Southeast Asian Bulletin of Mathematics (2022) 46: 415–424

Southeast Asian
Bulletin of
Mathematics

A Class of Sub-Almost Distributive Lattices Through Intervals in an Almost Distributive Lattice

S. Ramesh
Department of Mathematics, Institute of Science, GITAM (Deemed to be University), Visakhapatnam-530 045, India
Email: ramesh.sirisetti@gmail.com

G. Jogarao
Department of Mathematics, GITAM School of Science, GITAM (Deemed to be University), Bangalore, India
Email: jogarao.gunda@gmail.com

Received 27 November 2019
Accepted 26 April 2022

Communicated by Xueming Ren

AMS Mathematics Subject Classification(2020): 06D99, 06D75

Abstract. In this paper, we acquaint a class of sub-almost distributive lattices through intervals in an almost distributive lattice and prove an outnumbered of algebraic properties on them. We observe that this class forms an almost distributive lattice. We accomplish peer conditions for an almost distributive lattice to become weakly relatively complemented (distributive)(Boolean algebra) in terms of this class of sub-almost distributive lattices.

Keywords: Almost distributive lattice; Dense element; Intervals; Weakly relatively complemented ADL.

1. Introduction

In [1], Boole put forward a special class of algebraic constructions in connection with his research work in mathematical logic, and 2-valued propositional calculus, which undergos to the concept of Boolean algebras (complemented, distributive lattices). Several algebraists studied and generalized the concept of distributive lattices in various aspects. In this context, Swamy and Rao [7] originated the abstraction of almost distributive lattice L along with two binary operations $\vee$ and $\wedge$, which satisfies almost all the conditions of a distributive lattice with the zero element exempting commutativity of $\wedge$, $\vee$ and right distributivity of $\vee$ over $\wedge$. More recently, Ramesh and Rao [2] imported the set $B_F(L) = \{a \in L \mid \ a \wedge b = 0$ and $a \vee b \in F$, for some element $b \in L\}$, for a filter F in an associate almost distributive lattice L and proved that the set $B_F(L)$ is a sub-almost distributive lattice of L.

In this paper, for any element x in an almost distributive lattice L (the operation join is not necessarily associate), we define the set $B_D([0, x]) = \{a \in [0, x] \mid a \wedge b = 0$ and $a \vee b$ is dense in $[0, x]$ for some element $b \in [0, x]\}$, and show that the set is a sub-almost distributive lattice of L. We establish definite algebraic properties on the class $\{B_D([0, x]) \mid x \in L\}$. We observe that the class of sub-almost distributive lattices forms an almost distributive lattice. We obtain needed and acceptable conditions for an almost distributive lattice to become a weakly relatively complemented almost distributive lattice or a distributive lattice or a Boolean algebra in terms of the above class.

2. Preliminaries

In this section we present a few needed definitions and results that are subsequently used for ready reference from [5, 6, 4, 7].

Definition 2.1. [7] *By an almost distributive lattice (abbreviated: ADL) we mean an algebra* $(L, \wedge, \vee, 0)$ *of type* $(2, 2, 0)$*, if it satisfies the following conditions;*

(i) $0 \wedge a = 0$

(ii) $a \vee 0 = a$

(iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(iv) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(v) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(vi) $(a \vee b) \wedge b = b$,

for all $a, b, c \in L$.

Example 2.2. [7] If X is a non-empty set, fix $x_0 \in X$ and for any $x, y \in X$, define

$$x \wedge y = \begin{cases} x_0 & \text{if } x = x_0, \\ y & \text{if } x \neq x_0 \end{cases} \quad \text{and} \quad x \vee y = \begin{cases} y & \text{if } x = x_0, \\ x & \text{if } x \neq x_0. \end{cases}$$

Then $(X, \wedge, \vee, x_0)$ is an ADL, in which x_0 as its zero element. Clearly, this ADL is not a lattice and is called discrete ADL.

Everywhere of this paper by L mean an almost distributive lattice L with two binary operations $\vee, \wedge$ and 0 as its zero element unless if not intimated.

Given $a, b \in L$, we read that a is less than or equal to b and write $a \leq b$, if $a \wedge b = a$ or, equivalently $a \vee b = b$. It can be easy to confirm that " $\leq$ " is a partial ordering on L.

