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Left Invariant Measures on Locally Compact Nonassociative Core Quasigroups

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Abstract. In this article left invariant measures and functionals on locally compact nonassociative core quasigroups are investigated. For this purpose necessary properties of topological core quasigroups, estimates and approximations of functions on such quasigroups are studied. An existence of nontrivial left invariant measures on locally compact core quasigroups is proved. Examples of not necessarily locally compact core quasigroups are provided by taking different types of products of such quasigroups.

Keywords: Measure; Left invariant; Quasigroup; Locally compact.

1. Introduction

Left invariant measures or Haar measures on locally compact groups play a very important role in measure theory, harmonic analysis, representation theory, geometry, mathematical physics, etc. (see, for example, [6, 11, 17] and references therein). On the other hand, in nonassociative algebra, in noncommutative geometry, field theory, topological algebra there frequently appear binary systems which are nonassociative generalizations of groups and related with quasigroups, quasi-groups, Moufang quasigroups, IP-quasigroups, etc. (see [8, 29, 30, 31] and references therein). An arbitrary IP-quasigroup Y is a quasigroup with a restriction: for each $x \in Y$ there exist elements x_1 and x_2 in Y such that for each y in Y the identities are satisfied $x_1(xy) = y$ and $(yx)x_2 = y$, where x_1 and x_2 are also denoted by ${}^{-1}x$ and x^{-1} and called left and right inverses of x respectively.

It was investigated and proved in the 20th century that a nontrivial geometry exists if and only if there exists a corresponding quasigroup.

A very important role in mathematics and quantum field theory is played by octonions and generalized Cayley-Dickson algebras [1, 2, 9]. A multiplicative law of their canonical bases is nonassociative and leads to a more general notion of a metagroup instead of a group [27]. They are used not only in algebra and geometry, but also in noncommutative analysis and PDEs, particle physics, mathematical physics (see [2, 9, 12]-[16, 18]-[26] and references therein). The preposition "meta" is used to emphasize that such an algebraic object has properties milder than a group. By their axiomatic metagroups are quasigroups with weak relations. They were used in [27] for investigations of automorphisms and derivations of nonassociative algebras.

In this article more general binary systems such as core quasigroups are studied (see Definition 2.1). They are also more general than IP-quasigroups, because in core quasigroups G left and right inverses ${}^{-1}x$ and x^{-1} of nonunit elements x in G may not exist.

This article is devoted to left invariant measures (see Definition 3.18) on locally compact core quasigroups. Necessary preliminary results about core quasigroups are given in Section 2. Specific algebraic and topological features of core quasigroups are studied in Formulas (1)-(35) and Formulas (42)-(44). A quotient of a core quasigroup by its core is investigated in Formulas (36)-(41). A uniform continuity of maps on topological core quasigroups is studied in Theorem 2.14 and Corollary 2.15.

Left invariant functionals and measures are investigated in Section 3. These properties are more complicated than for groups and IP-quasigroups, because of the nonassociativity of core quasigroups and absence of left and right inverses in general. The main results can be found in Theorems 3.15, 3.16, 3.19, 3.20. For their proofs estimates of nonnegative functions with compact supports in core quasigroups are investigated in Lemmas 3.2, 3.4, 3.6. Functionals on a space of nonnegative functions with compact supports in a core quasigroup are studied in Lemmas 3.7, 3.8, 3.10, 3.13 (estimates (132)-(147)) and Theorem 3.9. In Theorem 3.11 approximations of nonnegative functions with compact supports in the core quasigroup are described.

In an appendix abundant families of core quasigroups are provided with the help of a direct product and smashing products (see Remark 4.3 and Definition 4.5). For this purpose Theorems 4.1 and 4.4 are proved.

All main results of this paper are obtained for the first time. They can be used in harmonic analysis on nonassociative algebras and metagroups and quasigroups, representation theory, geometry, mathematical physics, quantum field theory, particle physics, PDEs, etc.

2. Core Quasigroups

To avoid misunderstandings we give necessary definitions. For short it will be written core quasigroup instead of nonassociative core quasigroup.

Definition 2.1. *Let G be a set with a multiplication (that is a single-valued binary operation) $G \times G \ni (a,b) \mapsto ab \in G$ defined on G satisfying the conditions:*

(i) *for each a and b in G there is a unique $x \in G$ with $ax = b$ and*

(ii) *a unique $y \in G$ exists satisfying $ya = b$, which are denoted by $x = a \setminus b = Div_l(a,b)$ and $y = b/a = Div_r(a,b)$ correspondingly,*

(iii) *there exists a neutral (i.e. unit) element $e_G = e \in G$: $eg = ge = g$ for each $g \in G$.*

We consider subsets in G:

(iv) $Com(G) := \{a \in G : \ \forall b \in G, \ ab = ba\}$;

(v) $N_l(G) := \{a \in G : \ \forall b \in G, \ \forall c \in G, \ (ab)c = a(bc)\}$;

(vi) $N_m(G) := \{a \in G : \ \forall b \in G, \ \forall c \in G, \ (ba)c = b(ac)\}$;

(vii) $N_r(G) := \{a \in G : \ \forall b \in G, \ \forall c \in G, \ (bc)a = b(ca)\}$;

(viii) $N(G) := N_l(G) \cap N_m(G) \cap N_r(G)$; $Z(G) := Com(G) \cap N(G)$.

Then $N(G)$ is called a nucleus of G and $Z(G)$ is called the center of G.

We call G a core quasigroup if a set G possesses a multiplication and satisfies Conditions (i)-(iii) above and

(ix) *$(ab)c = t(a,b,c)a(bc)$ and $(ab)c = a(bc)p(a,b,c)$ for each a, b and c in G, where $t(a,b,c) = t_G(a,b,c) \in N(G)$ and $p(a,b,c) = p_G(a,b,c) \in N(G)$.*

Then G will be called a central core quasigroup if in addition to Condition (ix) above it satisfies the condition:

(x) *$ab = \mathsf{t}_2(a,b)ba$ for each a and b in G, where $\mathsf{t}_2(a,b) \in Z(G)$.*

There, for given a, b, c in G, the elements $t_G(a,b,c)$, $p_G(a,b,c)$ and $\mathsf{t}_2(a,b)$ are unique such that $t_G : G \times G \times G \to N(G)$, $p_G : G \times G \times G \to N(G)$, $\mathsf{t}_2 : G \times G \to Z(G)$ are mappings.

Let τ be a topology on G such that the multiplication $G \times G \ni (a,b) \mapsto ab \in G$ and the mappings $Div_l(a,b)$ and $Div_r(a,b)$ are jointly continuous relative to τ. Then (G,τ) is called a topological core quasigroup. Henceforth, it will be assumed that τ is a $T_1 \cap T_{3.5}$ topology, unless something else is specified.

A minimal closed subgroup $N_0(G)$ in the topological core quasigroup G containing $t(a,b,c)$ and $p(a,b,c)$ for each a, b and c in G will be called a core of G.

Elements of the core quasigroup G will be denoted by small letters, subsets of G will be denoted by capital letters. If A and B are subsets in G, then $A - B$ means the difference of them $A - B = \{a \in A : \ a \notin B\}$. Henceforward, maps and functions on core quasigroups are supposed to be single-valued, unless something else is specified.

Lemma 2.2. *If G is a core quasigroup, then for each a, b and c in G the following identities are fulfilled:*

$$b \setminus e = t(e/b, b, b \setminus e)(e/b); \tag{1}$$

$$b \setminus e = (e/b)p(e/b, b, b \setminus e); \tag{2}$$

$$(a \setminus e)b = t(e/a, a, a \setminus e)[t(e/a, a, a \setminus b)]^{-1}(a \setminus b); \tag{3}$$

$$(a \setminus b) = (a \setminus e)bp(a, a \setminus e, b); \quad (4)$$
$$(bc) \setminus a = (c \setminus (b \setminus a))[p(b, c, (bc) \setminus a)]^{-1}; \quad (5)$$
$$(a \setminus b)c = (a \setminus (bc))[p(a, a \setminus b, c)]^{-1}; \quad (6)$$
$$(ab) \setminus e = (b \setminus e)(a \setminus e)[t(a, b, b \setminus e)]^{-1}t(ab, b \setminus e, a \setminus e); \quad (7)$$
$$b(e/a) = (b/a)p(b/a, a, a \setminus e)[p(e/a, a, a \setminus e)]^{-1}; \quad (8)$$
$$(b/a) = [t(b, e/a, a)]^{-1}b(e/a); \quad (9)$$
$$a/(bc) = t(a/(bc), b, c)((a/c)/b); \quad (10)$$
$$c(b/a) = t(c, b/a, a)(cb)/a; \quad (11)$$
$$e/(ab) = [p(e/b, e/a, ab)]^{-1}p(e/a, a, b)(e/b)(e/a). \quad (12)$$

Proof. Note that $N(G)$ is a subgroup in G due to Conditions (v)-(viii) in Definition 2.1 (see also [8]). Then Conditions (i)-(iii) in Definition 2.1 imply that

$$b(b \setminus a) = a, \quad b \setminus (ba) = a; \quad (13)$$
$$(a/b)b = a, \quad (ab)/b = a \quad (14)$$

for each a and b in any quasigroup G (see also [8, 31]). Using Condition (ix) in Definition 2.1 and Identities (13) and (14) we deduce that $e/b = (e/b)(b(b \setminus e)) = [t(e/b, b, b \setminus e)]^{-1}(b \setminus e)$ which leads to (1).

Let $c = a \setminus b$. Then from Identities (1) and (13) it follows that $(a \setminus e)b = t(e/a, a, a \setminus e)(e/a)(ac) = t(e/a, a, a \setminus e)[t(e/a, a, a \setminus b)]^{-1}((e/a)a)(a \setminus b)$ which taking into account (14) provides (3).

On the other hand, $b \setminus e = ((e/b)b)(b \setminus e) = (e/b)(b(b \setminus e))p(e/b, b, b \setminus e)$ that gives (2).

Now let $d = b/a$. Then Identities (2) and (14) imply that $b(e/a) = (da)(a \setminus e)[p(e/a, a, a \setminus e)]^{-1} = (b/a)p(b/a, a, a \setminus e)[p(e/a, a, a \setminus e)]^{-1}$ which demonstrates (8).

Next we infer from (ix) in Definition 2.1 and (13) that $b(c((bc) \setminus a)) = (bc)((bc) \setminus a)[p(b, c, (bc) \setminus a)]^{-1} = a[p(b, c, (bc) \setminus a)]^{-1}$, hence $c((bc) \setminus a) = (b \setminus a)[p(b, c, (bc) \setminus a)]^{-1}$ that implies (5).

Symmetrically it is deduced that $(a/(bc))b)c = t(a/(bc), b, c)a$, consequently, $(a/(bc))b = t(a/(bc), b, c)(a/c)$. From the latter identity it follows (10).

Evidently, formulas $a((a \setminus b)c) = (a(a \setminus b))c[p(a, a \setminus b, c)]^{-1} = bc[p(a, a \setminus b, c)]^{-1}$ and $(c(b/a))a = t(c, b/a, a)cb$ imply (6) and (11) correspondingly.

From (ix) in Definition 2.1 we infer that $(ab)((b \setminus e)(a \setminus e)) = [t(ab, b \setminus e, a \setminus e)]^{-1}t(a, b, b \setminus e)$, since by (13) $(a(b(b \setminus e)))(a \setminus e) = e$. This together with (i) and (ii) in Definition 2.1 implies (7).

Analogously from (ix) in Definition 2.1 we deduce that $((e/b)(e/a))(ab) = [p(e/a, a, b)]^{-1}p(e/b, e/a, ab)$, since by (14) $(e/b)(((e/a)a)b) = e$. Finally applying (i) and (ii) in Definition 2.1 we get Identity (12). ∎

Lemma 2.3. *Assume that G is a core quasigroup. Then for every a, a_1, a_2, a_3*

in G and z_1, z_2, z_3 in $Z(G)$, $b \in N(G)$:

$$t(z_1a_1, z_2a_2, z_3a_3) = t(a_1, a_2, a_3); \tag{15}$$
$$p(z_1a_1, z_2a_2, z_3a_3) = p(a_1, a_2, a_3); \tag{16}$$
$$t(a, a \setminus e, a)a = ap(a, a \setminus e, a); \tag{17}$$
$$t(a, e/a, a)a = ap(a, e/a, a); \tag{18}$$
$$p(a, a \setminus e, a)t(e/a, a, a \setminus e) = e; \tag{19}$$
$$t(a_1, a_2, a_3b) = t(a_1, a_2, a_3); \tag{20}$$
$$p(ba_1, a_2, a_3) = p(a_1, a_2, a_3); \tag{21}$$
$$t(ba_1, a_2, a_3) = bt(a_1, a_2, a_3)b^{-1}; \tag{22}$$
$$p(a_1, a_2, a_3b) = b^{-1}p(a_1, a_2, a_3)b. \tag{23}$$

Proof. Let the elements a, a_1, a_2, a_3 belong to G, the elements z_1, z_2, z_3 be in $Z(G)$. Since we have $(a_1a_2)a_3 = t(a_1, a_2, a_3)a_1(a_2a_3)$ with $t(a_1, a_2, a_3) \in N(G)$ for every a_1, a_2, a_3 in G, it follows that

$$t(a_1, a_2, a_3) = ((a_1a_2)a_3)/(a_1(a_2a_3)). \tag{24}$$

In addition, for each $q \in Z(G)$, a and b in G, we have

$$b/(qa) = q^{-1}b/a \text{ and } b/q = q \setminus b = bq^{-1}, \tag{25}$$

because $Z(G)$ is the commutative group satisfying Conditions (iv) and (viii) in Definition 2.1. From (24) and (25) we infer that

$$\begin{aligned} t(z_1a_1, z_2a_2, z_3a_3) &= (((z_1a_1)(z_2a_2))(z_3a_3))/((z_1a_1)((z_2a_2)(z_3a_3))) \\ &= ((z_1z_2z_3)((a_1a_2)a_3))/((z_1z_2z_3)(a_1(a_2a_3))) \\ &= ((a_1a_2)a_3)/(a_1(a_2a_3)). \end{aligned}$$

Thus $t(z_1a_1, z_2a_2, z_3a_3) = t(a_1, a_2, a_3)$.

Symmetrically we get

$$p(a_1, a_2, a_3) = (a_1(a_2a_3)) \setminus ((a_1a_2)a_3) \tag{26}$$

and $p(z_1a_1, z_2a_2, z_3a_3) = ((z_1a_1)((z_2a_2)(z_3a_3))) \setminus (((z_1a_1)(z_2a_2))(z_3a_3)) = ((z_1z_2z_3)(a_1(a_2a_3))) \setminus ((z_1z_2z_3)((a_1a_2)a_3)) = (a_1(a_2a_3)) \setminus ((a_1a_2)a_3)$ that provides (16).

From Formulas (24) and (1) it follows that $t(a, a \setminus e, a) = ((a(a \setminus e))a)/(a((a \setminus e)a)) = a/[at(e/a, a, a \setminus e)]$ and consequently,

$$t(a, a \setminus e, a)at(e/a, a, a \setminus e) = a. \tag{27}$$

Then from Formulas (26), (13) and Condition (ix) in Definition 2.1 we deduce that $p(a, a \setminus e, a) = (a((a \setminus e)a)) \setminus ((a(a \setminus e))a) = \{[t(a, a \setminus e, a)]^{-1}a\} \setminus a$, which implies (17). Identities (17) and (27) lead to (19). Next using (26) and (ix) in Definition 2.1 we infer that $p(a, e/a, a) = [a((e/a)a)] \setminus [(a(e/a))a] = a \setminus [t(a, e/a, a)a]$

that implies (18). From (ix) in Definition 2.1 we get that $((a_1a_2)a_3)b = (a_1a_2)(a_3b) = (t(a_1,a_2,a_3b)a_1(a_2a_3))b$, from which together with (14) and (24) Identity (20) follows, because $b \in N(G)$. Then $b((a_1a_2)a_3) = ((ba_1)a_2)a_3 = b(a_1(a_2a_3)p(ba_1,a_2,a_3))$ and (13) and (26) imply Identity (21). Symmetrically we deduce $b((a_1a_2)a_3) = t(ba_1,a_2,a_3))b(a_1(a_2a_3))$ and $((a_1a_2)a_3)b = (a_1(a_2a_3))bp(a_1,a_2,a_3b)$ which together with (24) and (26) imply Identities (22) and (23). ■

Lemma 2.4. *If (G,τ) is a topological quasigroup, then the functions $t(a_1,a_2,a_3)$ and $p(a_1,a_2,a_3)$ are jointly continuous in a_1, a_2, a_3 in G.*

Proof. This follows immediately from Formulas (24), (26) and Definition 2.1. ■

Lemma 2.5. *Assume that (G,τ) is a topological quasigroup and U is an open subsets in G. Then for each $b \in G$ the sets Ub and bU are open in G.*

Proof. Take any $c \in Ub$ and consider the equation

$$xb = c. \tag{28}$$

Then from Condition (ii) in Definition 2.1 it follows that

$$x = c/b. \tag{29}$$

Thus $x = \psi_b(c)$, where $\psi_b(c) = c/b$ is a continuous bijective function in the variable c due to Identity (9) and Lemma 2.4. On the other hand, the right shift mapping

$$R_bu := ub \tag{30}$$

from G into G is continuous and bijective in u (see Definition 1). Moreover, $\psi_b(R_bu) = u$ and $R_b(\psi_b(c)) = c$ for each fixed $b \in G$ and all $u \in G$ and $c \in G$ by Identities (14). Thus R_b and ψ_b are open mappings, consequently, Ub is open in G.

