then $\forall \tilde{q} \in Q_{\mathfrak{F}^\triangle}(A^\triangle)$, there exists $q \in Q_{\mathfrak{F}}(A)$ such that $\psi(q) = \tilde{q}$. Since $Q_{\mathfrak{F}^\triangle}(A^\triangle)$ is ring, there exists $\tilde{t} \in Q_{\mathfrak{F}^\triangle}(A^\triangle)$ such that $\tilde{q} + \tilde{t} = \tilde{0}$ where $\tilde{t} = [\tilde{t'}], \tilde{t'} : J^\triangle \to A^\triangle$. Then $\psi(q+t) = \widetilde{q+t} = \tilde{q} + \tilde{t} = 0$. Since ψ is one-to-one implies that $q + t = 0$. This contradicts to $Q_{\mathfrak{F}}(A)$ is a proper semiring. Hence ψ is not onto.

Proposition 3.14. $(Q_{\mathfrak{F}}(A))^\triangle$ *is embedded in* $Q_{\mathfrak{F}^\triangle}(A^\triangle)$.

Proof. Let $\phi : (Q_{\mathfrak{F}}(A))^\triangle \to Q_{\mathfrak{F}^\triangle}(A^\triangle)$ defined by $\phi(q) = \tilde{q_1} - \tilde{q_2}$, where $q = q_1 - q_2$, $q_1, q_2 \in Q_{\mathfrak{F}}(A)$.

First we prove that to preserve addition, we prove that $\phi(q_1 + q_2) = \phi(q_1) + \phi(q_2)$, where $q_1, q_2 \in (Q_{\mathfrak{F}}(A))^\triangle$. Let $q_1 = q_1' - q_1''$, $q_2 = q_2' - q_2''$, where $q_1' = [f_1'], f_1' : I_1 \to A, q_2'' = [f_1''], f_1'' : I_2 \to A, q_2' = [f_2'], f_2' : I_3 \to A, q_2'' = [f_2''], f_2'' : I_4 \to A$ are in $Q_{\mathfrak{F}}(A)$. Since $q_1 + q_2 = (q_1' + q_2') - (q_1'' + q_2'')$, $\phi(q_1 + q_2) = \widetilde{(q_1' + q_2')} - \widetilde{(q_1'' + q_2'')}$ where $\widetilde{(q_1' + q_2')} = [\widetilde{f_1' + f_2'}], \widetilde{f_1' + f_2'} :$ $(I_1 \cap I_3)^\triangle \to A^\triangle$ defined by $(a - b)(\widetilde{f_1' + f_2'}) = a(f_1' + f_2') - b(f_1' + f_2')$, $\forall a, b \in I_1 \cap I_3$ and $\widetilde{(q_1'' + q_2'')} = [\widetilde{f_1'' + f_2''}], (\widetilde{f_1'' + f_2''}) : (I_2 \cap I_4)^\triangle \to A^\triangle$ defined by $(a - b)(\widetilde{f_1'' + f_2''}) = a(f_1'' + f_2'') - b(f_1'' + f_2'')$, $\forall a, b \in I_2 \cap I_4$. On the other hand $\phi(q_1) + \phi(q_2) = (\tilde{q_1'} - \tilde{q_1''}) + (\tilde{q_2'} - \tilde{q_2''}) = (\tilde{q_1'} + \tilde{q_2'}) - (\tilde{q_1''} + \tilde{q_2''}) = \widetilde{(q_1' + q_2')} - \widetilde{(q_1'' + q_2'')}$, and hence ϕ is additive.

Next we prove that $\phi(q_1 q_2) = \phi(q_1)\phi(q_2)$, where $q_1, q_2 \in (Q_{\mathfrak{F}}(A))^\triangle$ i.e.,$\widetilde{q_1 q_2} = \widetilde{q_1}\widetilde{q_2}$. Let $q_1 q_2 = (q_1' - q_1'')(q_2' - q_2'') = (q_1' q_2' + q_1'' q_2'') - (q_1' q_2'' + q_1'' q_2')$. Then $\widetilde{q_1 q_2} = (\widetilde{q_1' q_2' + q_1'' q_2''}) - (\widetilde{q_1' q_2'' + q_1'' q_2'})$. On the other hand $\widetilde{q_1}\widetilde{q_2} = (\tilde{q_1'} - \tilde{q_1''})(\tilde{q_2'} - \tilde{q_2''}) = (\tilde{q_1'}\tilde{q_2'} + \tilde{q_1''}\tilde{q_2''}) - (\tilde{q_1'}\tilde{q_2''} + \tilde{q_1''}\tilde{q_2'}) = (\widetilde{q_1' q_2' + q_1'' q_2''}) - (\widetilde{q_1' q_2'' + q_1'' q_2'})$. This proves ϕ is multiplicative.