Lemma 2.3. [7] *For any $a, b, c \in L$, we have*

(i) $a \wedge 0 = 0$ *and* $0 \vee a = a$
(ii) $a \wedge a = a \vee a = a$
(iii) $a \vee (b \vee a) = a \vee b$
(iv) $\wedge$ *is associative*
(v) $a \wedge b \wedge c = b \wedge a \wedge c$
(vi) $a \wedge b = 0 \iff b \wedge a = 0$
(vii) $a \wedge b \leq b$ *and* $a \leq a \vee b$
(viii) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(ix) $a \vee b = b \vee a \Longleftrightarrow a \wedge b = b \wedge a$
(x) $(a \wedge b) \vee b = b$, $a \vee (a \wedge b) = a$ *and* $a \wedge (a \vee b) = a$.

An aggregate I of L is called an ideal of L, if for any $a, b \in I$ and $x \in L$, $a \vee b$, $a \wedge x \in I$. Notably, for any $a \in L$, $(a] = \{a \wedge x \mid x \in L\}$ is the principal ideal generated by a. The set $\mathcal{I}(L)$ ideals of L forms a bounded distributive lattice, where $I \cap J$ is the infimum and $I \vee J = \{i \vee j \mid i \in I \text{ and } j \in J\}$ is the supremum of I and J in $\mathcal{I}(L)$. The set $\mathcal{PI}(L)$ principal ideals of L designs a sublattice of $\mathcal{I}(L)$, where $(a] \wedge (b] = (a \wedge b]$ and $(a] \vee (b] = (a \vee b]$, for any $a, b \in L$. A non-empty subset F of L is called a filter of L, if for any $a, b \in F$ and $x \in L$, $a \wedge b$, $x \vee a \in F$.

Give aggregate A of L, the set $A^* = \{x \in L \mid a \wedge x = 0, \text{ for all } a \in A\}$ is an ideal of L. Especially, for any $a \in L$, $\{a\}^* = (a)^*$, where $(a) = (a]$ is the principal ideal generated by a.

Lemma 2.4. [6] *For any $a, b \in L$, we have*

(i) $a \leq b$ *implies* $(b)^* \subseteq (a)^*$
(ii) $(a \vee b)^* = (a)^* \cap (b)^*$
(iii) $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$

An element $d \in L$ is called dense [5], if $(d)^* = \{0\}$. It can be easily seen that the set D of all dense elements in L is a filter provided D is non-empty. An element $m \in L$ is called maximal, if for any $a \in L$, $m \leq a$ implies $m = a$. It is easy to examine that every maximal element is dense.

Theorem 2.5. [7] *For any $m \in L$, the following are equivalent:*

(i) m *is maximal*
(ii) $m \wedge x = x$, *for all* $x \in L$
(iii) $m \vee x = m$, *for all* $x \in L$.

Definition 2.6. [7] *An aggregate S of L is called a sub-ADL of L, if it is closed under $\wedge$, $\vee$ and S contains zero element.*

In [4], Ramesh and Rao made known the set

$$B_D(L) = \{a \in L \ \mid a \wedge b = 0 \text{ and } a \vee b \text{ is dense, for some element } \ b \in L \ \},$$

in an ADL L with dense elements and proven that $B_D(L)$ is a sub-ADL of L.

Definition 2.7. [4] *L is called weakly relatively complemented, if for any $a, b \in L$ there exists $x \in L$ so that $a \wedge x = 0$ and $(a \vee x)^* = (aveeb)^*$.*

3. A Class of Sub-ADLs Through Intervals in an ADL

In this section, we present a class of sub-ADLs via an interval in an ADL with dense elements and we attain a few algebraic properties on them. We tested that the class of sub-ADLs patterns an almost distributive lattice. We determine needed and acceptable conditions for an almost distributive lattice to be weakly relatively complemented (distributive)(Boolean algebra) in terms of the class of sub-ADLs.

Lemma 3.1. *If $a, b, c \in L$ in which b is dense and $a \leq b \leq c$, then b is dense in $[a, c]$.*

Remark 3.2. The converse of the above lemma may not be true.

For, let $L = \{0, a, b, 1\}$ whose Hasse-diagram is Fig. 1.

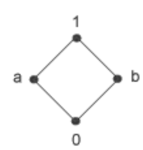

Fig. 1. Hasse-diagram

Then L is an ADL and also a distributive lattice. Clearly a is dense in [0, a], but a is not dense in L. Because, $a \wedge b = 0$ and $b \neq 0$.