Similarly for the equation

$$by = c \tag{31}$$

the unique solution is

$$y = b \setminus c \tag{32}$$

by Condition (i) in Definition 2.1.

Therefore, $y = \theta_b(c)$, where $\theta_b(c) = b \setminus c$ is a continuous bijective function in c according to Lemma 2.4 and Formula (4). Next we consider the left shift mapping

$$L_bu = bu \tag{33}$$

for each fixed $b \in G$ and any $u \in G$. This mapping L_b is continuous, since the multiplication on G is continuous. Then $L_b(\theta_b(c)) = c$ and $\theta_b(L_b u) = u$ for every fixed $b \in G$ and all $u \in G$ and $c \in G$ by Identities (13). Therefore θ_b and L_b are open mappings. Thus the subset bU is open in G. ∎

Lemma 2.6. *Let (G, τ) be a topological quasigroup.*

(i) *Let also U and V be subsets in G such that either U or V is open. Then UV is open in G.*

(ii) *If A and B are compact subsets in G, then AB is compact.*

(iii) *For each open neighborhood U of e in G there exists an open neighborhood V of e such that*

(a) $\check{V} \subseteq U$, *where*

(b) $\check{V} = V \cup Inv_l(V) \cup Inv_r(V)$, *where* $Inv_l(a) = Div_l(a, e)$, $Inv_r(a) = Div_r(a, e)$ *for each* $a \in G$,

(c) $DQ = \{x = ab : \ a \in D, \ b \in Q\}$,

(d) $Inv_l(D) = \{x = a \setminus e : \ a \in D\}$,

(e) $Inv_r(D) = \{x = e/a : \ a \in D\}$

for any subsets D and Q in G.

Proof. (i). In view of Lemma 2.5 the subsets Ub and aV are open in G for each $a \in U$ and $b \in V$, consequently, $UV = \{x : \ x = uv, \ u \in U, \ v \in V\} = \bigcup_{b \in V} Ub = \bigcup_{a \in U} aV$ is open in G.

(ii). Let A and B be compact subsets of G. Then the subset $AB = \{c : \ c = ab, \ a \in A, \ b \in B\}$ is a continuous image of a compact subset $A \times B$ in $G \times G$, where $G \times G$ is supplied with the product (i.e. Tychonoff) topology, consequently, AB is a compact subset in G (see Theorem 3.1.10 and the Tychonoff Theorem 3.2.4 in [10]).

(iii). The mappings Inv_l and Inv_r are homeomorphisms of G onto itself as a topological space, since they are bijective, continuous and

$$Inv_l(Inv_r(b)) = b \text{ and } Inv_r(Inv_l(b)) = b \tag{34}$$

for each b in G by (a), (b). Therefore for each open neighborhood U of e there exists an open neighborhood of e of the form

$$V := \hat{U}, \tag{35}$$

where $\hat{U} := U \cap Inv_l(U) \cap Inv_r(U)$.

From (c) we infer that $Inv_r(Inv_l(U)) = U$ and $Inv_l(Inv_r(U)) = U$, hence $Inv_l(V) \subseteq U \cap Inv_l(U) \cap Inv_l(Inv_l(U)) \subseteq U \cap Inv_l(U)$ and $Inv_r(V) \subset U \cap Inv_r(U)$, consequently, $V \cup Inv_l(V) \cup Inv_r(V) \subseteq U$. ∎

Definition 2.7. *A subquasigroup H of a quasigroup G is called normal if it satisfies*

(i) $xH = Hx$ *and*

(ii) $(xy)H = x(yH)$ *and* $(xH)y = x(Hy)$ *and* $H(xy) = (Hx)y$

for each x *and* y *in* G.

A family of cosets $\{bH : \ b \in G\}$ *will be denoted by* $G/\cdot/H$.

Theorem 2.8. *If* G *is a* T_1 *topological core quasigroup, then its core* N_0 *is a normal subgroup and its quotient* $G/\cdot/N_0$ *is a* $T_1 \cap T_{3.5}$ *topological group.*

Proof. Let τ be a T_1 topology on G relative to which G is a topological quasigroup. Then each point x in G is closed, since G is the T_1 topological space (see Section 1.5 in [10]). From the joint continuity of the multiplication and the mappings Div_l and Div_r it follows that the nucleus $N = N(G)$ is closed in G. Therefore the subgroup N_0 is the closure of a subgroup $N_{0,0}(G)$ in N generated by elements $t(a, b, c)$ and $p(a, b, c)$ for all a, b and c in G (see Definition 2.1). According to Conditions (v)-(viii) in Definition 2.1 one gets that N and hence N_0 are subgroups in G satisfying Condition (ii) in Definition 2.7, because $N_0 \subseteq N$ (see also [8, 31]).

Let a and b belong to N and $x \in G$. Then $x(x \setminus (ab)) = ab$ and $x((x \setminus a)b) = (x(x \setminus a))b = ab$, consequently,

$$x \setminus (ab) = (x \setminus a)b \tag{36}$$

for each a and b in $N(G)$ and every $x \in G$.

Similarly it is deduced

$$(ab)/x = a(b/x) \tag{37}$$

for each a and b in $N(G)$, $x \in G$.

Therefore from (ix) in Definition 2.1, (13) and (36) it follows that $((x \setminus a)x)((x \setminus b)x) = (x \setminus a)(x((x \setminus b)x))p(x \setminus a, x, (x \setminus b)x) = (x \setminus (ab))x[p(x, x \setminus b, x)]^{-1}p(x \setminus a, x, (x \setminus b)x)$, since $(x \setminus a)(bx) = ((x \setminus a)b)x = (x \setminus (ab))x$. Thus

$$(x \setminus (ab))x = ((x \setminus a)x)((x \setminus b)x)[p(x \setminus a, x, (x \setminus b)x)]^{-1}p(x, x \setminus b, x) \tag{38}$$

for each a and b in $N(G)$, $x \in G$.

From Identities (5) and (6) it follows that

$$x \setminus ((u \setminus v)y) = ((ux) \setminus (vy))p(u, x, (ux) \setminus (vy))[p(u, u \setminus v, x)]^{-1} \tag{39}$$

for each u, v, x and y in G, since $x \setminus ((u \setminus v)y) = x \setminus (u \setminus (vy))[p(u, u \setminus v, y)]^{-1}$.

In particular for $u = a(bc)$ and $v = (ab)c$ with any a, b and c in G we infer using (ix) in Definition 2.1 that $ux = (a(b(cx)))p(b, c, x)p(a, bc, x)$ and $vx = (ab)(cx)p(ab, c, x)$. Hence from (39) and (26) it follows that

$$x \setminus (p(a, b, c)x) = [p(b, c, x)p(a, bc, x)]^{-1}p(a, b, cx)p(u, x, (ux) \setminus (vx)), \tag{40}$$

since $x \setminus (p(a,b,c)x) = [(a(b(cx)))p(b,c,x)p(a,bc,x)] \setminus [(ab)(cx)p(ab,c,x)]$ $p(u,x,(ux) \setminus (vx))[p(u,u \setminus v,x)]^{-1}$, because $u \setminus v = p(a,b,c) \in N(G)$ and $p(u,u \setminus v,x) = e$.

Notice that (i), (ii) and (ix) in Definition 2.1 imply $u \setminus (tu) = p$, where $t = t(a,b,c)$, $p = p(a,b,c)$, $u = a(bc)$ for any a, b and c in G. Let $z \in G$. Then there exists $x \in G$ such that $z = ux$, that is $x = u \setminus z$. Therefore we deduce that

$$z \setminus (tz) = [x \setminus (px)]p(u,u \setminus (tu),x)[p(u,x,(ux) \setminus (tux))]^{-1}, \tag{41}$$

since $t \in N(G)$, $p \in N(G)$, $(u \setminus (tu))x = (u \setminus (tux))[p(u,u \setminus (tu),x)]^{-1}$ by (6). From the equality (5) by taking $c = x$, $b = u$, $a = tux$ we infer $x \setminus (u \setminus (tux)) = [(ux) \setminus (tux)]p(u,x,(ux) \setminus (tux))$. Thus from Identities (38), (40) and (41) it follows that the group $N_{0,0} = N_{0,0}(G)$ generated by $\{p(a,b,c),\ t(a,b,c) : a \in G,\ b \in G,\ c \in G\}$ satisfies Condition (i) in Definition 2.7. From the joint continuity of the multiplication and the mappings Div_l and Div_r it follows that the closure N_0 of $N_{0,0}$ also satisfies (i) in Definition 2.7. Thus N_0 is a closed normal subgroup in G. In view of Theorem 1.1 in Ch. IV, Section 1 in [8] a quotient quasigroup $G/\cdot/N_0$ exists consisting of all cosets aN_0, where $a \in G$.

Then from Conditions (ix) in Definition 2.1, (i) and (ii) in Definition 2.7 it follows that for each a, b, c in G the following identities are valid:

$$\begin{aligned} (aN_0)(bN_0) &= (ab)N_0, \\ ((aN_0)(bN_0))(cN_0) &= (aN_0)((bN_0)(cN_0)), \\ eN_0 &= N_0 \end{aligned}$$

because $p(a,b,c) \in N_0$ and $t(a,b,c) \in N_0$ for all a, b and c in G.

In view of Lemmas 2.2 and 2.3 $(aN_0) \setminus e = e/(aN_0)$ and consequently, for each $aN_0 \in G/\cdot/N_0$ a unique inverse $(aN_0)^{-1}$ exists. Thus the quotient $G/\cdot/N_0$ of G by N_0 is a group. Since the topology τ on G is T_1 and N_0 is closed in G, the quotient topology τ_q on $G/\cdot/N_0$ is also T_1. By virtue of Theorem 8.4 in [17] this implies that τ_q is a $T_1 \cap T_{3.5}$ topology on $G/\cdot/N_0$. ∎

Proposition 2.9. *Assume that G is a T_1 topological core quasigroup and functions t and p on G are defined by Formulas* (ix) *in Definition 2.1. Then for each compact subset S in G and each open neighborhood V of e there exists an open neighborhood U of e in G such that*

(i) $t((u_1a)v_1,(u_2b)v_2,(u_3c)v_3) \in (Vt(a,b,c)) \cap (t(a,b,c)V)$ *and*

(ii) $p((u_1a)v_1,(u_2b)v_2,(u_3c)v_3) \in (Vp(a,b,c)) \cap (p(a,b,c)V)$

for every a, b, c in S and u_j, v_j in $\check{U}$ for each $j \in \{1,2,3\}$.

Proof. Take arbitrary fixed elements f, g and h in S. From the joint continuity of the maps $t(a,b,c)$ and $p(a,b,c)$ in the variables a, b and c in G it follows that there exists an open neighborhood $U_{f,g,h}$ of e in G and an open neighborhood $W_{f,g,h}$ of $(f,g,h) \in S \times S \times S$ in $G \times G \times G$ such that (i) and (ii) are valid for each u_j, v_j in $\check{U}_{f,g,h}$, $j \in \{1,2,3\}$, and $(a,b,c) \in W_{f,g,h}$ (see Lemmas 2.4 and

2.6). Notice that $S \times S \times S$ is compact in the Tychonoff product $G \times G \times G$ of G as the topological space (see Section 2.3 and Theorem 3.2.4 in [10]). Hence an open covering $\{W_{f,g,h} : \ f \in S, \ g \in S, \ h \in S\}$ of $S \times S \times S$ has a finite subcovering $\{W_{f_i,g_i,h_i} : \ i = 1, ..., n\}$, where n is a natural number, $n \geq 1$. That is $S \times S \times S \subseteq \bigcup_{i=1}^n W_{f_i,g_i,h_i}$. Then $\bigcap_{i=1}^n U_{f_i,g_i,h_i} =: U$ is an open neighborhood of e in G. Therefore, Properties (i) and (ii) are satisfied for every a, b, c in S and u_j, v_j in $\check{U}$ for each $j \in \{1, 2, 3\}$. ∎

We remind the following definition.

Definition 2.10. *Let G be a topological quasigroup. For a subset U in G it is put:*

(i) $\mathcal{L}_{U,G} := \{(x, y) \in G \times G : \ x \setminus y \in U\}$ *and*

(ii) $\mathcal{R}_{U,G} := \{(x, y) \in G \times G : \ y/x \in U\}$.

The family of all subsets $\mathcal{L}_{U,G}$ (or $\mathcal{R}_{U,G}$) with U being an open neighborhood of e will be denoted by $\mathcal{L}_G$ (or $\mathcal{R}_G$ correspondingly).

Proposition 2.11. *Let G be a T_1 topological locally compact core quasigroup. Then the family $\mathcal{L}_G$ (or $\mathcal{R}_G$) induces a uniform structure on G. A topology τ_1 on G provided by $\mathcal{L}_G$ (or $\mathcal{R}_G$ respectively) is $T_1 \cap T_{3.5}$ and equivalent to the initial topology τ on G.*

Proof. Let (G, τ) be a topological quasigroup and let $\mathcal{B}_e$ denote a base of its open neighborhoods at e. In view of Lemma 2.5 $\mathcal{C}_l(U) := \{xU : \ x \in G\}$ is an open covering of G for each $U \in \mathcal{B}_e$. We put $\mathcal{C}_l^0 = \{\mathcal{C}_l(U) : \ U \in \mathcal{B}_e\}$ and $\mathcal{C}_l$ to be a family of all coverings for each of which there exists a refinement of the type $\mathcal{C}_l^0$.

Below it is verified, that the family $\mathcal{C}_l$ satisfies Conditions $(UC1)$-$(UC4)$ of Section 8.1 in [10]. If $\mathcal{A} \in \mathcal{C}_l$, $\mathcal{E}$ is a covering of G and $\mathcal{A}$ refines $\mathcal{E}$, then there exists $U \in \mathcal{B}_e$ such that $\mathcal{C}_l(U)$ refines $\mathcal{A}$ and hence $\mathcal{C}_l(U)$ refines $\mathcal{E}$. Thus $(UC1)$ is satisfied.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ belong to $\mathcal{C}_l$. Then there are U_1 and U_2 in $\mathcal{B}_e$ such that $\mathcal{C}_l(U_j)$ refines $\mathcal{A}_j$ for each $j \in \{1, 2\}$. We put $U = U_1 \cap U_2$, consequently, $U \in \mathcal{B}_e$ and hence $\mathcal{C}_l(U)$ refines both $\mathcal{C}_l(U_1)$ and $\mathcal{C}_l(U_2)$. Therefore $\mathcal{C}_l(U)$ refines $\mathcal{A}_1$ and $\mathcal{A}_2$. Thus $(UC2)$ also is satisfied.

Condition $(UC3)$ means that for each $\mathcal{A} \in \mathcal{C}_l$ there exists $\mathcal{E} \in \mathcal{C}_l$ such that $\mathcal{E}$ is a star refinement of $\mathcal{A}$. In order to prove it, it evidently is sufficient to prove that for each $U \in \mathcal{B}_e$ there exists $U_1 \in \mathcal{B}_e$ such that

$$St(xU_1, \mathcal{C}_l(U_1)) \subset xU \text{ for each } x \in G, \tag{42}$$

where $St(M, \mathcal{A})$ denotes the star of a set M with respect to $\mathcal{A}$ (see its definition in [10, Section 5.1]).