Finally we shall prove that ϕ is one to one:

$$\begin{aligned} Ker(\phi) &= \{q_1 - q_2 \in (Q_{\mathfrak{F}}(A))^{\triangle} | \phi(q_1 - q_2) = 0\} \\ &= \{q_1 - q_2 \in (Q_{\mathfrak{F}}(A))^{\triangle} | \tilde{q_1} - \tilde{q_2} = 0\} \\ &= \{q_1 - q_2 \in (Q_{\mathfrak{F}}(A))^{\triangle} | \tilde{q_1} = \tilde{q_2}\} \\ &= \{q_1 - q_2 \in (Q_{\mathfrak{F}}(A))^{\triangle} | q_1 = q_2\} \ (\because \psi \text{ is one to one}) \\ &= \{0\} \end{aligned}$$

Hence, ϕ is one to one. ■

Remark 3.15. The map ϕ in the above Proposition is onto if for every $\tilde{q} : I^{\triangle} \to A^{\triangle}$, the restriction q of $\tilde{q}$ to I is a map from I to A. That is, the map ϕ is onto if $\tilde{q}(a) \in A$, for every $a \in I$.

Corollary 3.16. *The following diagram commutes.*

$$\begin{tikzcd} Q_{\mathfrak{F}}(A) \arrow[r, "\psi"] \arrow[d, hook, "i"'] & Q_{\mathfrak{F}^{\triangle}}(A^{\triangle}) \\ (Q_{\mathfrak{F}}(A))^{\triangle} \arrow[ur, dashed, "\Phi"] & \end{tikzcd}$$

4. Action of H on $Q_{\mathfrak{F}}(A)$:

Let H be Hopf algebra in which S^{-1} exists, and let A be an $H-$semimodule semialgebra. Now let us define an action of H on $Q_{\mathfrak{F}}(A)$. Let $q \in Q_{\mathfrak{F}}(A)$ and say $q : I \to A$. Then, define $h \cdot q : I \to A$, by

$$(a)(h \cdot q) = \sum_{(h)} h_2 \cdot (S^{-1}(h_1) \cdot a)q, \forall a \in I.$$

Theorem 4.1. *Let H be Hopf algebra with S^{-1} exists, and let A be an $H-$semimodule semialgebra. Then the above action extends the action of H on A to $Q_{\mathfrak{F}}(A)$ and makes $Q_{\mathfrak{F}}(A)$ into a $H-$semimodule semialgebra.*

Proof. (i) First we prove $h \cdot q \in Q_{\mathfrak{F}}(A)$, that is, $h \cdot q$ is left $A-$semimodule homomorphism. Let $a \in I$ and $x \in A$. Then

$$\begin{aligned} &(xa)(h \cdot q) \\ &= \sum_{(h)} h_2 \cdot \Big(\big(S^{-1}(h_1) \cdot (xa)\big) q \Big) \\ &= \sum_{(h)} h_3 \cdot \Bigg(\Big(\big(S^{-1}(h_2) \cdot x\big) \big(S^{-1}(h_1) \cdot a\big) \Big) q \Bigg) \ (\text{by Def. 3.1 (ii)}) \end{aligned}$$

$$= \sum_{(h)} h_3 \cdot \Big(\big(S^{-1}(h_2) \cdot x\big)\Big(\big(S^{-1}(h_1) \cdot a\big)q\Big)\Big) \ (\because q \text{ is left homomorphism})$$

$$= \sum_{(h)} \Big(h_3 \cdot \big(S^{-1}(h_2) \cdot x\big)\Big) \cdot \Big(h_4 \cdot \big(S^{-1}(h_1) \cdot a\big)q\Big) \ (\text{by Def. 3.1 (ii)})$$