Lemma 3.3. *If d is a dense element in L and $a \in [0, d]$, then a is dense in $[0, d]$ implies a is dense in L.*

Proof. Let d be a dense element in L. Suppose that a is dense in $[0, d]$. For any $t \in L$,

$$\begin{aligned} t \in (a)^* &\Rightarrow t \wedge a = 0 \\ &\Rightarrow t \wedge d \wedge a = 0 && (\text{since } a = d \wedge a) \\ &\Rightarrow t \wedge d = 0 && (\text{since } a \text{ is dense in } [0, d]) \\ &\Rightarrow t = 0 && (\text{since } d \text{ is dense in } L). \end{aligned}$$

Therefore $(a)^* = \{0\}$. Hence a is dense in L. ∎

Definition 3.4. *For any $x \in L$ (x need not be dense), define the set $B_D([0,x]) = \{a \in [0,x] \mid$ there exists $b \in [0,x]$ such that $a \wedge b = 0$ and $a \vee b$ is dense in $[0,x]\}$. Here the element x is the greatest element in $[0,x]$ such that $0 \wedge x = 0$ and $0 \vee x$ is dense in $[0,x]$. Hence $0, x \in B_D([0,x]), B_D([0,x]) \neq \phi$.*

Theorem 3.5. *$B_D([0,d]) \subseteq B_D(L)$, where d is a dense elements in L.*

Proof. Let $a \in B_D([0,d])$. Then $a \wedge b = 0$ and $a \vee b$ is dense in $[0,d]$, for some $b \in [0,d]$. By Lemma 3.3, $a \wedge b = 0$ and $a \vee b$ is dense in L. Therefore $a \in B_D(L)$. Hence $B_D([0,d]) \subseteq B_D(L)$. ■

Theorem 3.6. *$B_D([0,d]) = \{a \wedge d \mid a \in B_D(L)\}$, where d is a dense element in L.*

Proof. Let d be a dense element in L. Then $d \in B_D(L)$ (since $D \subseteq B_D(L)$) and by Theorem 3.5, $B_D([0,d]) \subseteq B_D(L)$. For $x \in L$,

$$\begin{aligned} x \in B_D([0,d]) &\Rightarrow x \in [0,d] && \text{(since } B_D([0,d]) \subseteq [0,d])\\ &\Rightarrow x = x \wedge d \\ &\Rightarrow x \in \{a \wedge d \mid a \in B_D(L)\}. && \text{(since } x \in B_D([0,d]) \subseteq B_D(L)) \end{aligned}$$

Therefore $B_D([0,d]) \subseteq \{a \wedge d \mid a \in B_D(L)\}$. Let $x \in \{a \wedge d \mid a \in B_D(L)\}$. Then $x = a \wedge d$, for some $a \in B_D(L)$. Therefore $x \in [0,d]$. For this $a \in B_D(L)$, we have $a \wedge b = 0$ and $a \vee b$ is dense in L, for some $b \in L$. Now,

$$\begin{aligned} x \wedge b &= (a \wedge d) \wedge b && \text{(since } x = a \wedge d)\\ &= d \wedge (a \wedge b) && \text{(by Lemma 2.3 (iv) (v))}\\ &= 0. && \text{(since } a \wedge b = 0) \end{aligned}$$

For $t \in [0,d]$,

$$\begin{aligned} t \in (x \vee b)^* &\Rightarrow (x \vee b) \wedge t = 0 \\ &\Rightarrow \{(a \wedge d) \vee b\} \wedge t = 0 && \text{(since } x = a \wedge d)\\ &\Rightarrow \{b \vee (a \wedge d)\} \wedge t = 0 && \text{(by Lemma 2.3 (viii))}\\ &\Rightarrow \{(b \vee a) \wedge (b \vee d)\} \wedge t = 0 && \text{(by Definition 2.1 (v))}\\ &\Rightarrow \{(a \vee b) \wedge (d \vee b)\} \wedge t = 0 && \text{(by Lemma 2.3 (viii))}\\ &\Rightarrow (a \vee b) \wedge t = 0 && \text{(since } d \vee b \text{ is dense)}\\ &\Rightarrow t = 0. && \text{(since } a \vee b \text{ is dense)} \end{aligned}$$

Therefore $(x \vee b)$ is dense in $[0,d]$ and hence $x \in B_D([0,d])$. Thus $B_D([0,d]) = \{a \wedge d \mid a \in B_D(L)\}$. ■

Theorem 3.7. *Given an element $a \in L$, $a \in B_D(L)$ if and only if $a \wedge x \in B_D([0,x])$, for all $x \in L$.*