Note that a map $f(x_1, x_2, x_3) = (x_1/x_2)x_3$ is the composition of jointly continuous maps $G \times G \ni (x_1, x_2) \mapsto x_1/x_2 \in G$ and $G \times G \ni (y, x_3) \mapsto yx_3 \in G$, hence it is jointly continuous from $G \times G \times G$ into G and $f(e,e,e) = e$, because G is the topological quasigroup (see Definition 2.1). The quasigroup G is locally compact. Notice that for each open neighborhood Q_1 of e in G there exists an open neighborhood Q_2 of e such that its closure $cl_G(Q_2)$ is compact and $cl_G(Q_2) \subset Q_1$ by the corresponding Theorem 3.3.2 in [10] for topological spaces. Hence for each open neighborhood W of e in G there exists an open neighborhood U_0 of e in G with the compact closure $cl_G\check{U}_0$ such that $cl_G\check{U}_0$ is contained in W (see Lemma 2.6).

Therefore for each $U \in \mathcal{B}_e$ there exists $V_1 \in \mathcal{B}_e$ such that $f(V_1, V_1, V_1) \subset U$ and $cl_G(V_1)$ is compact. If for an arbitrary fixed element $x \in G$ and some $x_1 \in G$ the intersection $xV_1 \cap x_1V_1 \neq \emptyset$ is non void, then there are h_0 and h_1 in V_1 such that $x_1 = (xh_0)/h_1$. On the other hand, $x_1h \in x_1V_1$ for each $h \in V_1$ and for each $y \in x_1V_1$ there exists $h \in V_1$ with $y = x_1h$, consequently, $x_1h = ((xh_0)/h_1)h \in ((xV_1)/V_1)V_1$.

Using Identities (8), (9) and Condition (ix) in Definition 2.1 we get that

$$x_1h = (x(h_0(e/h_1))p(x, h_0, e/h_1) \tag{43}$$

$p(e/h_1, h_1, h_1 \setminus e)[p((xh_0)/h_1, h_1, h_1 \setminus e)]^{-1})h$. We choose open neighborhoods V and W of e in G such that $\check{V}^2 \subset W$ and $\check{W}^2 \subset V_1$ by Lemma 2.6. In view of the inclusion (ii) of Proposition 2.9 and Formula (43) there exists $U_1 \in \mathcal{B}_e$ such that $\check{U}_1 \subset V$ and

$$p((u_1a)v_1, (u_2b)v_2, (u_3c)v_3) \in (Vp(a,b,c)) \cap (p(a,b,c)V) \tag{44}$$

for every a, b, c in $cl_G(V_1)$ and u_j, v_j in $\check{U}_1$ for each $j \in \{1,2,3\}$. This implies (42) and hence $(UC3)$, since $p(a,b,c) = e$ if either $a = e$ or $b = e$ or $c = e$.

It remains to prove that $\mathcal{C}_l$ also satisfies the condition $(UC4)$. That is for each $x \neq y$ in G there exists $\mathcal{A} \in \mathcal{C}_l$ such that $\{x,y\} \cap V \neq \{x,y\}$ for each $V \in \mathcal{A}$. It is sufficient to find an open neighborhood U of e in G such that $x/U \cap y/U = \emptyset$, because this implies $x_0U \cap \{x,y\} \neq \{x,y\}$ for each $x_0 \in G$. The quasigroup G is T_1. By virtue of Lemmas 2.5 and 2.6 and the joint continuity of the multiplication and Div_r in G there is $U_1 \in \mathcal{B}_e$ such that $y \notin (xU_1)/U_1$, that is $xU_1 \cap yU_1 = \emptyset$ by (14). In view of Proposition 2.9 there exists $U \in \mathcal{B}_e$ such that $(e/U)p(e/U, U, U \setminus e)[p(a/U, U, U \setminus e)]^{-1} \subset U_1$ for each $a \in \{x,y\}$, since the two-point set $\{x,y\}$ is compact in G, for each $W \in \mathcal{B}_e$ there exists $W_1 \in \mathcal{B}_e$ such that $e/W_1 \subset W$. From (8) it follows that $x/U \cap y/U = \emptyset$. Therefore $\{x,y\} \cap V \neq \{x,y\}$ for every $V \in \mathcal{C}_l(U)$.

By virtue of Theorem 8.1.1 in [10] the uniformity $\mathcal{C}_l$ induces a T_1 topology τ_1 on G. Note that the family $\mathcal{C}_l$ consists of open coverings of G and that for each $x \in G$ and each open neighborhood V of x in the initial topology τ there exists $U \in \mathcal{B}_e$ such that $xU \subset V$. Therefore from the latter inclusion and (42) it follows that the topology τ_1 induced by $\mathcal{C}_l$ coincides with the initial topology τ on G. In view of Corollary 8.1.13 in [10] (G,τ) is a Tychonoff space, that is (G,τ) is a completely regular space, $T_1 \cap T_{3.5}$. Finally note that $\mathcal{C}_l^0 = \mathcal{L}_G$. Symmetrically the case $\mathcal{C}_r^0 = \mathcal{R}_G$ is proved. ∎

Lemma 2.12. *Suppose that (G,τ) is a T_1 topological quasigroup, S is a compact subset in G, q is a fixed element in G, V is an open neighborhood of the unit element e. Then there are elements $b_1, ..., b_m$ in G and an open neighborhood U of e such that $\check{U} \subset V$ and $\{b_1 \setminus (qU), ..., b_m \setminus (qU)\}$ is an open covering of S.*

Proof. The multiplication is continuous on G, hence the left shift mapping $L_b(x) = bx$ is continuous on G in the variable x. On the other hand, the mapping Inv_l is continuous on G.

In view of (i), (ii) in Definition 2.1, Lemmas 2.5 and 2.6 and the compactness of S for each open neighborhood U of e in G with $\check{U} \subset V$ there are $b_1, ..., b_m$ in G such that $\{b_1 \setminus (qU), ..., b_m \setminus (qU)\}$ is an open covering of S. ∎

Corollary 2.13. *Let G be a T_1 topological quasigroup. Then for each open neighborhood W of e in G there exists an open neighborhood U of e such that $\check{U} \subset W$ and*

(i) $(\forall x\ \forall y\ ((x \in G)\&(y \in G)\&(x \setminus y \in U))) \Rightarrow (y \in xW)$ *and*
(ii) $(\forall x\ \forall y\ ((x \in G)\&(y \in G)\&(y/x \in U))) \Rightarrow (y \in Wx)$.

Proof. This follows from Lemmas 2.6 and 2.12, (i), (ii) in Definition 2.1. ∎

Theorem 2.14. *Let G and H be T_1 topological core quasigroups (see Definition 2.1) and let $f : G \to H$ be a continuous map so that for each open neighborhood V of a unit element e_H in H a compact subset K_V in G exists such that $f(G - K_V) \subset V$. Then f is uniformly $(\mathcal{L}_G, \mathcal{L}_H)$ continuous and uniformly $(\mathcal{R}_G, \mathcal{R}_H)$ continuous (see also Definition 2.10).*

Proof. Since the multiplication in H is continuous, for each open neighborhood Y of e_H there exists an open neighborhood X of e_H such that $X^2 \subset Y$. In view of Lemma 2.6 there exists an open neighborhood V_1 of e_H in H such that $\check{V}_1^2 \subset V$, where $A^2 = AA$ for a subset A in H. By the conditions of this theorem there exists a compact subset K_{V_1} in G such that $f(G - K_{V_1}) \subset V_1$.

For a subset A of the quasigroup G, let

$$P(A) = (P_0(A) \cup \{e\})(P_0(A) \cup \{e\}), \tag{45}$$

where $P_0(A) = A \cup Inv_l(A) \cup Inv_r(A)$, hence $A \subset P_0(A)$ and $P_0(A) \cup \{e\} \subset P(A)$. We have $S_1 = P(K_{V_1})$ is a compact subset in G, since the mappings Inv_l and Inv_r are continuous on G and the multiplication is jointly continuous on $G \times G$ (see Theorems 3.1.10, 8.3.13-8.3.15 in [10]), hence $R_1 = P(f(S_1))$ is compact in H.

By virtue of Proposition 2.9 there exists an open neighborhood V_2' of e_H in H such that

$$[t_H((V_2a)V_2, (V_2b)V_2, (V_2c)V_2)V_2] \cup [V_2t_H((V_2a)V_2, (V_2b)V_2, (V_2c)V_2)]$$
$$\subset (V_3t_H(a,b,c)) \cap (t_H(a,b,c)V_3),$$

$$\begin{aligned}&[p_H((V_2a)V_2,(V_2b)V_2,(V_2c)V_2)V_2]\cup[V_2p_H((V_2a)V_2,(V_2b)V_2,(V_2c)V_2)]\\ \subset\ &(V_3p_H(a,b,c))\cap(p_H(a,b,c)V_3)\end{aligned} \tag{46}$$

for every a, b, c in R_1, where $\check{V}_3^2 \subset V_1$, $V_2 = \check{V}_2{}'$, and V_3 is an open neighborhood of e in H. For V_2 there exists a compact subset K_{V_2} in G such that $f(G - K_{V_2}) \subset V_2$ by the conditions of this theorem. If A and B are compact subsets in G, then their union $A \cup B$ is also compact. Therefore it is possible to choose K_{V_2} such that $K_{V_1} \subset K_{V_2}$, since $V_2 \subset V_1$ and $(G - A) - B = G - (A \cup B) \subset G - A$. We take $S_2 = P(K_{V_2})$ by Formula (45), consequently, $S_1 \subset S_2$, since $K_{V_1} \subset K_{V_2}$.

From the continuity of the map f and Lemmas 2.5, 2.6 it follows that for each $x \in G$ open neighborhoods $W_{x,l}$ and $W_{x,r}$ of e in G exist such that $f(x\check{W}_{x,l}^2) \subset (f(x)V_2)$ and $f(\check{W}_{x,r}^2 x) \subset (V_2 f(x))$, consequently,

$$f(x\check{W}_x^2) \subset (f(x)V_2) \text{ and } f(\check{W}_x^2 x) \subset (V_2 f(x)) \tag{47}$$

for an open neighborhood $W_x = W_{x,l} \cap W_{x,r}$ of e in G. The compactness of S_2 implies that the coverings $\{xW_x :\ x \in S_2\}$ and $\{W_y y :\ y \in S_2\}$ of S_2 have finite subcoverings $\{x_j W_{x_j} :\ x_j \in S_2,\ j = 1, ..., n\}$ and $\{W_{y_i} y_i :\ y_i \in S_2,\ i = 1, ..., m\}$. Hence

$$W = \bigcap_{j=1}^{n} W_{x_j} \cap \bigcap_{i=1}^{m} W_{y_i} \tag{48}$$

is an open neighborhood of e in G. Therefore according to Proposition 2.9 there exists an open neighborhood U' of the unit element e in G such that

$$\begin{aligned}&[t_G((Ua)U,(Ub)U,(Uc)U)U]\cup[Ut_G((Ua)U,(Ub)U,(Uc)U)]\\ \subset\ &[W_3t_G(a,b,c)]\cap[t_G(a,b,c)W_3],\\ &[p_G((Ua)U,(Ub)U,(Uc)U)U]\cup[Up_G((Ua)U,(Ub)U,(Uc)U)]\\ \subset\ &[W_3p_G(a,b,c)]\cap[p_G(a,b,c)W_3]\end{aligned} \tag{49}$$

for every a, b, c in S_2, where $U = \check{U}'$, and where W_0 and W_3 are open neighborhoods of e in G such that $\check{W}_3^2 \subset W_0$ and $\check{W}_0^2 \subset W$.

Now let x and y in G be such that $x \setminus y \in U$. Then Formula (13) implies that

$$y \in xU. \tag{50}$$

There are several options. Consider at first the case $x \in K_{V_2}$. From Formulas (48)-(50) and Corollary (2.13) it follows that there exists $j \in \{1, ..., n\}$ such that $x \in x_j W_{x_j}$ and $y \in x_j W_{x_j}^2$. Therefore, Formulas (46) and (47) imply that $f(x) \setminus f(y) \in V$.

From $x \setminus y \in U$ and Identities (13) it follows that $y = xu$ for a unique $u \in U$. Hence

$$x = [t(y, e/u, u)]^{-1} y(e/u) \tag{51}$$

according to Identities (9), (14).

If $y \in K_{V_2}$, then similarly from Formulas (48)- (51) and Corollary (2.13) it follows that there exists $k \in \{1, ..., n\}$ such that $y \in x_k W_{x_k}$ and $x \in x_k W^2_{x_k}$, since $t(a,b,e) = t(a,e,b) = t(e,a,b) = e$ for each a and b in G. Therefore, $f(x) \setminus f(y) \in V$ by Formulas (46) and (47), since $S_2 = P(K_{V_2})$ (see Formula (45)).

It remains the case $x \in G - K_{V_2}$ and $y \in G - K_{V_2}$. Therefore $f(x) \in V_2$ and $f(y) \in V_2$. According to the choice of R_1 we have $e_H \in R_1$. From Condition (46), Identity (13) and the inclusion $\check{V}_1^2 \subset V$, it follows that $f(x) \setminus f(y) \in V$. Taking into account the inclusion $K_{V_1} \subset K_{V_2}$ we get that f is uniformly $(\mathcal{L}_G, \mathcal{L}_H)$ continuous.

The uniform $(\mathcal{R}_G, \mathcal{R}_H)$ continuity is proved analogously using the finite sub-covering $\{W_{y_i} y_i : \ y_i \in S_2, \ i = 1, ..., m\}$ and Corollary 2.13. ∎

Corollary 2.15. *Let G be a T_1 topological locally compact core quasigroup and let $f \in C_0(G)$ and let $H = (\mathbf{C}, +)$ be the complex field $\mathbf{C}$ considered as an additive group. Then f is uniformly $(\mathcal{L}_G, \mathcal{L}_H)$ continuous and uniformly $(\mathcal{R}_G, \mathcal{R}_H)$ continuous.*

3. Left Invariant Measures

Notation 3.1. For a completely regular topological space X by $C_b(X)$ is denoted the Banach space of all continuous bounded functions f from X into the complex field $\mathbf{C}$ supplied with the norm

$$\|f\|_X = \sup_{x \in X} |f(x)| < \infty. \tag{52}$$

We put

$$\begin{aligned} C_0(X) &:= \{f \in C_b(X) : \ \forall \ \epsilon > 0, \ \exists \ S \subset X, \ S \text{ is compact}, \\ &\qquad \forall \ x \in X - S, \ |f(x)| < \epsilon\}, && (53) \\ C_{0,0}(X) &:= \{f \in C_b(X) : \ \exists S \subset X, \ S \text{ is compact}, \\ &\qquad \forall x \in X - S, \ f(x) = 0\}, && (54) \\ C^+_{0,0}(X) &= \{f \in C_{0,0}(X) : \ \forall x \in X, \ f(x) \geq 0\}. && (55) \end{aligned}$$

Let G be a quasigroup. For a function $f : G \to \mathbf{C}$ and an element $b \in G$ let $L_b f(x) = {}_b f(x) = f(bx)$ and $R_b f(x) = f_b(x) = f(xb)$ for each $x \in G$. Consider a support $S_f := cl_G\{x \in G : \ f(x) \neq 0\}$ of $f \in C_b(G)$, where $cl_G(A)$ denotes the closure of a subset A in G.

Lemma 3.2. *Let (G, τ) be a T_1 topological locally compact core quasigroup. Let also f and ϕ belong to $C^+_{0,0}(G)$ and ϕ be not identically zero (see Notation 3.1, Formulas (52)-(55)). Then there exist a natural number $m > 0$, elements*

$b_1,...,b_m$ in G and positive constants $c_1,...,c_m$ such that

$$\forall x\in G,\ f(x)\le\sum_{j=1}^m c_jL_{b_j}\phi(x). \tag{56}$$

Proof. Since $f\in C^+_{0,0}(G)$, the support S_f is compact. The function ϕ is not null, hence there exists $q\in G$ such that $\phi(q)>0$. From Lemma 2.5 and from the continuity of the function ϕ it follows that there exists an open neighborhood qV of q such that $\phi(x)>\phi(q)/2$ for reach $x\in qV$, where V is an open neighborhood of the unit element e. By virtue of Lemma 2.12 there exists an open neighborhood U of e and elements $b_1,...,b_m$ in G such that $\check{U}\subset V$ and for each $x\in S_f$ there exists $j\in\{1,...,m\}$ such that $x\in b_j\setminus(qU)$.