$$= \sum_{(h)} \big(h_3 S^{-1}(h_2) \cdot x\big) \cdot \Big(h_4 \cdot \big(S^{-1}(h_1) \cdot a\big)q\Big)$$

$$= \sum_{(h)} (\epsilon(h_2)x) \cdot \Big(h_3 \cdot \big(S^{-1}(h_1) \cdot a\big)q\Big)$$

$$= x \sum_{(h)} h_2 \cdot \Big(\big(S^{-1}(h_1) \cdot a\big)q\Big)$$

$$= x\big(a(h \cdot q)\big)$$

(ii) Let $h, h' \in H$. Then

$$\begin{aligned}
(a)(hh' \cdot q) &= \sum_{(h)} h_2 h'_2 \cdot \Big(\big(S^{-1}(h_1 h'_1) \cdot a\big)q\Big) \\
&= \sum_{(h)} h_2 \cdot \Big(\Big(h'_2 \cdot \big(S^{-1}(h'_1) \cdot S^{-1}(h_1)\big) \cdot a\Big)q\Big) \\
&= \sum_{(h)} h_2 \cdot \Big(h'_2 \cdot \Big(S^{-1}(h'_1) \cdot \big(S^{-1}(h_1) \cdot a\big)\Big)q\Big) \\
&= \sum_{(h)} h_2 \cdot \Big(\big(S^{-1}(h_1) \cdot a\big)(h' \cdot q)\Big) \\
&= (a)\big(h \cdot (h' \cdot q)\big)
\end{aligned}$$

Clearly, $(a)(1 \cdot q) = (a)q$.

(iii) Let $h \in H, q, q' \in Q_{\mathfrak{F}}(A)$. Then $\forall a \in I$,

$$\begin{aligned}
\sum_{(h)} (a)\big((h_1 \cdot q)(h_2 \cdot q')\big) &= \sum_{(h)} \Big(a(h_1 \cdot q)\Big)(h_2 \cdot q') \\
&= \sum_{(h)} \Big(h_2 \cdot \big(S^{-1}(h_1) \cdot a\big)q\Big)\big(h_3 \cdot q'\big) \\
&= \sum_{(h)} h_4 \cdot \Big(S^{-1}(h_3) \cdot \Big(h_2 \cdot \big(S^{-1}(h_1) \cdot a\big)q\Big)\Big)q' \\
&= \sum_{(h)} h_4 \cdot \Big(S^{-1}(h_3)h_2 \cdot \big(S^{-1}(h_1) \cdot a\big)\Big)qq' \\
&= \sum_{(h)} h_3 \cdot \Big(\epsilon(h_2) \cdot \big(S^{-1}(h_1) \cdot a\big)\Big)qq'
\end{aligned}$$

$$= \sum_{(h)} h_3\epsilon(h_2) \cdot \Big(\big(S^{-1}(h_1) \cdot a\big)qq'\Big)$$
$$= \sum_{(h)} h_2 \cdot \Big(\big(S^{-1}(h_1) \cdot a\big)qq'\Big)$$
$$= a\big(h \cdot (qq')\big) \qquad \blacksquare$$

Proposition 4.2. *If A is a $H-$semimodule semialgebra then $A\#H$ is embedded in $Q_{\mathfrak{F}}(A)\#H$.*

Proof. The proof follows from Proposition 3.11. $\blacksquare$

Proposition 4.3. *Let $\psi' : Q_{\mathfrak{F}}(A)\#H \to Q_{\mathfrak{F}^{\triangle}}(A^{\triangle})\#H$ be the map defined by $\psi'(\sum_i q_i\#h_i) = \sum_i \tilde{q}_i\#h_i$, where for each i, $q_i = [f_i], f_i : I_i \to A$ and $\tilde{q} = [\tilde{f}_i], \tilde{f}_i : I_i^{\triangle} \to A^{\triangle}$, given by $(a-b)\tilde{f}_i = af_i - bf_i, \forall a-b \in I_i^{\triangle}$. Then ψ' is injective homomorphism.*

Proof. The proof follows from Proposition 3.10. $\blacksquare$

Proposition 4.4. *$(Q_{\mathfrak{F}}(A))^{\triangle}\#H$ is embedded in $Q_{\mathfrak{F}^{\triangle}}(A^{\triangle})\#H$.*

Proof. The proof follows from Proposition 3.14. $\blacksquare$

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