Proof. Suppose that $a \in B_D(L)$. Then $a \wedge b = 0$ and $a \vee b$ is dense, for some element $b \in L$. Let $x \in L$. Then $a \wedge x,\ b \wedge x \in [0,x]$ and $a \wedge x \wedge b \wedge x = a \wedge b \wedge x = 0$

(by Lemma 2.3 (iv) (v)). For $t \in [0, x]$,

$$\begin{aligned} t \in \{(a \wedge x) \vee (b \wedge x)\}^* &\Rightarrow t \in \{(a \vee b) \wedge x\}^* && \text{(by Definition 2.1 (iv))} \\ &\Rightarrow t \wedge (a \vee b) \wedge x = 0 \\ &\Rightarrow t \wedge x = 0 && \text{(since } a \vee b \text{ is dense)} \\ &\Rightarrow t = 0. && \text{(since } t \wedge x = t\text{)} \end{aligned}$$

Therefore $(a \wedge x) \vee (b \wedge x)$ is dense in $[0, x]$. Hence $a \wedge x \in B_D([0, x])$. Conversely, suppose that $a \wedge x \in B_D([0, x])$, for all $x \in L$. Let d be a dense element in L. Then $a \wedge d \in B_D([0, d])$. Therefore there exists $b \in [0, d]$ such that $(a \wedge d) \wedge b = 0$ and $(a \wedge d) \vee b$ is dense in $[0, d]$. By Lemma 3.3, $(a \wedge d) \vee b$ is dense in L and $a \wedge b = 0$ (since $b \wedge d = b$). For $t \in L$,

$$\begin{aligned} t \in (a \vee b)^* &\Rightarrow t \wedge (a \vee b) = 0 \\ &\Rightarrow t \wedge a = 0 = t \wedge b && \text{(by Definition 2.1 (iii))} \\ &\Rightarrow a \wedge d \wedge t = 0 = b \wedge t && \text{(by Lemma 2.3 (v))} \\ &\Rightarrow \{(a \wedge d) \vee b\} \wedge t = 0 && \text{(by Definition 2.1 (iv))} \\ &\Rightarrow t = 0. && \text{(since } (a \wedge d) \vee b \text{ is dense in } L\text{)} \end{aligned}$$

Therefore $(a \vee b)^* = \{0\}$ and hence $a \vee b$ is dense in L. Thus $a \in B_D(L)$. ∎

Theorem 3.8. *$a \in B_D(L)$ if and only if $b \in B_D(L)$, where $a, b \in L$ so that $(a] = (b]$.*

Proof. Choose $a, b \in L$ in which $(a] = (b]$ and $a \in B_D(L)$. Then $a \wedge x = 0$ and $a \vee x$ is dense, for some $x \in L$. Now,

$$\begin{aligned} (a] = (b] &\Rightarrow a = b \wedge a \text{ and } b = a \wedge b && \text{(since } a \in (b] \text{ and } b \in (a]\text{)} \\ &\Rightarrow b \wedge x = (a \wedge b) \wedge x && \text{(since } b = a \wedge b\text{)} \\ &\Rightarrow b \wedge x = (b \wedge a) \wedge x && \text{(by Lemma 2.3 (v))} \\ &\Rightarrow b \wedge x = b \wedge (a \wedge x) && \text{(by Lemma 2.3 (iv))} \\ &\Rightarrow b \wedge x = 0. && \text{(since } a \wedge x = 0\text{)} \end{aligned}$$

For $t \in L$,

$$\begin{aligned} & t \in \{(b \vee x) \wedge (a \vee x)\}^* \\ \Rightarrow\ & t \in \{[(b \vee x) \wedge a] \vee [(b \vee x) \wedge x]\}^* \text{ by Definition 2.1 (iii))} \\ \Rightarrow\ & t \in \{[(b \vee x) \wedge a] \vee x\}^* \text{ (since } (b \vee x) \wedge x = x\text{)} \\ \Rightarrow\ & t \in \{[(b \wedge a) \vee (x \wedge a)] \vee x\}^* \text{ (by Definition 2.1 (iv))} \\ \Rightarrow\ & t \in \{(b \wedge a)^* \cap (x \wedge a)^* \cap (x)^*\} \text{ (by Lemma 2.4 (iii))} \\ \Rightarrow\ & t \in \{(b \wedge a)^* \cap [x \vee (x \wedge a)]^*\} \text{ (by Lemma 2.4 (iii))} \\ \Rightarrow\ & t \in \{(b \wedge a) \vee x\}^* \text{ (by Lemma 2.3 (x))} \\ \Rightarrow\ & t \in (a \vee x)^* \text{ (since } b \wedge a = a\text{)} \\ \Rightarrow\ & t = 0 \text{ (since } a \vee x \text{ is dense).} \end{aligned}$$

Therefore $(b \vee x) \wedge (a \vee x)$ is dense in L. So that $b \vee x$ is dense in L. Hence $b \in B_D(L)$. Similarly we can prove $a \in B_D(L)$. ∎

Theorem 3.9. $B_D(\mathcal{PI}(L)) = \{(a] \mid a \in B_D(L)\}$.