Therefore,

$$f(x)\le\|f\|_G(2/\phi(q))\sum_{j=1}^m\phi(b_jx)$$

for each $x\in G$ according to (13), so it is sufficient to take $c_j\ge\|f\|_G(2/\phi(q))$ for each $j=1,...,m$. This implies Inequality (56). ∎

Corollary 3.3. *Let the conditions of Lemma 3.2 be satisfied and let*

$$\begin{aligned}(f:\phi):=\inf\Big\{\sum_{j=1}^m c_j:\ \exists\,\{b_1,...,b_m\}\subset G,\ \exists\,\{c_1,...,c_m\}\subset(0,\infty),\\ \forall\ x\in G,\ f(x)\le\sum_{j=1}^m c_jL_{b_j}\phi(x)\Big\}.\end{aligned} \tag{57}$$

Then $(f:\phi)\le 2m\|f\|_G/\phi(q)$ in the notation of Lemma 3.2.

Lemma 3.4. *Assume that the conditions of Lemma 3.2 are fulfilled. Then for each $b\in G$*

$$({}_bf:\phi)\ =\ (f:\phi^b), \tag{58}$$
$$(f:{}_b\phi)\ =\ (f^b:\phi), \tag{59}$$

where $f^b(x)=f(b\setminus x)$ for each $x\in G$; particularly,

$$({}_\gamma f:\phi)\ =\ (f:\phi), \tag{60}$$
$$(f:{}_\gamma\phi)\ =\ (f:\phi)\ \textit{for each}\ \gamma\in N(G), \tag{61}$$
$$(\alpha f:\phi)\ =\ \alpha(f:\phi)\ \textit{for each}\ \alpha\ge 0, \tag{62}$$
$$((f_1+f_2):\phi)\ \le\ (f_1:\phi)+(f_2:\phi)\ \textit{for every}\ f_1\ \textit{and}\ f_2\ \textit{in}\ C^+_{0,0}(G). \tag{63}$$

If $f(x)\le f_1(x)$ for each $x\in G$, then

$$(f:\phi)\le(f_1:\phi). \tag{64}$$

Proof. Let $c_1, ..., c_m$ in $(0, \infty)$ and $b_1, ..., b_m$ in G be such that

$$ {}_b f(x) \le \sum_{j=1}^{m} c_j L_{b_j} \phi(x) \tag{65} $$

for each $x \in G$. From Formulas (13) and (65) by changing of a variable $y = bx$ it follows that

$$ f(y) \le \sum_{j=1}^{m} c_j L_{b_j} \phi(b \setminus y) \tag{66} $$

for each $y \in G$. From (66) it follows (58). Similarly from the inequality

$$ f(x) \le \sum_{j=1}^{m} c_j L_{b_j} (L_b \phi(x)) \tag{67} $$

for each $x \in G$ we infer that

$$ f(b \setminus y) \le \sum_{j=1}^{m} c_j L_{b_j} \phi(y) \tag{68} $$

for each $y \in G$. Thus (68) implies Equality (59).

In particular, if $\gamma \in N(G)$, then $b_j(\gamma \setminus y) = (b_j \gamma^{-1}) y$ and $b_j(\gamma y) = (b_j \gamma) y$ for each y and b_j in G by Condition (viii) and Formulas (3), (4) and (15). Hence (66) transforms into to

$$ f(y) \le \sum_{j=1}^{m} c_j L_{b_j \gamma^{-1}} \phi(y) $$

and (67) into

$$ f(x) \le \sum_{j=1}^{m} c_j L_{b_j \gamma} \phi(x) $$

with $\gamma \in N(G)$ instead of b. This implies Equalities (60), (61).

Properties (62) and (63) evidently follow from Formula (57).

For proving Property (64) note that if $f(x) \le f_1(x)$ for each $x \in G$, then from $f_1(x) \le \sum_{j=1}^{m} c_j L_{b_j} \phi(x)$ for each $x \in G$ it follows that $f(x) \le \sum_{j=1}^{m} c_j L_{b_j} \phi(x)$ for each $x \in G$, consequently, $(f : \phi) \le (f_1 : \phi)$. ∎

Notation 3.5. Let ϕ, f_0 and f belong to $C^+_{0,0}(G)$ and ϕ and f_0 be not null, where G is a T_1 topological locally compact core quasigroup. We consider a functional

$$ J_{\phi, f_0}(f) := \frac{(f : \phi)}{(f_0 : \phi)}. \tag{69} $$

Assume that there exists a compact subgroup $N_0 = N_0(G)$ in $N(G)$ such that

$$t(a,b,c) \in N_0 \text{ and } p(a,b,c) \in N_0 \tag{70}$$

for every a, b and c in G.

Then we denote by $\Upsilon(G, N_0)$ the family of all non null functions h in $C^+_{0,0}(G)$ such that

$$h(\gamma a) = h(a) \tag{71}$$

for each $a \in G$ and $\gamma \in N_0$.

Evidently, for $h \in C^+_{0,0}(G)$, Condition (71) is equivalent to

$$h(a\gamma) = h(a) \tag{72}$$

for each $a \in G$ and $\gamma \in N_0$, since $aN_0 = N_0 a$ for each $a \in G$ according to Theorem 2.8.

Lemma 3.6. *Let G be a T_1 topological locally compact core quasigroup satisfying Condition* (70)*, f and ϕ be in $C^+_{0,0}(G)$ and $\omega \in \Upsilon(G, N_0)$ (see Condition* (71)*), ϕ be non null. Then*

$$(f : \phi) \leq (f : \omega)(\omega : \phi). \tag{73}$$

Proof. If b is a fixed element in G and there are elements $b_1, ..., b_m$ in G and positive constants $c_1, ..., c_m$ such that

$$_b\omega(x) \leq \sum_{j=1}^{m} c_j \phi(b_j x) \tag{74}$$

for each $x \in G$, then

$$_b\omega(x) \leq \sum_{j=1}^{m} c_j \phi(b_j x\gamma) \tag{75}$$

for each $x \in G$ and $\gamma \in N_0$, since $N_0 \subset N(G)$ and $_b\omega(x\gamma) = {}_b\omega(x)$ for each $x \in G$ and $\gamma \in N_0$ by (72) equivalent to (71).

By the conditions of this lemma N_0 is a compact group. Therefore there exists a Haar measure λ on the Borel σ-algebra $\mathcal{B}(N_0)$ of N_0 and with values in the unit segment $[0,1]$ such that $\lambda(N_0) = 1$, $\lambda(sA) = \lambda(A)$ and $\lambda(As) = \lambda(A)$ for each $s \in N_0$ and $A \in \mathcal{B}(N_0)$ (see Theorems 15.5, 15.9 and 15.13 and Subsection 15.8 in [17]). In view of this, Conditions (54) and (55) and Corollary 2.15 the function

$$\phi^{[\lambda]}(x) := \int_{N_0} \phi(\gamma x)\lambda(d\gamma) \tag{76}$$

on G is nonzero and belongs to $C^+_{0,0}(G)$, since N_0S_ϕ is a compact subset in G by Lemma 2.6, where S_ϕ is a compact support of ϕ. From Formula (76) it follows that

$$\phi^{[\lambda]}(\beta x) = \phi^{[\lambda]}(x) \tag{77}$$

for each $\beta \in N_0$ and $x \in G$, since the measure λ is left and right invariant $\lambda(\beta A) = \lambda(A) = \lambda(A\beta)$ for each $\beta \in N_0$ and each Borel subset A in N_0. Hence $\phi^{[\lambda]} \in \Upsilon(G, N_0)$, since $S_\phi N_0$ is compact, and since Conditions (71) and (72) are equivalent, where S_ϕ is the support of ϕ (see Subsection 3.2). From (76), (77), (71), (72) and the Fubini theorem it follows that

$$\phi^{[\lambda]}(x) = \int_{N_0} \phi(x\beta)\lambda(d\beta), \tag{78}$$

since

$$\begin{aligned} \phi^{[\lambda]}(x) &= \int_{N_0} (\int_{N_0} \phi(\gamma x\beta)\lambda(d\gamma))\lambda(d\beta) \\ &= \int_{N_0} (\int_{N_0} \phi(\gamma x\beta)\lambda(d\beta))\lambda(d\gamma) = \int_{N_0} \phi(x\beta)\lambda(d\beta)), \end{aligned}$$

because $\int_{N_0} \phi(x\gamma\beta)\lambda(d\beta) = \int_{N_0} \phi(x\beta)\lambda(d\beta)$ for each $\gamma \in N_0(G)$.

Integrating both sides of Inequality (75) and utilizing Formulas (76), (78) we infer that

$${}_b\omega(x) \le \sum_{j=1}^m c_j \phi^{[\lambda]}(b_j x) \tag{79}$$

for each $x \in G$. On the other hand,

$$\int_{N_0} (\sum_{j=1}^m c_j \ {}_{b_j}\phi)(x\gamma)\lambda(d\gamma) = (\sum_{j=1}^m c_j \ {}_{b_j}\phi)^{[\lambda]}(x) = \sum_{j=1}^m c_j \ {}_{b_j}\phi^{[\lambda]}(x),$$

hence for each $x \in G$ there exists $\gamma \in N_0$ such that

$$(\sum_{j=1}^m c_j \ {}_{b_j}\phi)(x\gamma) \ge \sum_{j=1}^m c_j \ {}_{b_j}\phi^{[\lambda]}(x).$$

Thus vice versa from $\omega \in \Upsilon(G, N_0)$ and (79) it follows (75) and hence (74), consequently,

$$({}_b\omega : \phi^{[\lambda]}) = ({}_b\omega : \phi). \tag{80}$$

Let $a_1, ..., a_n$ in G and positive constants $q_1, ..., q_n$ be such that

$${}_b\omega(x) \le \sum_{j=1}^n q_j \phi^{[\lambda]}(a_j x) \tag{81}$$

for each $x \in G$ (see Lemma 3.2). From Formulas (77), (81) and Conditions (70), (71), (72) we deduce that

$$\omega(y) \leq \sum_{j=1}^{n} q_j \phi^{[\lambda]}((a_j(b \setminus e))y[p(a_j, b \setminus e, y)]^{-1} p(b, b \setminus e, y)) = \sum_{j=1}^{n} q_j \phi^{[\lambda]}(d_j y) \tag{82}$$

for each $y \in G$, where $d_j = a_j(b \setminus e)$ for each j. Therefore $({}_b\omega : \phi^{[\lambda]}) \leq (\omega : \phi^{[\lambda]})$ for each $b \in G$. Notice that

$$L_c L_{c \setminus e} \omega(x) = \omega(x) \tag{83}$$

for each c and x in G by Lemmas 2.2, 2.3 and Condition (71). Therefore we analogously get $(\omega : \phi^{[\lambda]}) \leq ({}_c\omega : \phi^{[\lambda]})$ for each $c \in G$. Thus

$$({}_b\omega : \phi^{[\lambda]}) = (\omega : \phi^{[\lambda]}) \tag{84}$$

for each $b \in G$.

From (80)-(84), it follows that

$$({}_b\omega : \phi) = (\omega : \phi) \tag{85}$$

for each $b \in G$.

If $c_1, ..., c_n$, $h_1, ..., h_k$ in $(0, \infty)$ and $a_1, .., a_k$, $g_1, ..., g_n$ in G are such that

$$f(x) \leq \sum_{j=1}^{k} h_j L_{a_j} \omega(x), \tag{86}$$

$$\omega(x) \leq \sum_{i=1}^{n} c_i L_{g_i} \phi(x) \tag{87}$$

for each $x \in G$ (see Lemma 3.2). Then from (71), (80), (85)-(87) and Lemma 2.2 we infer that

$$f(x) \leq \sum_{j=1}^{k} h_j \sum_{i=1}^{n} c_i L_{g_i} L_{a_j} \phi(x) = \sum_{j=1}^{k} h_j \sum_{i=1}^{n} c_i \phi((g_i a_j)x). \tag{88}$$

Apparently (88) implies (73). ∎

Lemma 3.7. *Let G be a T_1 topological locally compact core quasigroup, and let ϕ, f_0 be nonzero functions belonging to $C^+_{0,0}(G)$. Then for all functions f, f_1 in $C^+_{0,0}(G)$ and $\alpha \geq 0$*

$$J_{\phi, f_0}(\alpha f) = \alpha J_{\phi, f_0}(f), \tag{89}$$

$$J_{\phi,f_0}(f+f_1) \leq J_{\phi,f_0}(f) + J_{\phi,f_0}(f_1). \tag{90}$$

If $f(x) \leq f_1(x)$ *for each* $x \in G$, *then*

$$J_{\phi,f_0}(f) \leq J_{\phi,f_0}(f_1). \tag{91}$$

Moreover, if G *satisfies Condition* (70) *and* $f_0 \in \Upsilon(G, N_0)$ *(see Condition* (71)*), then*

$$J_{\phi,f_0}(f) \leq (f : f_0). \tag{92}$$

Proof. Properties (89) and (90) follow immediately from (62) and (63). Property (91) follows from Property (64).

Applying Inequality (73) and Formula (69) we infer Inequality (92), since $J_{\phi,f_0}(f_0) = 1$. ■

Lemma 3.8. *Assume that* G *is a* T_1 *topological locally compact core quasigroup, and suppose that functions* ϕ, f_0 *and* f *belong to* $C^+_{0,0}(G)$ *and that* ϕ *and* f_0 *are not null. Then mappings* $J_{\phi,f_0}({}_bf)$ *and* $J_{\phi,f_0}(f_b)$ *are continuous in the variable* b *in* G.

Proof. For each x, b_1 and b_2 in G we have ${}_{b_1}f(x) - {}_{b_2}f(x) = f(b_1x) - f(b_2x)$. In view of Corollary 2.15 for each $\epsilon > 0$ there exists an open of the form (a) in Lemma 2.6 neighborhood U of e in G with a compact closure $cl_G(U)$ for which

$$|f(b_1x) - f(b_2x)| < \epsilon \tag{93}$$

for each x, b_1 and b_2 in G such that $(b_2x) \setminus (b_1x) \in U$.

On the other hand, the support S_f of f is compact, consequently, $bS_f = L_bS_f$ is compact for each $b \in G$. Let b_1 be fixed. For each $x \in G$ with $b_1x \in S_f$ there exists an open neighborhood W_x of e in G of the form (a) in Lemma 2.6 such that $(b_2x) \setminus (b_1x) \in U$ for each $b_2x \in (b_1W_x)x \cap b_1(xW_x)$ according to Lemmas 2.2, 2.4, 2.5, Proposition 2.9 and Formula (47). For an open covering $\{(b_1W_x)x \cap b_1(xW_x) : \ b_1x \in S_f, \ x \in G\}$ of S_f there exists a finite subcovering $\{(b_1W_{x_j})x_j \cap b_1(x_jW_{x_j}) : \ b_1x_j \in S_f, \ x_j \in G, \ j = 1, ..., m\}$ (see also Lemma 2.5), since the subset S_f is compact.

We take $W_0 = U \cap \bigcap_{j=1}^m W_{x_j}$ and choose an open neighborhood W of e in G of the form (a) in Lemma 2.6 with compact closure $cl_G(W)$ contained in W_0 (see Theorem 3.3.2 in [10] and Formula (47)), because G is locally compact.

In view of Proposition 2.9 and Lemma 2.6 there exists an open neighborhood V' of e in G with $V = \check{V}'$ and compact closure $cl_G(V)$ such that

$$\begin{aligned} & [t((Va)V, (Vb)V, (Vc)V)V] \cup [Vt((Va)V, (Vb)V, (Vc)V)] \\ \subset \ & [t(a, b, c)W_1] \cap [W_1t(a, b, c)], \\ & [p((Va)V, (Vb)V, (Vc)V)V] \cup [Vp((Va)V, (Vb)V, (Vc)V)] \end{aligned}$$

$$\subset [p(a,b,c)W_1] \cap [W_1 p(a,b,c)] \tag{94}$$

for each a, b and c in S, where $\check{W}_1^2 \subset W$, W_1 is an open neighborhood of e in G, $S = P(S_1)$, $S_1 = S_2 \cup cl_G(U)$, where $b_1 \in G$ is as above, $S_2 = \{y \in G : y = (b_1 u)x, u \in cl_G(U), x \in G, b_1 x \in S_f\}$ (see Formula (45)), since S is compact, $t(a,b,c) = e$ and $p(a,b,c) = e$ if $e \in \{a,b,c\}$. For $b_1 x \notin S_f$ and $b_2 x \notin S_f$ certainly $f(b_1 x) - f(b_2 x) = 0$. So remain two cases either $b_1 x \in S_f$ or $b_2 x \in S_f$ which are similar to each other up to a notation. From Formulas (14) it follows that $b_2 x \in (b_1 V)x$ is equivalent to $b_2 \in b_1 V$. Hence Lemma 2.2 and Inclusion (94) provide that $(b_2 x) \setminus (b_1 x) \in U$ for each $b_2 \in b_1 V$ and $b_1 x \in S_f$.