Proof. Let $(a] \in B_D(\mathcal{PI}(L))$, for some $a \in L$. Then $(a] \cap (b] = (0]$ and $(a] \vee (b]$ is a dense element in $\mathcal{PI}(L)$, for some $b \in L$. Therefore $(a \wedge b] = (0]$ and $[(a] \vee (b]]^* = [(a \vee b]]^* = (a \vee b)^* = \{0\}$. So that $a \wedge b = 0$ and $a \vee b$ is dense. So that $a \in B_D(L)$. Hence $B_D(\mathcal{PI}(L)) \subseteq \{(a] \mid a \in B_D(L)\}$. Let $(a] \in \{(a] \mid a \in B_D(L)\}$. For this $a \in B_D(L)$, there is an element $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is dense. Therefore $(a] \cap (b] = (a \wedge b] = \{0\}$ and $(a] \vee (b] = (a \vee b]$ is a dense element in $\mathcal{PI}(L)$. Hence $(a] \in B_D(\mathcal{PI}(L))$. Thus $B_D(\mathcal{PI}(L)) = \{(a] \mid a \in B_D(L)\}$. ∎

Theorem 3.10. *L is weakly relatively complemented if and only if for each $a \in L$, there exists a dense element d in L such that $a \in B_D[0, d]$.*

Proof. Assume that L is weakly relatively complemented ADL. Let $x \in L$. Then $a \wedge x = 0$ and $a \vee x$ is dense, for some $x \in L$. Since $x, a \in [0, a \vee x], a \in B_D[0, a \vee x]$ and $a \vee x$ is dense in L. Conversely, let $a \in L$. Then there is a dense element d in L such that $a \in B_D[0, d]$. For this $a \in B_D[0, d]$, $a \wedge x = 0$ and $a \vee x$ is dense in $[0, d]$, for some $x \in [0, d]$. By Theorem 3.3, $a \vee x$ is dense in L. Thus L is weakly relatively complemented. ∎

Lemma 3.11. *$B_D([0, a])$ is a sub-ADL of L, for all elements a in L.*

Proof. Let $a_1, a_2 \in B_D([0, a])$. Then there are $b_1, b_2 \in [0, a]$ such that $a_1 \wedge b_1 = 0 = a_2 \wedge b_2$ and $a_1 \vee b_1$ & $a_2 \vee b_2$ are dense in $[0, a]$. Now, $(a_1 \wedge a_2) \wedge (b_1 \vee b_2) = (a_1 \wedge a_2 \wedge b_1) \vee (a_1 \wedge a_2 \wedge b_2) = 0$ (by Definition 2.1 (iii)) and, for $x \in [0, a]$,

$$
\begin{aligned}
& x \in [(a_1 \wedge a_2) \vee (b_1 \vee b_2)]^* \\
\Rightarrow\ & x \wedge [(a_1 \wedge a_2) \vee (b_1 \wedge b_2)] = 0 \quad \text{(by Definition 2.1 (iii))} \\
\Rightarrow\ & x \wedge a_1 \wedge a_2 = x \wedge b_1 = 0 = x \wedge b_2 \\
\Rightarrow\ & x \wedge a_1 \wedge a_2 = 0 = x \wedge a_2 \wedge b_1 \\
\Rightarrow\ & x \wedge a_2 \wedge (a_1 \vee b_1) = 0 \quad \text{(by Definition 2.1 (iii))} \\
\Rightarrow\ & x \wedge a_2 = 0 \quad \text{(since } a_1 \vee b_1 \text{ is dense in } [0, a]) \\
\Rightarrow\ & x \wedge (a_2 \vee b_2) = 0 \quad \text{(by Definition 2.1 (iii))} \\
\Rightarrow\ & x = 0 \quad \text{(since } a_2 \vee b_2 \text{ is dense in } [0, a]).
\end{aligned}
$$

Therefore $(a_1 \wedge a_2) \vee (b_1 \vee b_2)$ is dense in $[0, a]$ and hence $a_1 \wedge a_2 \in B_D([0, a])$. $(a_1 \vee a_2) \wedge b_1 \wedge b_2 = (a_1 \wedge b_1 \wedge b_2) \vee (a_2 \wedge b_1 \wedge b_2) = 0$ (by Definition 2.1 (iv)) and, for $x \in [0, a]$,

$$
\begin{aligned}
& x \in [(a_1 \vee a_2) \vee (b_1 \wedge b_2)]^* \\
\Rightarrow\ & x \wedge (a_1 \vee a_2) = 0 = x \wedge (b_1 \wedge b_2) \quad \text{(by Definition 2.1 (iii))} \\
\Rightarrow\ & x \wedge a_1 = x \wedge a_2 = x \wedge b_1 \wedge b_2 = 0 \quad \text{(by Definition 2.1 (iii))} \\
\Rightarrow\ & x \wedge b_2 \wedge a_1 = 0 = x \wedge b_1 \wedge b_2
\end{aligned}
$$

$\Rightarrow (x \wedge b_2) \wedge (a_1 \vee b_1) = 0$ (by Definition 2.1 (iii))
$\Rightarrow x \wedge b_2 = 0$ (since $a_1 \vee b_1$ is dense in $[0, a]$)
$\Rightarrow x \wedge (a_2 \vee b_2) = 0$ (by Definition 2.1 (iii))
$\Rightarrow x = 0$ (since $a_2 \vee b_2$ is dense in $[0, a]$).

Therefore $(a_1 \vee a_2) \vee (b_1 \wedge b_2)$ is dense in $[0, a]$ and hence $a_1 \vee a_2 \in B_D([0, a])$. Thus $B_D([0, a])$ is a sub-ADL of L. ∎

Lemma 3.12. *For any $x \in L$, the following statements hold:*

(i) *x is the greatest element in L if and only if $B_D[0, x] = L$.*
(ii) *$B_D[0, x] = \{0\}$ if and only if $x = 0$.*

Proof. (i) Suppose that x be the greatest element in L, say 1. Then $[0, 1] = L$. Therefore $B_D[0, 1] = B_D(L) = L$ (since $B_D(L) = L$). Conversely suppose that $B_D[0, x] = L$, for some $x \in L$. For any $a \in L$, $a \in B_D[0, x] \subseteq [0, x]$. Therefore $a \leq x$ if and only if $x \wedge a = a$. Hence x is the greatest element in L.

(ii) Suppose that $B_D[0, x] = \{0\}$, for some $x \in L$. Then clearly we know that x is the greatest(dense) element in $[0, x]$. Therefore $x \in B_D[0, x] = \{0\}$. So that $x = 0$. Conversely suppose that $x = 0$. Assume that $B_D[0, x] \neq \{0\}$. Then there exists a non-zero element $a \in B_D[0, x]$ such that $a \wedge y = 0$ and $a \vee y$ is dense in $[0, x]$, for some $y \in [0, x]$. Therefore $0 < a \vee y \leq x(= 0)$. So that $a \vee y = 0$. Which is contradiction (since $a \vee y$ is dense). Hence $B_D[0, x] = \{0\}$. ∎

Lemma 3.13. *Given $x, y \in L$, $B_D[0, x] = B_D[0, y]$ if and only if $x = y$.*

Proof. Let $B_D[0, x] = B_D[0, y]$. We know that the elements x, y are the greatest elements in the intervals $[0, x]$ and $[0, y]$ respectively. Therefore $x \in B_D[0, x]$ and $y \in B_D[0, y]$. So that $x \in B_D[0, y] \subseteq [0, y]$ and $y \in B_D[0, x] \subseteq [0, x]$. We get that $x \leq y$ and $y \leq x$. Hence $x = y$. The converse is clear. ∎

Theorem 3.14. *Given $x, y \in L$, $x \leq y$ implies $B_D([0, x]) \subseteq B_D([0, y])$.*