Let $w \in C^+_{0,0}(G)$ be a function such that $w(y) = 1$ for each $y \in (cl_G(U)S_f)cl_G(U)$. Using (93), we deduce that $|f(b_1 x) - f(b_2 x)| < \epsilon w(x)$ for each x, b_1 and b_2 in G such that $b_2 \in b_1 V$ and with $b_1 x \in S_f$.

Therefore for each $\epsilon > 0$ there exists an open neighborhood V of e in G such that $|({}_{b_1}f : \phi) - ({}_{b_2}f : \phi)| < \epsilon (w : \phi)$ for each $b_2 \in b_1 V$, consequently,

$$|J_{\phi,f_0}({}_{b_1}f) - J_{\phi,f_0}({}_{b_2}f)| < \epsilon J_{\phi,f_0}(w) \tag{95}$$

according to Formula (69), since $(f_0 : \phi) > 0$. Thus the mapping $J_{\phi,f_0}({}_b f)$ is continuous in the parameter $b \in G$, since $0 < J_{\phi,f_0}(w) < \infty$ (see Lemmas 3.2, 3.7 and Corollary 3.3).

The case $J_{\phi,f_0}(f_b)$ is proved symmetrically. ∎

Theorem 3.9. *Assume that G is a T_1 topological locally compact core quasigroup satisfying Condition* (70)*, ϕ, f and f_1 are nonzero functions belonging to $C^+_{0,0}(G)$ and $f_0 \in \Upsilon(G, N_0)$ (see* (71)*). Then the following inequalities are true:*

$$(f_0 : f)^{-1} \le J_{\phi,f_0}(f) \le (f : f_0), \tag{96}$$

$$(f_1 : f_0)^{-1}(f_0 : f)^{-1} \le J_{\phi,f_1}(f) \le (f : f_0)(f_0 : f_1). \tag{97}$$

Proof. The right inequality in (96) follows from the inequality (92).

Formulas (80) and (85) imply that

$$({}_b f_0 : f) = (f_0 : f^{[\lambda]}) \text{ and } ({}_b f^{[\lambda]} : \phi) = (f^{[\lambda]} : \phi^{[\lambda]}) \tag{98}$$

for each $b \in G$.

Let $c_1, ..., c_k$, $h_1, ..., h_n$ in $(0, \infty)$ and $a_1, .., a_k$, $g_1, ..., g_n$ in G be such that

$$f_0(x) \le \sum_{j=1}^{k} c_j f^{[\lambda]}(a_j x) \text{ and} \tag{99}$$

$$f^{[\lambda]}(x) \le \sum_{i=1}^{n} h_i \phi^{[\lambda]}(g_i x) \tag{100}$$

for each $x \in G$ (see Lemma 3.2). Then from Identity (ix) in Definition 2.1, Inequalities (99), (100) and Conditions (71), (72) we deduce that

$$f_0(x) \leq \sum_{j=1}^{k} c_j \sum_{i=1}^{n} h_i \phi^{[\lambda]}((g_i a_j) x [p(g_i, a_j, x)]^{-1})$$
$$= \sum_{j=1}^{k} c_j \sum_{i=1}^{n} h_i \phi^{[\lambda]}((g_i a_j) x). \tag{101}$$

Suppose that there are $y_1, ..., y_k \in G$ and $q_1, ..., q_k \in (0, \infty)$ such that

$$f(x) \leq \sum_{i=1}^{k} q_i \phi(y_i x) \tag{102}$$

for each $x \in G$. Taking the integral $\int_{N_0} f(x\gamma)\lambda(d\gamma)$ and similarly for the right side (see Formulas (76) and (78)), we get from Inequality (102) that

$$f^{[\lambda]}(x) \leq \sum_{i=1}^{k} q_i \phi^{[\lambda]}(y_i x)$$

for each $x \in G$ (see Lemma 3.2). Hence

$$(f^{[\lambda]} : \phi^{[\lambda]}) \leq (f : \phi). \tag{103}$$

Utilizing Formulas (73), (98), (101) and (103) we infer that

$$(f_0 : \phi) \leq (f_0 : f)(f^{[\lambda]} : \phi^{[\lambda]}) \leq (f_0 : f)(f : \phi) \tag{104}$$

for each $f_0 \in \Upsilon(G, N_0)$ and nonzero functions f and ϕ in $C^+_{0,0}(G)$.

Using (69) and (104) we infer that

$$(f_0 : f) J_{\phi, f_0}(f) = \frac{(f_0 : f)(f : \phi)}{(f_0 : \phi)} \geq \frac{(f_0 : \phi)}{(f_0 : \phi)} = 1,$$

consequently, $J_{\phi, f_0}(f) \geq (f_0 : f)^{-1}$. Thus the left inequality in (96) is also proved.

From Inequalities (96) for $J_{\phi, f_0}(f)$ and $J_{\phi, f_0}(f_1)$ and Formula (69) it follows (97). ∎

Lemma 3.10. *Let G be a T_1 topological locally compact core quasigroup satisfying Condition* (70)*, let $f_0 \in \Upsilon(G, N_0)$ (see Condition* (71)*) and let f_1,...,f_m be nonzero functions belonging to $C^+_{0,0}(G)$, let also $0 < \delta < \infty$, $0 < \delta_1 < \infty$. Then there exists an open neighborhood V of e in G such that for each nonzero function ϕ in $C^+_{0,0}(G)$ with a support S_ϕ contained in V and $0 \leq q_j \leq \delta_1$ for each $j = 1, ..., m$ the following inequality is satisfied:*

$$\sum_{j=1}^{m} q_j J_{\phi, f_0}(f_j) \leq J_{\phi, f_0}(\sum_{j=1}^{m} q_j f_j) + \delta. \tag{105}$$

Proof. The quasigroup G is locally compact. Let $S_{f_0,...,f_m} = \bigcup_{j=0}^m S_{f_j}$ be a common compact support of these functions, where S_{f_j} denotes a closed support of f_j (see also Subsection 3.1). We choose any function g_1 in $C^+_{0,0}(G)$ such that $g_1 : G \to [0,1]$ and $g_1(S_{f_0,...,f_m} cl_G(Y_1)) = \{1\}$, where Y'_1 is an open neighborhood of e in G with $Y_1 = \check{Y'}_1$ and a compact closure $cl_G(Y_1)$ (see Lemma 2.6). Consider arbitrary fixed positive numbers $0 < \delta < \infty$, $0 < \delta_1 < \infty$ and $0 < \epsilon < M$ such that $\epsilon\delta_1 \sum_{j=1}^m (f_j : f_0) + \epsilon(1+\epsilon)(g_1 : f_0) \le \delta$, where $M = \delta_1 m \max_{j=1,...m} \|f_j\|_G$. By virtue of Corollary 2.15 the functions $f_0, ..., f_m$ are uniformly $(\mathcal{L}_G, \mathcal{L}_H)$ continuous, where $H = (\mathbf{C}, +)$. Therefore there exists an open neighborhood W' of e with $W = \check{W}'$ and with compact closure $cl_G(W)$ in G and $W \subset Y_1$, since G is locally compact, such that

$$|f_j(s) - f_j(x)| < \epsilon^3[4Mm\delta_1]^{-1} \tag{106}$$

for each $s \setminus x \in W$. Next we take a function $g \in C^+_{0,0}(G)$ such that $g : G \to [0,1]$ and $g(S_{f_0,...,f_m} cl_G(W)) = \{1\}$ and $g(x) \le g_1(x)$ for each $x \in G$, because $W \subset Y_1$. Hence $(g : f_0) \le (g_1 : f_0)$ by Inequality (64).

Let $S = P((S_{f_0,...,f_m} \cup S_g) cl_G(W))$ (see Formula (45)). Since $cl_G(V)$, $S_{f_0,...,f_m}$ and S_g are compact, S is a compact subset in G. For each open neighborhood Y of e in G there exists an open neighborhood X of e in G such that $X^2 \subset Y$, since the multiplication in G is continuous. In view of Proposition 2.9 and Corollary 2.15 there exist open neighborhoods U'_k of e in G such that $U_k = \check{U'}_k$ and such that

$$\begin{aligned} &[t((U_k a)U_k, (U_k b)U_k, (U_k c)U_k)U_k] \cup [U_k t((U_k a)U_k, (U_k b)U_k, (U_k c)U_k)] \\ \subset\ &[t(a,b,c)W_{k-1}] \cap [W_{k-1} t(a,b,c)], \\ &[p((U_k a)U_k, (U_k b)U_k, (U_k c)U_k)U_k] \cup [U_k p((U_k a)U_k, (U_k b)U_k, (U_k c)U_k)] \\ \subset\ &[p(a,b,c)W_{k-1}] \cap [W_{k-1} p(a,b,c)] \end{aligned} \tag{107}$$

for every a, b, c in S and $k \in \{1,2\}$ with $U_0 = W$ and an open neighborhood W_{k-1} of e in G of the form (a) in Lemma 2.6 such that $\check{W}^2_{k-1} \subset U_{k-1}$ and

$$|g(s) - g(x)| < \epsilon^2[4M]^{-1} \tag{108}$$

for each s and x in G such that $s \setminus x \in U_1$, where $t = t_G$.

Take any $0 \le q_j \le \delta_1$ for each $j = 1, ..., m$ and put

$$\Psi = \epsilon g + \sum_{j=1}^m q_j f_j, \tag{109}$$

$$h_j(x) = q_j f_j(x)[\Psi(x)]^{-1} \tag{110}$$

for each $x \in S_{f_1,...,f_m}$ and $h_j(x) = 0$ for each $x \in G - S_{f_1,...,f_m}$, where $S_{f_1,...,f_m} = \bigcup_{j=1}^m S_{f_j}$. Therefore the function Ψ belongs to $C^+_{0,0}(G)$ and $\sum_{j=1}^m h_j(x) \le 1$ for each $x \in G$.

From Inequalities (106) and (108) it follows that

$$|\Psi(s) - \Psi(x)| \le \epsilon^3[2M]^{-1} \tag{111}$$

for each s and x in G such that $s \setminus x \in U_1$. Moreover, $\|\Psi\|_G \le M + \epsilon < 2M$.

Let s and x belong to $S_{f_1,...,f_m} cl_G(W)$ and $s \setminus x \in U_1$. The latter inclusion is equivalent to $x \in sU_1$ and also to $s \in x/U_1$. Then from (106), (110) and (111) we deduce that

$$|h_j(s) - h_j(x)| \le \epsilon/m. \tag{112}$$

Next we consider the following case: $s \setminus x \in U_1$ and $x \notin S_{f_1,...,f_m} cl_G(W)$. Suppose that $s \in S_{f_1,...,f_m}$. Then Condition (107), Lemmas 2.2, 2.3 imply that $x \in S_{f_1,...,f_m} cl_G(W)$ contradicting the assumption $x \notin S_{f_1,...,f_m} cl_G(W)$. Hence $s \notin S_{f_1,...,f_m}$ and consequently, $h_j(s) = 0$ and $h_j(x) = 0$. Thus Inequality (112) is true in this case as well.

In the case $s \setminus x \in U_1$ and $s \notin S_{f_1,...,f_m} cl_G(W)$ Condition (107), Lemmas 2.2, 2.3 imply that $x \notin S_{f_1,...,f_m}$. Therefore the inequality (112) is fulfilled in this case too. Thus the estimate (112) is satisfied for each s and x in G such that $s \setminus x \in U_1$.

Next we choose any fixed function $\phi \in C^+_{0,0}(G)$ such that ϕ is not identically zero on G and $\phi(y) = 0$ for each $y \in G - {U'}_2$. By virtue of Lemma 3.2 there are $m \in \mathbf{N}$, $c_j > 0$ and $b_j \in G$ for each $j \in \{1, ..., m\}$ such that

$$\Psi(x) \le \sum_{j=1}^m c_j \phi(b_j x) \tag{113}$$

for every $x \in G$ and

$$-\epsilon + \sum_{j=1}^m c_j \le (\Psi : \ \phi) \le \sum_{j=1}^m c_j. \tag{114}$$

Then Formulas (107), (112), (113) and Lemma 2.2 imply that for each $x \in G$

$$\Psi(x) h_l(x) \le \sum_{j=1}^m c_j \phi(b_j x)[h_l(b_j \setminus e) + \epsilon/m]$$

for each l. Hence for each $x \in G$ we get

$$q_l f_l(x) = \Psi(x) h_l(x) \le \sum_{j=1}^m c_j [h_l(b_j \setminus e) + \epsilon/m] \phi(b_j x)$$

and consequently, $(q_l f_l : \ \phi) \le \sum_{j=1}^m c_j [h_l(b_j \setminus e) + \epsilon/m]$. From $\sum_{l=1}^m h_l \le 1$ we deduce that $\sum_{l=1}^m (q_l f_l : \ \phi) \le (1 + \epsilon) \sum_{j=1}^m c_j$. Together with Inequalities (114) this leads to the following estimate:

$$\sum_{j=1}^m (q_i f_j : \ \phi) \le (1 + \epsilon)(\Psi : \ \phi).$$

Dividing both of sides by $(f_0 : \ \phi)$ we get the inequality

$$\sum_{j=1}^{m} q_j J_{\phi,f_0}(f_j) \le (1+\epsilon) J_{\phi,f_0}(\Psi). \tag{115}$$

Then from (89), (90), (109) and (115) we infer that

$$\sum_{j=1}^{m} q_j J_{\phi,f_0}(f_j) \le J_{\phi,f_0}(\sum_{j=1}^{m} q_j f_j) + \epsilon \sum_{j=1}^{m} q_j J_{\phi,f_0}(f_j) + \epsilon(1+\epsilon) J_{\phi,f_0}(g). \tag{116}$$

Therefore from Inequalities (96), (116), (64) and for ϵ as above it follows that

$$\begin{aligned} \sum_{j=1}^{m} q_j J_{\phi,f_0}(f_j) &\le J_{\phi,f_0}(\sum_{j=1}^{m} q_j f_j) + \epsilon\delta_1 \sum_{j=1}^{m} (f_j : \ f_0) + \epsilon(1+\epsilon)(g : \ f_0) \\ &\le J_{\phi,f_0}(\sum_{j=1}^{m} q_j f_j) + \delta. \end{aligned}$$

This implies the estimate (105) with $V = {U_2}'$. ∎

Theorem 3.11. *Let G be a T_1 topological locally compact core quasigroup, $0 < \epsilon$ and f in $C^+_{0,0}(G)$ be a nonzero function, $S_f = cl_G\{x \in G : \ f(x) \ne 0\}$. Let also V' be an open neighborhood of e in G and let*

$$|f(x) - f(y)| < \epsilon \tag{117}$$

for each x and y in G with $x \setminus y \in V$, where $V = \check{V}'$. Let $g \in C^+_{0,0}(G)$ be a nonzero function such that $g(x) = 0$ for each $x \in G - V'$. Then for each $\delta > \epsilon$ and each open neighborhood ${W_e}'$ of e in G with $W_e = \check{W_e}'$ and a compact closure $cl_G(W_e)$ contained in V there is an open neighborhood U' of e in G such that $U = \check{U}'$ and for each nonzero function ϕ in $C^+_{0,0}(G)$ with a support S_ϕ contained in U' there are positive constants $c_1, ..., c_n$ and elements $b_1, ..., b_n$ in $S_f cl_G(W_e)$ such that for each $x \in G$ and $\gamma \in N(G)$:

$$|f(\gamma x) - \sum_{j=1}^{n} \frac{c_j}{J^v_{\phi,f_0}(g(v \setminus x))} g(b_j \setminus \gamma x)| \le \delta, \tag{118}$$

where an expression $J^v_{\phi,f_0}(g(v \setminus x))$ means that a functional J_{ϕ,f_0} is taken in the v variable.

Proof. The continuous functions f and g are with compact supports, hence they are uniformly $(\mathcal{L}_G, \mathcal{L}_H)$ continuous and uniformly $(\mathcal{R}_G, \mathcal{R}_H)$ continuous on G by Corollary 2.15, where $H = (\mathbf{C}, +)$. For each $y \in G$ the right translation operator R_y is the homeomorphism of G as the topological space onto itself (see also Section 2). Therefore the function $\nu(y) := (f(x) : \ g(x \setminus y))$ is continuous on the quasigroup G and consequently, uniformly continuous on the compact subset S_f, hence $\sup_{y\in S_f} \nu(y) < \infty$, where $(f(x) : \ g(x \setminus y)) = (f : z)$ is calculated in the x variable with $z(x) = g(x \setminus y)$ for a fixed parameter y. We take any fixed δ such that $\epsilon < \delta < \infty$. Evidently there exists $0 < \eta$ such that

$$\eta \sup_{y\in S_f} \nu(y) < \delta - \epsilon. \tag{119}$$

Therefore take any fixed open neighborhood ${W_e}'$ of e in G such that $W_e = \check{W_e}'$ and $cl_G(W_e)$ is compact and $cl_G(W_e) \subset V$ (see Lemma 2.6). By virtue of Corollary 2.15 the functions g and h are uniformly $(\mathcal{L}_G, \mathcal{L}_H)$ continuous and uniformly $(\mathcal{R}_G, \mathcal{R}_H)$ continuous. Hence there exists an open neighborhood ${W_1}'$ of e in G such that $W_1 = \check{W_1}'$ and $cl_G(W_1)$ is compact and $cl_G(W_1) \subset {W_e}'$ and for each x and y in G with $x \setminus y \in W_1$:

$$|g(x) - g(y)| < \eta. \tag{120}$$

Therefore, a subset $S_f cl_G(W_1)$ is compact in G (see Theorems 3.1.10, 8.3.13-8.3.15 in [10], Lemma 2.6). Then we take compact subsets $S_1 = S_f cl_G(W_1)$ and $S = P(S_f cl_G(W_1))$ in G (see Formula (45)). In view of Lemma 2.6 they contain open subsets $S_f W_1$ and $P(S_f W_1)$ respectively, since W_1 is open in G. Recall that the topological spaces S_1 and S are normal, since they are compact and $T_1 \cap T_{3.5}$ (see Theorem 3.1.9 in [10]). Using Proposition 2.9 we take an open neighborhood ${W_2}'$ of e in G with $W_2 = \check{W_2}'$ such that

$$\begin{aligned} &[t((W_2a)W_2, (W_2b)W_2, (W_2c)W_2)W_2] \cup [W_2t((W_2a)W_2, (W_2b)W_2, (W_2c)W_2)] \\ \subset\ &[t(a, b, c)W_3] \cap [W_3t(a, b, c)], \\ &[p((W_2a)W_2, (W_2b)W_2, (W_2c)W_2)W_2] \cup [W_2p((W_2a)W_2, (W_2b)W_2, (W_2c)W_2)] \\ \subset\ &[p(a, b, c)W_3] \cap [W_3p(a, b, c)] \end{aligned} \tag{121}$$

for every a, b, c in S, where W_3 is an open neighborhood of e in G such that $\check{W}_3^2 \subset W_1$.

In view of the Dieudonné theorem 3.1 in [17] there exists a partition of unity on S_1. Together with Theorem 3.3.2 in [10] and Lemma 2.5 this implies that there are functions $q_1, ..., q_n$ in $C^+_{0,0}(G)$ and elements $w_1, ..., w_n$ in S_1 such that $S_1 \subset \bigcup_{j=1}^n w_j W_2$ and

$$\sum_{j=1}^n q_j(x) = 1 \text{ for each } x \in S_1, \tag{122}$$

$$q_j(y) = 0 \text{ for each } y \in G - (w_j W_2). \tag{123}$$

The conditions of this theorem imply that for each x and y in G with $y\setminus x \in V$ the following inequalities are satisfied:

$$[f(x) - \epsilon]g(y \setminus x) \le f(y)g(y \setminus x) \le [f(x) + \epsilon]g(y \setminus x), \tag{124}$$

since for $y \setminus x \in V$ Inequality (117) is fulfilled; for $u = y \setminus x \notin V$ the function g is nil, $g(u) = 0$.

Certainly $y \in w_j W_2$ if and only if there exists $b \in W_2$ such that $y = w_j b$. Then $(y \setminus x) \setminus (w_j \setminus x) \in W_1$ if and only if there exists $c \in W_1$ such that $w_j \setminus x = ((w_j b) \setminus x)c$. For $w_j \setminus x = v \in V$ this gives $c = ((w_j b) \setminus (w_j v)) \setminus v$. In view of (5), (13), (viii) and (ix) in Definition 2.1 $((w_j b) \setminus (w_j v)) \setminus v = p(w_j, b, (w_j b) \setminus (w_j v))((b \setminus v) \setminus v)$.

Therefore, from Conditions (120)-(123) it follows that for each x and y in G and $j = 1, ..., n$:

$$\begin{aligned} q_j(y)f(y)[g(y \setminus x) - \eta] &\leq q_j(y)f(y)g(w_j \setminus x) \\ &\leq q_j(y)f(y)[g(y \setminus x) + \eta]. \end{aligned} \tag{125}$$

Summing by j in (125), using (124) we infer that for each x and y in G:

$$\begin{aligned} &[f(x) - \epsilon]g(y \setminus x) - \eta f(y) \\ \leq &\sum_{j=1}^{n} q_j(y)f(y)g(w_j \setminus x) \leq [f(x) + \epsilon]g(y \setminus x) + \eta f(y). \end{aligned} \tag{126}$$

Next we take any ϕ and f_0 in $C^+_{0,0}(G)$ such that ϕ and f_0 are not identically zero. From Inequalities (126) after dividing by $J^y_{\phi,f_0}(g(y \setminus x))$ and using Lemma 3.7 it follows that for each x in G:

$$\begin{aligned} [f(x) - \epsilon] - \eta \frac{J_{\phi,f_0}(f)}{J^y_{\phi,f_0}(g(y \setminus x))} &\leq J^y_{\phi,f_0}(\frac{\sum_{j=1}^n g(w_j \setminus x)q_j(y)f(y)}{J^v_{\phi,f_0}(g(v \setminus x))}) \\ &\leq [f(x) + \epsilon] + \eta \frac{J_{\phi,f_0}(f)}{J^y_{\phi,f_0}(g(y \setminus x))}, \end{aligned} \tag{127}$$

where $J^y_{\phi,f_0}(g(y \setminus u)) = J_{\phi,f_0}(z)$ means that the functional J^y_{ϕ,f_0} is taken in the y variable in G, where $z(y) = g(y \setminus u)$ for each $y \in G$ and a fixed parameter u in G.

Notice that the function $g(y \setminus x)$ is jointly continuous in $(x, y) \in G \times G$. On the other hand, in view of Lemmas 2.2, 2.4, 2.6 $\{u = y \setminus x : \ x \in S_f, \ u \in S_g\}$ is a compact subset in G, since $Inv_l(S_f)$, S_g, $S_f S_g$ and $t(S_f, Inv_l(S_f), S_f S_g)$ are compact subsets in G. By virtue of Lemma 3.8 a mapping $\psi(x) := J^y_{\phi,f_0}(g(y \setminus x))$ is continuous in the variable $x \in S_f$, $\ \psi : S_f \to (0, \infty)$. Hence

$$0 < K_0 = \inf_{x \in S_f} \psi(x) \leq \sup_{x \in S_f} \psi(x) = K_1 < \infty. \tag{128}$$

Apparently in Formula (119) the parameter $\eta > 0$ can be taken sufficiently small, because Inequalities (119) and (128) are independent. Then from (127) and (128) we deduce that for each $\beta > \epsilon$ there exist q_j and w_j (see above) such that $\eta J_{\phi,f_0}(f) < (\beta - \epsilon) \min(1, K_0)$, consequently,

$$f(x) - \beta \leq J^y_{\phi,f_0}\left(\frac{\sum_{j=1}^n g(w_j \setminus x)q_j(y)f(y)}{J^v_{\phi,f_0}(g(v \setminus x))}\right) \leq f(x) + \beta \tag{129}$$

for each $x \in G$.

In view of Lemmas 3.7 and 3.10 for each $\delta > \delta_1 > \beta > \epsilon$ there exists an open neighborhood U of e in G of the form (a) in Lemma 2.6 such that $U \subset W_2$ and

$$|J^y_{\phi,f_0}\left(\frac{\sum_{j=1}^n g(w_j \setminus x)q_j(y)f(y)}{J^v_{\phi,f_0}(g(v\setminus x))}\right) - \sum_{j=1}^n \frac{J_{\phi,f_0}(q_j f)}{J^v_{\phi,f_0}(g(v\setminus x))} g(w_j \setminus x)| < \delta_1 - \beta \tag{130}$$

for each $x \in S_f$. We put $c_j = J_{\phi,f_0}(q_j f)$ and $b_j = w_j$ for each $j = 1, ..., n$. Thus the estimates (129) and (130) and Formula (60) imply the assertion (118) of this theorem. ∎

Definition 3.12. *Let W be an open neighborhood of e in a locally compact quasigroup G and a nonzero function $\phi_W \in C^+_{0,0}(G)$ be such that $\phi_W(x) = 0$ for each $x \in G - W$. A family $\{\phi_W\}$ of these functions will be directed by:*

(i) *$\phi_{W_1} \preceq \phi_{W_2}$ if and only if $W_2 \subseteq W_1$ and $\phi_{W_2}(x) = 0$ implies $\phi_{W_1}(x) = 0$.*

(ii) *If $\phi_{W_1} \preceq \phi_{W_2}$ and ϕ_{W_1} and ϕ_{W_2} are different functions, then it will be written $\phi_{W_1} \prec \phi_{W_2}$.*

Lemma 3.13. *Let G be a T_1 topological locally compact core quasigroup satisfying Condition* (70) *and let a family of nonzero functions $\{\phi_U\}$ in $C^+_{0,0}(G)$ be directed by Condition (i) in Definition 3.12. In addition choose $f_0 \in \Upsilon(G, N_0)$ (see* (71)*) and $f \in C^+_{0,0}(G)$. Then the limit exists:*

$$\lim_{\{\phi_U\}} J_{\phi_U,f_0}(f) =: J_{f_0}(f). \tag{131}$$

Proof. Let the net of functions $\{\phi_U\}$ in $C^+_{0,0}(G)$ be directed as in Condition (i) in Definition 3.12. It suffices to prove that the net $\{J_{\phi_U,f_0}(f) : \phi_U\}$ is fundamental (i.e. Cauchy) in $\mathbf{R}$. We take any fixed open neighborhood $U_0{}'$ of e in G with $U_0 = \check{U_0}{}'$ and a compact closure $cl_G(U_0)$. Let $A = S_{f+f_0} cl_G(U_0)$, where $S_{f+f_0} = cl_G\{x \in G : \ f(x) + f_0(x) \ne 0\}$. Therefore, a subset $S = P(A)$ is compact (see Formula (45) and Lemma 2.6), since S_{f+f_0} is compact.

We choose any function $z \in C^+_{0,0}(G)$ such that $z|_A = 1$. Let $0 < \epsilon < 1$ and $\xi_1 = \epsilon(16[1 + (z : f_0)][1 + (f : f_0)])^{-1}$. From Corollary 2.15 it follows that there exists an open neighborhood W' of e in G such that with $W = \check{W}'$:

$$|f(x) - f(y)| < \xi_1/2, \tag{132}$$

$$|f_0(x) - f_0(y)| < \xi_1/2 \tag{133}$$

for each x and y in G with $x \setminus y \in W$.

In view of Proposition 2.9 there exists an open neighborhood $U_2{}'$ of e in G with $U_2 = \check{U_2}{}'$ such that

$$[t((U_2a)U_2, (U_2b)U_2, (U_2c)U_2)U_2] \cup [U_2t((U_2a)U_2, (U_2b)U_2, (U_2c)U_2)]$$

$$\subset [t(a,b,c)B_1] \cap [B_1 t(a,b,c)],$$
$$[p((U_2a)U_2,(U_2b)U_2,(U_2c)U_2)U_2] \cup [U_2 p((U_2a)U_2,(U_2b)U_2,(U_2c)U_2)]$$
$$\subset [p(a,b,c)B_1] \cap [B_1 p(a,b,c)] \tag{134}$$

for every a, b, c in S, where B_1 is an open neighborhood of e in G such that $\check{B}_1^2 \subset U_1$, $U_1 = {U_0}' \cap W'$ (see Lemma 2.6). Next we take a nonzero function $g \in C^+_{0,0}(G)$ such that $g(x) = 0$ for each $x \in G - {U_2}'$.

By virtue of Theorem 3.11 for any fixed $0 < \delta < \xi_1$ and each open neighborhood ${W_e}'$ of e in G with $W_e = \check{W_e}'$ and a compact closure $cl_G(W_e)$ contained in ${U_2}'$ there is an open neighborhood $U'_{3,f}$ of e in G with $U_{3,f} = \check{U}'_{3,f}$ such that for each nonzero function ϕ in $C^+_{0,0}(G)$ with a support S_ϕ contained in $U'_{3,f}$ there are positive constants $c_1, ..., c_n$ and elements $b_1, ..., b_n$ in $S_f cl_G(W_e)$ such that for each $x \in G$ and $\gamma \in N(G)$:

$$|f(\gamma x) - \sum_{j=1}^{n} \frac{c_j}{J^v_{\phi,f_0}(g(v \setminus x))} g(b_j \setminus \gamma x)| \le \delta. \tag{135}$$

Taking $U_{3,f} \subset {U_2}'$ we get $f(x) = 0$ and $g(b_j \setminus x) = 0$ for each $x \in G - A$ according to the choice of b_j in the proof of Theorem 3.11, consequently,

$$|f(\gamma x) - \sum_{j=1}^{n} \frac{c_j}{J^v_{\phi,f_0}(g(v \setminus x))} g(b_j \setminus \gamma x)| \le \delta z(\gamma x) \tag{136}$$

for each $x \in G$ and $\gamma \in N(G)$. From the latter estimate and Lemma 3.7 we infer that

$$|J_{\phi,f_0}(f) - K_{\phi,f_0}(f;g)| \le \delta J_{\phi,f_0}(z) \le \delta(z : f_0), \tag{137}$$

where $K_{\phi,f_0}(f;g) = J^x_{\phi,f_0}(\sum_{j=1}^n \frac{c_j}{J^v_{\phi,f_0}(g(v\setminus x))} g(b_j \setminus x))$.