Proof. Let $a \in B_D([0, x])$. Then $a \wedge x_1 = 0$ and $a \vee x_1$ is dense in $[0, x]$, for some $x_1 \in [0, x]$. If $a \vee x_1$ is dense in $[0, y]$, then $x_1 \in B_D([0, y])$ (since $x \leq y$ implies $[0, x] \subseteq [0, y]$). Therefore $B_D([0, x]) \subseteq B_D([0, y])$. If $a \vee x_1$ is non-dense in $[0, y]$, then there is an element $y_1 \in [0, y]$ such that $(a \vee x_1) \wedge y_1 = 0$. So that $a \wedge y_1 = 0 = x_1 \wedge y_1$. Since $[0, y]$ is bounded, we can choose a maximal element y_m in $[0, y]$ such that $a \wedge y_m = 0$. Hence $a \vee y_m \in [0, y]$. If $a \vee y_m$ is dense in $[0, y]$, then $B_D([0, x]) \subseteq B_D([0, y])$. Otherwise, there is a non-zero element $y^1 \in [0, y]$ such that $(a \vee y_m) \wedge y^1 = 0$. Therefore $a \wedge y^1 = 0 = y_m \wedge y^1$. So that the elements y_m and y^1 are incomparable elements. Hence $y_m < y_m \vee y^1$. Now, $a \wedge (y_m \vee y^1) = (a \wedge y_m) \vee (a \wedge y^1) = 0$ (since $a \wedge y_m = 0$ and $a \wedge y^1 = 0$). Which is contradiction to the maximality of y_m. So that $a \vee y_m$ is dense in $[0, y]$. Hence $a \in B_D([0, y])$. Thus $B_D([0, x]) \subseteq B_D([0, y])$. ∎

Corollary 3.15. *For any $x, y \in L$, $[0, x] \subseteq [0, y]$ implies $B_D([0, x]) \subseteq B_D([0, y])$.*

Lemma 3.16. *$\{B_D([0, x]) \mid x \in L\}$ is closed under the operations $B_D([0, x]) \wedge B_D([0, y]) = B_D([0, x \wedge y])$ and $B_D([0, x]) \vee B_D([0, y]) = B_D([0, x \vee y])$.*

Proof. Let $x, y \in L$. Then $x \wedge y \leq x, y$. Therefore $[0, x \wedge y] \subseteq [0, x], [0, y]$. By Corollary 3.15, $B_D([0, x \wedge y]) \subseteq B_D([0, x]), B_D([0, y])$. Therefore $B_D([0, x \wedge y])$ is a lower bound of $B_D([0, x])$ and $B_D([0, y])$. Assume that a set $B_D([0, z])$, for some $z \in L$ is a lower bound of $B_D([0, x])$ and $B_D([0, y])$. Since z is dense in $[0, z]$, $z \in B_D([0, z]) \subseteq B_D([0, x]), B_D([0, y])$. Therefore $z \leq x$ and $z \leq y$ (since $B_D([0, x]) \subseteq [0, x]$, for any $x \in L$). So that $z \in [0, x \wedge y]$ and $[0, z] \subseteq [0, x \wedge y]$. By Corollary 3.15, $B_D([0, z]) \subseteq B_D([0, x \wedge y])$. Thus $B_D([0, x \wedge y])$ is the greatest lower bound of $B_D([0, x])$ and $B_D([0, y])$. Let $x, y \in L$. Then $x, y \leq x \vee y$. Therefore $[0, x], [0, y] \subseteq [0, x \vee y]$. By Corollary 3.15, $B_D([0, x]), B_D([0, y]) \subseteq B_D([0, x \vee y])$. Therefore $B_D([0, x \vee y])$ is an upper bound of $B_D([0, x])$ and $B_D([0, y])$. Assume that a set $B_D([0, z])$, for some $z \in L$ is an upper bound of $B_D([0, x])$ and $B_D([0, y])$. Since x, y are dense elements in $[0, x]$ and $[0, y]$ respectively, $x \in B_D([0, x]) \subseteq B_D([0, z])$ and $y \in B_D([0, y]) \subseteq B_D([0, z])$. Therefore $x \vee y \in B_D([0, z]) \subseteq [0, z]$ (Since $B_D([0, z])$ is a sub-ADL of L). Hence $B_D([0, x \vee y]) \subseteq B_D([0, z])$ (Since $B_D([0, x]) \subseteq [0, x]$, for any $x \in L$). Thus the set $\{B_D([0, x]) \mid x \in L\}$ is closed under the operations. ∎

Theorem 3.17. *$(\{B_D([0, x]) \mid x \in L\}, \wedge, \vee, \{0\})$ forms an ADL with operations defined in Lemma 3.16, and hence $B_D([0, 0]) = \{0\}$ is the least element.*

Corollary 3.18. *L has a maximal element if and only if $(\{B_D([0, x]) \mid x \in L\}, \wedge, \vee, \{0\})$ has maximal element.*

Corollary 3.19. *L is bounded if and only if $(\{B_D([0, x]) \mid x \in L\}, \wedge, \vee, \{0\})$ is bounded.*