From Estimate (137) and the right Inequality (96) it follows that

$$\sup_{\{\phi_U\}} K_{\phi_U,f_0}(f;g) \le (1+\delta)(f : f_0) + \delta(z : f_0) < \infty. \tag{138}$$

Applying the proof above to f_0 instead of f we get an open neighborhood U'_{3,f_0} of e with $U_{3,f_0} = \check{U}'_{3,f_0}$ and $U_{3,f_0} \subset {U_2}'$ such that for each nonzero function ϕ in $C^+_{0,0}(G)$ with support S_ϕ contained in U'_{3,f_0} there are positive constants $d_1, ..., d_m$ and elements $v_1, ..., v_m$ in $S_{f_0} cl_G(W_e)$ such that

$$|f_0(\gamma x) - \sum_{j=1}^{m} \frac{d_j}{J^v_{\phi,f_0}(g(v \setminus x))} g(v_j \setminus \gamma x)| \le \delta z(\gamma x) \tag{139}$$

for each $x \in G$ and $\gamma \in N(G)$. Consequently, we see:

$$|1 - K_{\phi,f_0}(f_0;g)| \le \delta(z : f_0), \tag{140}$$

where $K_{\phi,f_0}(f_0;g) = J^x_{\phi,f_0}(\sum_{j=1}^m \frac{d_j}{J^v_{\phi,f_0}(g(v\setminus x))} g(v_j \setminus x))$, since $J_{\phi,f_0}(f_0) = 1$. Moreover,

$$\sup_{\{\phi_U\}} K_{\phi_U,f_0}(f_0;g) \leq (1+\delta) + \delta(z:f_0) < \infty. \tag{141}$$

Then $U'_3 = U'_{3,f} \cap U'_{3,f_0}$ is an open neighborhood of e in G. From (137), (140) and (141) we deduce that

$$|J_{\phi,f_0}(f) - \frac{K_{\phi,f_0}(f;g)}{K_{\phi,f_0}(f_0;g)}| \leq \delta_2 + [1+\delta+\delta_2]\delta_2(1-\delta_2)^{-1}, \tag{142}$$

where $\delta_2 = \delta(z:f_0) < \xi_1(z:f_0) < 1/16$. In view of Lemmas 3.7 and 3.10, Formulas (135) and (136) there exists an open neighborhood U'_4 of e with $U_4 = \check{U}'_4$ and U_4 contained in U'_3 such that for each nonzero ϕ in $C^+_{0,0}(G)$ with $S_\phi \subset U'_4$ there are the following inequalities:

$$|K_{\phi,f_0}(f;g) - \sum_{j=1}^n c_j J^x_{\phi,f_0}(\frac{g(b_j \setminus x)}{J^v_{\phi,f_0}(g(v\setminus \gamma x))})| \leq \delta, \tag{143}$$

$$|K_{\phi,f_0}(f_0;g) - \sum_{j=1}^m d_j J^x_{\phi,f_0}(\frac{g(v_j \setminus x)}{J^v_{\phi,f_0}(g(v\setminus \gamma x))})| \leq \delta \tag{144}$$

holding for every $\gamma \in N(G)$. On the other hand, Formulas (69), (76), (78), (85) and (62) imply that

$$\begin{aligned} J^x_{\phi,f_0}(\frac{g(b_j\setminus x)}{J^v_{\phi,f_0}(g(v\setminus\gamma x))}) &= \int_{N_0}(\frac{g(b_j\setminus x)}{(g(v\setminus\gamma x):\phi(v))}:\phi(x))\lambda(d\gamma) \\ &= \frac{(g(b_j\setminus x):\phi(x))}{(g^{[\lambda]}(v\setminus e):\phi(v))}. \end{aligned} \tag{145}$$

Then from Proposition 2.9 and Formulas (59), (61) it follows that for each $b \in G$ and each $0 < \delta_3 \leq \delta$ there exists an open neighborhood $U'_{5,b}$ of e in G with $U_{5,b} = \check{U}'_{5,b}$ such that for each nonzero $\phi_U \in C^+_{0,0}(G)$ with $S_{\phi_U} \subset U \subset U'_{5,b}$

$$\left|\frac{(g(b\setminus x):\phi_U(x))}{(g^{[\lambda]}(v\setminus e):\phi_U(v))} - \frac{(g(x):\phi_U(x))}{(g^{[\lambda]}(v\setminus e):\phi_U(v))}\right| < \delta_3, \tag{146}$$

since $S_{\phi_U} \subset U$ and $t(a,b,e) = t(a,e,b) = t(e,a,b) = e$ and $p(a,b,e) = p(a,e,b) = p(e,a,b) = e$ for each a and b in G. Therefore we take $U'_5 = \bigcap_{j=1}^n U'_{5,b_j} \cap \bigcap_{k=1}^m U'_{5,v_k} \cap U'_4$ and $\phi = \phi_Y$ with $Y = U'_5$. We put $c = \sum_{j=1}^n c_j$ and $d = \sum_{k=1}^m d_k$. From (142)-(146) and (96) it follows that

$$\frac{c}{d} < K_1, \text{ where } K_1 = 3[1+(f:f_0)](1+\delta)(1-\delta)^{-1} < 4[1+(f:f_0)].$$

Then we deduce from Formulas (142)-(146) for each ϕ_U with an open neighborhood U of e in G such that $U \subset U'_5$:

$$|J_{\phi_U,f_0}(f) - \frac{c}{d}| < \delta(1-\delta)^{-1}4[1+(f:f_0)] + \delta_2 + [1+\delta+\delta_2]\delta_2(1-\delta_2)^{-1},$$

consequently,

$$\begin{aligned} & |J_{\phi_{V_1},f_0}(f) - J_{\phi_{V_2},f_0}(f)| \\ & < 8\delta(1-\delta)^{-1}[1+(f:f_0)] + 2\delta_2 + 2[1+\delta+\delta_2]\delta_2(1-\delta_2)^{-1} < \epsilon \end{aligned} \tag{147}$$

for each open neighborhoods V_1 and V_2 of e in G such that $V_1 \subset U'_5$ and $V_2 \subset U'_5$. Thus the net $\{J_{\phi_U,f_0}(f) : \phi_U\}$ is fundamental, whenever the net $\{\phi_U\}$ is directed as described in Condition (i) in Definition 3.12. ∎

Remark 3.14. Suppose that G is a T_1 topological locally compact core quasigroup such that Condition (70) is fulfilled and choose $f_0 \in \Upsilon(G, N_0)$ (see (71)), functions f and g belong to $C^+_{0,0}(G)$ and let g be nonzero. Then in view of Lemma 3.13 the following functional exists

$$J_g(f) = J_{f_0}(f)/J_{f_0}(g). \tag{148}$$

As a consequence of Lemma 3.13 and Formulas (69) and (148) we get that

$$\text{the functional } \; J_g(f) \text{ is independent of } \; f_0. \tag{149}$$

Then Formula (97) and Lemma 3.13, Property (149) imply that

$$(g:f_0)^{-1}(f_0:f)^{-1} \le J_g(f) \le (f:f_0)(f_0:g) \tag{150}$$

for each $f_0 \in \Upsilon(G, N_0)$ and every nonzero function $f \in C^+_{0,0}(G)$.

Theorem 3.15. *Let G be a T_1 topological locally compact core quasigroup fulfilling Condition* (70) *and the functional $J = J_g$ be defined by Formula* (148)*. Then J possesses the following properties:*

$$J(f) \ge 0 \textit{ for each } \; f \in C^+_{0,0}(G); \tag{151}$$

and if a function $f \in C^+_{0,0}(G)$ is nonzero, then $J(f) > 0$;

$$J(\alpha_1 f_1 + ... + \alpha_n f_n) = \alpha_1 J(f_1) + ... + \alpha_n J(f_n) \tag{152}$$

for each $f_1, ..., f_n$ in $C^+_{0,0}(G)$ and $\alpha_1 \ge 0$,...,$\alpha_n \ge 0$;

$$J({}_b f) = J(f) \tag{153}$$

for each $b \in G$ and $f \in C^+_{0,0}(G)$.

Proof. Property (151) follows from Formula (150). On the other hand, Lemmas 3.7, 3.10, 3.13 imply Equality (152).

Then Formulas (76), (78), (85), (148) and Lemma 3.13 imply

$$J({}_b f^{[\lambda]}) = J(f^{[\lambda]}) \tag{154}$$

for each $b \in G$ and f in $C^+_{0,0}(G)$.

As a topological space G is locally compact. According to the measure theory on locally compact spaces (see Chapter 3, Section 11 in [17]) a functional J on $C^+_{0,0}(G)$ satisfying Conditions (151) and (152) induces a regular σ-additive measure μ on the Borel σ-algebra $\mathcal{B}(G)$ of G such that

$$\mu(U) = \sup\{\mu(X) : \ X \text{ is compact}, \ X \subset U\} \tag{155}$$

for each open subset U in G and

$$\mu(A) = \inf\{\mu(V) : \ V \text{ is open}, \ A \subset V \subset G\} \tag{156}$$

for each $A \in \mathcal{B}(G)$ and

$$J(f) = \int_G f(x)\mu(dx) \tag{157}$$

for each $f \in C^+_{0,0}(G)$ and the functional J has an extension $\bar{\bar{J}}$ such that

$$\bar{\bar{J}}(f) = \int_G f(x)\mu(dx) \tag{158}$$

for each nonnegative μ-measurable function f on G, where $\bar{\bar{J}}(f) = \inf\{\bar{J}(h) : \ h \ge f, \ h \text{ is lower semicontinuous }\}$, $\bar{J}(h) = \sup\{J(p) : \ p \in C^+_{0,0}(G), \ p \le h\}$ (see Theorems 11.22, 11.23, 11.36 and Corollary 11.37 in [17]).

On the other hand, for each $\gamma \in N(G)$ Formulas (60) and (61) give

$$({}_\gamma f : \phi_U) = (f : {}_\gamma\phi_U) = (f : \phi_U). \tag{159}$$

From Lemma 3.13, Formulas (148) and (159) we deduce that

$$J({}_\gamma f) = J(f) \tag{160}$$

for each $\gamma \in N_0(G)$.

By virtue of the Fubini theorem 13.8 in [17], (71), (72), and Formulas (154)-(157) and (160) above we infer that

$$\begin{aligned} J({}_b f) &= \int_{N_0} J({}_{b\gamma} f)\lambda(d\gamma) = \int_G \int_{N_0} {}_b f(\gamma x)\lambda(d\gamma)\mu(dx) \\ &= J({}_b f^{[\lambda]}) = J(f^{[\lambda]}) = \int_{N_0} J({}_\gamma f)\lambda(d\gamma) = J(f), \end{aligned}$$

since $\lambda(N_0) = 1$ and $N_0 \subset N(G)$. Thus the last assertion of this theorem is also proved. ∎

Theorem 3.16. *If G is a T_1 topological locally compact core quasigroup fulfilling Condition* (70)*, then there exists a regular σ-additive measure μ on a Borel σ-algebra $\mathcal{B}(G)$ of G, $\mu : \mathcal{B}(G) \to [0, \infty]$ such that*

(i) $\mu(U) > 0$ *for each open subset in* G*;*
(ii) $\mu(A) < \infty$ *for each compact subset* A *in* G*;*
(iii) $\mu(bB) = \mu(B)$ *for each* $B \in \mathcal{B}(G)$ *and* $b \in G$.

Such μ *can be chosen corresponding to a functional* J *satisfying Conditions* (151)-(153).

Proof. This is an immediate consequence of (151)-(153), (155)-(158). In particular $\mu(A) = \bar{\bar{J}}(\chi_A)$ for the characteristic function χ_A of a Borel subset A in G, where $\chi_A(x) = 1$ for each $x \in A$, $\chi_A(y) = 0$ for each $y \in G - A$. ∎

Remark 3.17. Each function f in $C_{0,0}(G)$ can be represented as $f = f^+ - f^-$, where $f^+(x) = \max(0, f(x))$, f^+ and f^- belong to $C^+_{0,0}(G)$. Therefore, a functional J satisfying Conditions (151) and (152) can be extended to a linear functional on $C_{0,0}(G)$ such that $J(f) = J(f^+) - J(f^-)$. Hence Property (153) extends onto $C_{0,0}(G)$.

Definition 3.18. *A linear functional* J *on* $C_{0,0}(G)$ *possessing Property* (153) *is called left invariant.*

A measure μ *on the Borel* σ*-algebra* $\mathcal{B}(G)$ *of a topological core quasigroup* G *such that* μ *satisfies Condition (iii) in Theorem 3.16 is called left invariant.*

Theorem 3.19. *Let* G *be a* T_1 *topological locally compact core quasigroup fulfilling Condition* (70) *and let* μ *be a measure possessing Properties (i)-(iii) in Theorem 3.16. Then* $\mu(G) < \infty$ *if and only if* G *is compact.*

Proof. If G is compact, then by (ii) in Theorem 3.16 $\mu(G) < \infty$.

Vice versa suppose that $\mu(G) < \infty$ and consider the variant that G is not compact and take an open neighborhood U' of e in G with $U = \check{U}'$ such that $U = N_0U$ and its closure $cl_G(U)$ is compact, hence $0 < \mu(U) < \infty$ (see also Condition (70)). By virtue of Theorem 2.8 there exists an open neighborhood V' of e in G with $V = \check{V}'$ such that $V = N_0V$ and $[cl_G(V)]^2 \subset U'$. In view of Lemma 2.5 a subset xU is open in G for each $x \in G$.

At first we take some fixed $x_1 \in G$. Then we construct a sequence $\{x_j : j \in \mathbf{N}\}$ by induction. Let $x_1, ..., x_n$ be constructed such that if $n \geq 2$, then $x_jV \cap x_kV = \emptyset$ for each $1 \leq j < k \leq n$. By Theorem 3.1.10 in [10] and Lemmas 2.4, 2.6 there exists $y \in G - \bigcup_{j=1}^n U_j$, where $U_j := x_jUp(x_jU, V, V)p(V, V, V)[p(x_jU, V, V)]^{-1}$, since G is not compact and U_j is open by Lemma 2.6 and $cl_G(U_j)$ is compact. Put $x_{n+1} = y$ with this y. Suppose that there is $z \in x_jV \cap x_{n+1}V$ for some $1 \leq j \leq n$. Therefore there would be v and u in V for which $z = x_jv = x_{n+1}u$, consequently, $(x_jv)/u = (x_{n+1}u)/u = x_{n+1}$ by Condition (ii) in Definition 2.1 and Formula (14). Therefore by Formulas (8), (21) and Condition (ix) in Definition 2.1 $x_{n+1} = x_j(v(e/u))p(x_j, v, e/u)p(e/u, u, u \setminus e)[p(x_j(v(e/u)), u, u \setminus e)]^{-1}$ contradicting the choice of x_{n+1}, since $[cl_G(V)]^2 \subset U'$. Thus $x_jV \cap x_kV = \emptyset$ for each $1 \le j < k \le n+1$. This would mean by (iii) in Theorem 3.16 that $\mu(G) \ge \sum_{j=1}^{n} \mu(x_j V) = n\mu(V)$ for each n, contradicting $0 < \mu(G) < \infty$. ■

Theorem 3.20. *Assume that G is a T_1 topological locally compact core quasigroup satisfying Condition* (70) *and let the functionals J and H on $C^+_{0,0}(G)$ satisfy Conditions* (151)-(153).

Then a positive constant κ exists such that

$$H(f) = \kappa J(f) \text{ for each } f \in C^+_{0,0}(G). \tag{161}$$

Proof. By virtue of Theorem 3.16 there exist two measures μ_1 and μ_2 corresponding to J and H. We consider a subalgebra $\mathcal{C}(G) := \theta^{-1}(\mathcal{B}(G/\cdot/N_0))$ in $\mathcal{B}(G)$, where $\theta : G \to G/\cdot/N_0$ is the quotient homomorphism, $\mathcal{B}(G)$ denotes the Borel σ-algebra on G. Put $\nu_j(A) = \mu_j(\theta^{-1}(A))$ for each j and $A \in \mathcal{B}(G/\cdot/N_0)$.

From Theorems 2.8 and 3.16 it follows that the measure ν_j on the group $G/\cdot/N_0$ is such that $\nu_j(V) > 0$ for each nonempty open subset V in $G/\cdot/N_0$, $\nu_j(A) < \infty$ for each compact subset A in $G/\cdot/N_0$, $\nu_j(cB) = \nu_j(B)$ for each $c \in G/\cdot/N_0$ and $B \in \mathcal{B}(G/\cdot/N_0)$, $j \in \{1,2\}$. By virtue of Theorem 15.6 in [17] there are positive constants p_j such that $\nu_j = p_j\eta$, where η is a left invariant Haar measure on $G/\cdot/N_0$. Thus $J(f^{[\lambda]}) = p_1 H(f^{[\lambda]})/p_2$ for each $f \in C^+_{0,0}(G)$.

We consider $\eta_1(b,f) = J({}_bf)/J(f^{[\lambda]})$ and $\eta_2(b,f) = H({}_bf)/H(f^{[\lambda]})$ for each $b \in G$ and a nonzero function f in $C^+_{0,0}(G)$. According to Property (153) we get the identities $\eta_j(b,f) = \eta_j(e, f^{[\lambda]}) = 1$ for each $j \in \{1,2\}$. This implies that for each nonzero function $f \in C^+_{0,0}(G)$ and $b \in G$:

$$J({}_bf)/H({}_bf) = p_1/p_2. \tag{162}$$

The measures μ_1 and μ_2 possess Properties (i)-(iii) in Theorem 3.16. In view of the Lebesgue-Radon-Nikodym theorem (see [17, Theorem (12.17)] or [5]) there exists a μ_1 measurable nonnegative function $h(x)$ such that $\int_G g(x)\mu_2(dx) = \int_G g(x)h(x)\mu_1(dx)$ for each $g \in C^+_{0,0}(G)$. Therefore from Formulas (158) and (162) it follows that $h(x)$ is a positive constant. Thus (161) is proved. ■

4. Appendix. Products of Core Quasigroups

The main subject of this paper are measures on core quasigroups. Nevertheless, in this section it is shortly demonstrated that there are abundant families of core quasigroups besides those which appear in areas described in the introduction.

Theorem 4.1. *Let (G_j, τ_j) be a family of topological T_1 core quasigroups (see Definition 2.1), where $j \in J$, J is a set. Then their direct product $G = \prod_{j\in J} G_j$ relative to the Tychonoff product topology τ is a topological T_1 core quasigroup* *and*

$$Z(G) = \prod_{j\in J} Z(G_j) \ \ and \ \ N(G) = \prod_{j\in J} N(G_j). \tag{163}$$

Proof. The direct product of topological quasigroups is a topological quasigroup (see [8, 10]). Thus conditions (i)-(iii) in Definition 2.1 are satisfied.