Theorem 3.20. *L is distributive if and only if $(\{B_D[0, x] \mid x \in L\}, \wedge, \vee, \{0\})$ is distributive.*

Proof. Suppose that L is a distributive lattice. For any $x, y, z \in L$,

$$\begin{aligned}
&(B_D[0, x] \wedge B_D[0, y]) \vee B_D[0, z] \\
\Rightarrow\ & B_D[0, x \wedge y] \vee B_D[0, z] \\
\Rightarrow\ & B_D[0, (x \wedge y) \vee z] \\
\Rightarrow\ & B_D[0, (x \vee z) \wedge (y \vee z)] \qquad \text{(since } L \text{ is distributive)} \\
\Rightarrow\ & B_D[0, (x \vee xz) \wedge B_D[0, (y \vee z)] \\
\Rightarrow\ & (B_D[0, x] \vee B_D[0, z]) \wedge (B_D[0, y] \vee B_D[0, z)]).
\end{aligned}$$

Therefore $(\{B_D[0,x] \mid x \in L\}, \wedge, \vee, \{0\})$ is distributive. Conversely,

$$\begin{aligned}
& B_D[0,(x\wedge y)\vee z] \\
= \ & B_D[0, x\wedge y] \vee B_D[0,z] \\
= \ & (B_D[0,x] \wedge B_D[0,y]) \vee B_D[0,z] \\
= \ & (B_D[0,x] \vee B_D[0,z]) \wedge (B_D[0,y)] \vee B_D[0,z]) \\
= \ & B_D[0,x\vee z] \wedge B_D[0,y\vee z] \qquad \text{(since } \{B_D[0,x] \mid x\in L\} \text{ is distributive)} \\
= \ & B_D[0,(x\vee z)\wedge(y\vee z)].
\end{aligned}$$

Therefore $B_D[0,(x\wedge y)\vee z] = B_D[0,(x\vee z)\wedge(y\vee z)]$. By Lemma 3.13, $(x\wedge y)\vee z = (x\vee z)\wedge(y\vee z)]$. Hence L is distributive. ■

Theorem 3.21. *L is a Boolean algebra if and only if $(\{B_D[0,x] \mid x \in L\}, \wedge, \vee, \{0\}, L)$ is a Boolean algebra.*

Proof. Suppose that L is a Boolean algebra. Let $B_D[0,a]$, for some $a \in L$. Then $a\wedge x = 0$ and $a \vee x = 1$(1 is the greatest element in L), for some element $x \in L$. By Lemma 3.13, $B_D[0,a\wedge x] = B_D[0,0]$ and $B_D[0,a\vee x] = B_D[0,1]$. Therefore $B_D[0,a]\wedge B_D[0,x] = \{0\}$ and $B_D[0,a]\vee B_D[0,x] = L$ (By Theorem 3.20). Hence $(\{B_D[0,x] \mid x\in L\}, \wedge, \vee, \{0\}, L)$ is a Boolean algebra. Conversely suppose, for any $a \in L$, there is an element $x \in L$ such that $B_D[0,a]\wedge B_D[0,x] = \{0\}$ and $B_D[0,a]\vee B_D[0,x] = L$. Therefore $B_D[0,a\wedge x] = \{0\}$ and $B_D[0,a\vee x] = L$. By Corollary 3.13, $a\wedge x = 0$ and $a\vee x = 1$. Hence L is Boolean algebra. ■

References

[1] G. Boole, *An Investigation into the Laws of Thought,* 1854, Reprinted by Open Court Publishing Co., Chesla, 1940.

[2] S. Ramesh and G. Joga Rao, A class of sub-almost distributive lattices in an associate almost distributive lattice through ideals and filters, *Asian-Eur J. Math.* **13** (2020) 1–9.

[3] S. Ramesh and G. Joga Rao, Weak relative complements in almost distributive lattice, *Discussiones Mathematicae General Algebra and Applications* **38** (2018) 1–14.

[4] S. Ramesh and G. Joga Rao, Weakly relatively complemented almost distributive lattice, *Palestine Journal of Mathematics* **6** (2017) 1–10.

[5] G.C. Rao and G. Nanaji Rao, Dense elements in almost distributive lattices, *Southeast Asian Bull. Math.* **27** (2004) 1081–1088.

[6] G.C. Rao and M. Sambasiva Rao, Annihilator ideal in almost distributive lattices, *Int. Math. Forum* **4** (2009) 733–746.

[7] U.M. Swamy and G.C. Rao, Almost distributive lattices, *J. Austral. Math. Soc.* **31** (1981) 77–91.