Each element $a \in G$ is written as $a = \{a_j : \ \forall j \in J, \ a_j \in G_j\}$. From (iv)-(vii) in Definition 2.1 we infer that

$$\begin{aligned} Com(G) &:= \{a \in G : \forall b \in G, \ ab = ba\} \\ &= \{a \in G : \ a = \{a_j : \forall j \in J, a_j \in G_j\}; \forall b \in G, \\ &\quad b = \{b_j : \forall j \in J, b_j \in G_j\}; \forall j \in J, \ a_j b_j = b_j a_j\} \\ &= \prod_{j\in J} Com(G_j), \end{aligned} \tag{164}$$

$$\begin{aligned} N_l(G) &:= \{a \in G : \ \forall b \in G, \ \forall c \in G, \ (ab)c = a(bc)\} \\ &= \{a \in G : \ a = \{a_j : \forall j \in J, a_j \in G_j\}; \\ &\quad \forall b \in G, b = \{b_j : \forall j \in J, b_j \in G_j\}; \\ &\quad \forall c \in G, \ c = \{c_j : \forall j \in J, c_j \in G_j\}; \\ &\quad \forall j \in J, \ (a_j b_j)c_j = a_j(b_j c_j)\} \\ &= \prod_{j\in J} N_l(G_j) \end{aligned} \tag{165}$$

and similarly

$$N_m(G) = \prod_{j\in J} N_m(G_j), \tag{166}$$

$$N_r(G) = \prod_{j\in J} N_r(G_j). \tag{167}$$

Therefore (165)-(167) and (viii) in Definition 2.1 imply that

$$N(G) = \prod_{j\in J} N(G_j). \tag{168}$$

Thus

$$Z(G) := Com(G) \cap N(G) = \prod_{j\in J} Z(G_j). \tag{169}$$

Let a, b and c be in G. Then $(ab)c = \{(a_j b_j)c_j : \ \forall j \in J, \ a_j \in G_j, b_j \in G_j, c_j \in G_j\} = \{t_{G_j}(a_j, b_j, c_j)a_j(b_j c_j) : \ \forall j \in J, \ a_j \in G_j, b_j \in G_j, c_j \in G_j\} = t_G(a,b,c)a(bc)$ and analogously $(ab)c = a(bc)p_G(a,b,c)$, where

$$t_G(a,b,c) = \{t_{G_j}(a_j, b_j, c_j) : \ \forall j \in J, \ a_j \in G_j, b_j \in G_j, c_j \in G_j\}, \tag{170}$$

$$p_G(a,b,c) = \{p_{G_j}(a_j,b_j,c_j) : \ \forall j \in J, \ a_j \in G_j, b_j \in G_j, c_j \in G_j\}. \quad (171)$$

Therefore, Formulas (169)-(171) imply that Conditions (ix) in Definition 2.1 also are satisfied. Thus G is a topological core quasigroup. By virtue of Theorem 2.3.11 in [10] a product of T_1 spaces is a T_1 space, hence G is the T_1 topological core quasigroup. ∎

Corollary 4.2.

(i) *Let conditions of Theorem 4.1 be satisfied and for each $j \in J$ a core quasigroup G_j satisfies Condition* (70). *Then the product core quasigroup G satisfies Condition* (70).

(ii) *Moreover, if G_j is compact for all $j \in J_0$ and locally compact for each $j \in J \setminus J_0$, where $J_0 \subset J$ and $J \setminus J_0$ is a finite set, then G is locally compact.*

Proof. Using Formulas (170) and (171) it is sufficient to take $N_0(G) = \prod_{j\in J} N_0(G_j)$, since the direct product of compact groups $N_0(G_j)$ is a compact group $N_0(G)$ (see the Tychonoff theorem 3.2.4 in [10] or [17]). The last assertion (2) follows from the known fact that G as a topological space is locally compact under the imposed above conditions (see Theorem 3.3.13 in [10]). ∎

Remark 4.3. Let A and B be two core quasigroups and let N be a group such that $N_0(A) \hookrightarrow N$, $N_0(B) \hookrightarrow N$, $N \hookrightarrow N(A)$ and $N \hookrightarrow N(B)$ and let N be normal in A and in B (see also Sections 2.1, 2.7 and 3.5).

Using direct products it is always possible to extend either A or B to get such a case. In particular, either A or B may be a group. On $A \times B$ an equivalence relation Ξ is considered such that $(v\gamma, b)\Xi(v, \gamma b)$ for every v in A, b in B and γ in N.

Let $\phi : A \to \mathcal{A}(B)$ be a single-valued mapping, where $\mathcal{A}(B)$ denotes a family of all bijective surjective single-valued mappings of B onto B subjected to the conditions given below. If $a \in A$ and $b \in B$, then it will be written shortly b^a instead of $\phi(a)b$, where $\phi(a) : B \to B$. Let also $\eta_\phi : A \times A \times B \to N$, $\kappa_\phi : A \times B \times B \to N$ and $\xi_\phi : ((A \times B)/\Xi) \times ((A \times B)/\Xi) \to N$ be single-valued mappings written shortly as η, κ and ξ correspondingly such that

(i) $(b^u)^v = b^{vu}\eta(v,u,b)$, $\gamma^u = \gamma$, $b^\gamma = b$;

(ii) $\eta(v,u,(\gamma_1 b)\gamma_2) = \eta(v,u,b)$; if $\gamma \in \{v,u,b\}$ then $\eta(v,u,b) = e$;

(iii) $(cb)^u = c^u b^u \kappa(u,c,b)$;

(iv) $\kappa(u,(\gamma_1 c)\gamma_2,(\gamma_3 b)\gamma_4) = \kappa(u,c,b)$ and if $\gamma \in \{u,c,b)$ then $\kappa(u,c,b) = e$;

(v) $\xi(((\gamma u)\gamma_1,(\gamma_2 c)\gamma_3),((\gamma_4 v)\gamma_5,(\gamma_6 b)\gamma_7)) = \xi((u,c),(v,b))$ and $\xi((e,e),(v,b)) = e$ and $\xi((u,c),(e,e)) = e$ for every u and v in A, b, c in B, γ, γ_1,...,γ_7 in N, where e denotes the neutral element in N and in A and B. We put $(a_1,b_1)(a_2,b_2) = (a_1a_2, b_1b_2^{a_1}\xi((a_1,b_1),(a_2,b_2)))$ for each a_1, a_2 in A, b_1 and b_2 in B.

The Cartesian product $A \times B$ supplied with such a binary operation in Remark 4.3 will be denoted by $A \bigotimes^{\phi,\eta,\kappa,\xi} B$.

Theorem 4.4. *Let the conditions of Remark 4.3 be fulfilled. Then the Cartesian product $A \times B$ supplied with a binary operation in Remark 4.3 is a core quasigroup.*

Proof. From the conditions of Remark 4.3 it follows that the binary operation in Remark 4.3 is single-valued. The group N is normal in the quasigroups A and B by Conditions of embeddings in Remark 4.4. Hence for each $a \in A$ and $\beta \in N$ there exists $(a\beta)/a \in N$ and $a \setminus (\beta a) \in N$, since $aN = Na$ for each $a \in A$. Similarly it is for B. Thus there are single-valued mappings $r_{A,a}(\beta) = (a\beta)/a$, $\check{r}_{A,a}(\beta) = a \setminus (\beta a)$, $r_{B,b}(\beta) = (b\beta)/b$, $\check{r}_{B,b}(\beta) = b \setminus (\beta b)$, $r_{A,a} : N \to N$, $\check{r}_{A,a} : N \to N$, $r_{B,b} : N \to N$, $\check{r}_{B,b} : N \to N$ for each $a \in A$ and $b \in B$. Evidently $r_{A,a}(\check{r}_{A,a}(\beta)) = \beta$ and $\check{r}_{A,a}(r_{A,a}(\beta)) = \beta$ for each $a \in A$ and $\beta \in N$, and similarly for B.

Let $I_1 = ((a_1, b_1)(a_2, b_2))(a_3, b_3)$ and $I_2 = (a_1, b_1)((a_2, b_2)(a_3, b_3))$, where a_1, a_2, a_3 belong to A, b_1, b_2, b_3 belong to B. Then we infer that $I_1 = ((a_1a_2)a_3, (b_1b_2^{a_1})\xi((a_1, b_1), (a_2, b_2))b_3^{a_1a_2}\xi((a_1a_2, b_1b_2^{a_1}), (a_3, b_3)))$ and $I_2 = (a_1(a_2a_3), b_1(b_2^{a_1}b_3^{a_1a_2})\beta)$ with $\beta = \eta(a_1, a_2, b_3)\kappa(a_1, b_2, b_3^{a_2})[\xi((a_2, b_2), (a_3, b_3))]^{a_1}\xi((a_1, b_1), (a_2a_3, b_2b_3^{a_2}))$. Hence $I_1 = (a, b\alpha)$ and $I_2 = (a, b\beta)$, where $a = a_1(a_2a_3)$ and $b = b_1(b_2^{a_1}b_3^{a_1a_2})$, $\alpha = \check{r}_{B,b}(p_A(a_1, a_2, a_3))$ $p_B(b_1, b_2^{a_1}, b_3^{a_1a_2})$ $\check{r}_{B,b_3^{a_1a_2}}(\xi((a_1, b_1), (a_2, b_2)))$ $\xi((a_1a_2, b_1b_2^{a_1}), (a_3, b_3)))$.

Therefore

$$
\begin{aligned}
I_1 &= I_2p \text{ with } p = p_{A \bigotimes^{\phi,\eta,\kappa,\xi} B}((a_1, b_1), (a_2, b_2), (a_3, b_3)),\\
I_1 &= tI_2 \text{ with } \ t = t_{A \bigotimes^{\phi,\eta,\kappa,\xi} B}((a_1, b_1), (a_2, b_2), (a_3, b_3)); \qquad (172)\\
p &= \beta^{-1}\alpha \text{ and } \ t = r_{A,a}(r_{B,b}(p)). \qquad (173)
\end{aligned}
$$

Apparently $t_{A \bigotimes^{\phi,\eta,\kappa,\xi} B}((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in N$ and $p_{A \bigotimes^{\phi,\eta,\kappa,\xi} B}((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in N$ for each $a_j \in A$, $b_j \in B$, $j \in \{1, 2, 3\}$, since α and β belong to the group N.

If $\gamma \in N$ and either (γ, e) or (e, γ) belongs to $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$, then from the conditions of Section 4.3 and Formulas (172) and (173) it follows that $p_{A \bigotimes^{\phi,\eta,\kappa,\xi} B}((a_1, b_1), (a_2, b_2), (a_3, b_3)) = e$ and $t_{A \bigotimes^{\phi,\eta,\kappa,\xi} B}((a_1, b_1), (a_2, b_2), (a_3, b_3)) = e$, consequently, $(N, e) \cup (e, N) \subset N(A \bigotimes^{\phi,\eta,\kappa,\xi} B)$.

Apparently (iii) in Definition 2.1 follows from (v) and multiplication (binary operation) in Remark 4.3.

Next we consider the following equation

$$(a_1, b_1)(a, b) = (e, e), \qquad (174)$$

where $a \in A$, $b \in B$.

From (ii) in Definition 2.1 for core quasigroups A and B, (v) and multiplication (binary operation) in Remark 4.3 we deduce that

$$a_1 = e/a, \tag{175}$$

consequently, $b_1 b^{(e/a)}\xi((e/a, b_1), (a, b)) = e$ and hence

$$b_1 = e/[b^{(e/a)}\xi((e/a, e/b^{(e/a)}), (a, b))]. \tag{176}$$

Thus $a_1 \in A$ and $b_1 \in B$ given by (175) and (176) provide a unique solution of (174).

Similarly from the following equation

$$(a, b)(a_2, b_2) = (e, e), \tag{177}$$

where $a \in A$, $b \in B$ we infer that

$$a_2 = a \setminus e, \tag{178}$$

consequently, $bb_2^a\xi((a, b), (a \setminus e, b_2)) = e$ and hence $b_2^a = b \setminus [\xi((a, b), (a \setminus e, b_2))]^{-1}$ by Conditions (i), (ii) in Definition 2.1 and the conditions on ϕ, η_ϕ and ξ_ϕ in Remark 4.3 for core A and B. On the other hand, $(b_2^a)^{e/a} = b_2\eta(e/a, a, b_2)$, consequently, by Lemmas 2.2, 2.3 and the conditions of Section 4.3

$$b_2 = (b \setminus [\xi((a, b), (a \setminus e, (b \setminus e)^{e/a}))]^{-1})^{e/a})/\eta(e/a, a, (b \setminus e)^{e/a}). \tag{179}$$

Thus Formulas (178) and (179) provide a unique solution of (177).

Next we put $(a_1, b_1) = (e, e)/(a, b)$ and $(a_2, b_2) = (a, b) \setminus (e, e)$ and

$$(a, b) \setminus (c, d) = ((a, b) \setminus (e, e))(c, d)p((a, b), (a, b) \setminus (e, e), (c, d)); \tag{180}$$

$$(c, d)/(a, b) = [t((c, d), (e, e)/(a, b), (a, b))]^{-1}(c, d)((e, e)/(a, b)) \tag{181}$$

and $e_G = (e, e)$, where $G = A \bigotimes^{\phi,\eta,\kappa,\xi} B$. ∎

Definition 4.5. *The core quasigroup $A \bigotimes^{\phi,\eta,\kappa,\xi} B$ provided by Theorem 4.4 we call a smashed product of core quasigroups A and B with smashing factors ϕ, η, κ and ξ.*

Corollary 4.6. *Suppose that the conditions of Remark 4.3 are fulfilled and A and B are topological T_1 core quasigroups and smashing factors ϕ, η, κ, ξ are jointly continuous by their variables. Suppose also that $A \bigotimes^{\phi,\eta,\kappa,\xi} B$ is supplied with a topology induced from the Tychonoff product topology on $A \times B$. Then $A \bigotimes^{\phi,\eta,\kappa,\xi} B$ is a topological T_1 core quasigroup.*

Corollary 4.7. *If the conditions of Corollary 4.6 are satisfied and quasigroups A and B are locally compact, then $A \bigotimes^{\phi,\eta,\kappa,\xi} B$ is locally compact. Moreover, if A and B satisfy Condition (70) and ranges of η, κ, ξ are contained in $N_0(A)N_0(B)$, then $A \bigotimes^{\phi,\eta,\kappa,\xi} B$ satisfies Condition (70).*

Proof. Corollaries 4.6 and 4.7 follow immediately from Theorems 2.3.11, 3.2.4, 3.3.13 in [10] and Theorems 2.8, 3.4 and Corollary 2.6, since $N_0(A)N_0(B) \subseteq N \subseteq N(A) \cap N(B)$ and because $N_0(A)N_0(B)$ is a compact subgroup in $A \bigotimes^{\phi,\eta,\kappa,\xi} B$. ■

Remark 4.8. From Theorems 4.1, 4.4 and Corollaries 4.2, 4.6, 4.7 it follows that taking nontrivial ϕ, η, κ and ξ and starting even from groups with nontrivial $N(G_j)$ or $N(A)$ and $G_j/\cdot/N(G_j)$ or $A/\cdot/N(A)$ it is possible to construct new core quasigroups with nontrivial $N_0(G)$ and ranges $t_G(G,G,G)$ and $p_G(G,G,G)$ of t_G and p_G may be infinite and nondiscrete. With suitable smashing factors ϕ, η, κ and ξ and with nontrivial core quasigroups or groups A and B it is easy to get examples of core quasigroups in which $e/a \neq a \setminus e$ for an infinite family of elements a in $A \bigotimes^{\phi,\eta,\kappa,\xi} B$.

It is worth to mention that under rather general conditions an existence of a nontrivial nonnegative left invariant measure on the Borel σ-algebra of a topological unital quasigroup implies that it is either locally compact or dense in some locally compact unital quasigroup [28]. In the latter article also examples of quasigroups are discussed.

Conclusion 4.9. The results of this article can be used for further studies of measures on homogeneous spaces and noncommutative manifolds related with quasigroups. Other applications of left invariant measures on quasigroups belong to mathematical coding theory and technics such as assessing of web structural logic and semantic analysis [4, 3, 32]. This is natural, because codings and parallel architectures are frequently based on binary systems and measures. Another very important applications are in representation theory of quasigroups and harmonic analysis on quasigroups, mathematical physics, quantum field theory, quantum gravity, gauge theory, etc. [6, 7, 11, 16, 23, 25].

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