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Some Properties of K-frames with Pro-C^*-Valued Bounds in Hilbert Pro-C^*-Modules

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Abstract. $*$-frames of multipliers on Hilbert pro-C^*-modules are typical of frames, with pro-C^*-valued bounds. In this paper, we introduce $*$-frames where the lower pro-C^*-valued bound just keeps for the elements in the range of adjointable operator K in Hilbert pro-C^*-modules. New $*$-frames are called K-$*$-frames of multipliers. Also, we establish some relations between K-$*$-frames and $*$-frames in Hilbert pro-C^*-modules. Finally, we study K-$*$-frames in super Hilbert modules over pro-C^*-algebras and investigate an example of K-$*$-frames in these spaces.

Keywords: Pro-C^*-algebras; Hilbert pro-C^*-module; Super Hilbert space; Standard $*$-frame of multipliers; K-$*$-frame of multipliers.

1. Introduction

Pro-C^*-algebras (under the name of locally C^*-algebras) was first introduced by Inoue [13].

A pro-C^*-algebra is a Hausdorff complete complex topological $*$-algebra $\mathcal{A}$ whose the topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_\lambda\}_{\lambda\in\Lambda}$ converges to 0 if and only if the net $\{\rho(a_\lambda)\}_{\lambda\in\Lambda}$ converges to 0 for all continuous C^*-seminorm ρ on $\mathcal{A}$, and we have

(i) $\rho(ab) \leq \rho(a)\rho(b)$,
(ii) $\rho(a^*a) = \rho(a)^2$,

for all $a, b \in \mathcal{A}$. The set of all continuous C^*-seminorms on $\mathcal{A}$ is denoted by $S(\mathcal{A})$.

In Hilbert spaces, by letting the inner product to take values in a C^*-algebra rather than the field of complex numbers, Hilbert C^*-modules were created. These spaces are generalizations of Hilbert spaces.

Some concepts as Hilbert C^*-module, compact operator, adjointable operator, representation are defined with manifest changes in the framework of pro-C^*-algebras. The concept of Hilbert modules over pro-C^*-algebras were considered by Phillips [19]. But the main study on Hilbert pro-C^*-modules has been done by Joita in [17]. They showed that most of the basic properties of Hilbert C^*-modules are valid for Hilbert modules over pro-C^*-algebras.

Suppose $\mathcal{A}$ is a pro-C^*-algebra. A complex vector space E which is also a right $\mathcal{A}$-module, compatible with the complex algebra structure, equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : E \times E \to \mathcal{A}$ which is $\mathbb{C}$-and $\mathcal{A}$-linear in its second variable is called a pre-Hilbert pro-C^*-module if it satisfies the following relations:

(i) $\langle x, y \rangle^* = \langle y, x \rangle$, for every $x, y \in E$.
(ii) $\langle x, x \rangle \geq 0$, for every $x \in E$.
(iii) $\langle x, x \rangle = 0$ iff $x = 0$, for every $x \in E$.

We say that E is a Hilbert $\mathcal{A}$-module (or Hilbert pro-C^*-module over $\mathcal{A}$) if E is complete concerning the topology determined by the family of seminorms

$$\bar{\rho}_E(x) = \sqrt{\rho(\langle x, x \rangle)}, \quad x \in E, \ \rho \in S(\mathcal{A}).$$

Let E be a pre-Hilbert $\mathcal{A}$-module. For every $\rho \in S(\mathcal{A})$ and for all $x, y \in E$, the following Cauchy-Schwarz inequality holds [17]

$$\rho(\langle x, y \rangle) \leq \overline{\rho}_E(x)\overline{\rho}_E(y).$$

Consequently, for each $\rho \in S(\mathcal{A})$, we have:

$$\bar{\rho}_E(ax) \leq \rho(a)\bar{\rho}_E(x), \quad a \in \mathcal{A}, \ x \in E.$$

The theory of frames was first introduced in 1952 by Duffin and Schaeffer [6], to study some problems in the nonharmonic Fourier series and developed in 1986 by Daubechies et al. [4]. The notion of frames for Hilbert spaces had been extended by Frank and Larson [8], to the Hilbert C^*-modules.

In 2001, Feichtinger and Werther introduced the atomic systems for subspaces [7]. K-frames in Hilbert spaces were first introduced by Gavruta in [9], to study atomic decomposition systems. K-frames are frames where the lower frame bound just keeps for the elements in the range of bounded linear operator K in Hilbert spaces.

Let H and K be two Hilbert spaces. Then $H \oplus K$ is called super Hilbert space [3, 10, 12]. Balan [3] presented the concept of super frames and offered some density results for Weyl-Heisenberg super frames. In [12], Han and Larson deduced necessary and sufficient conditions for the direct sum of two frames to be a super frame. Recently, Rashidi-Kouchi investigated frames in super Hilbert modules [20].

Since topological $*$-algebras, in particular pro-C^*-algebras is applied to relativistic quantum mechanics [5], Khosravi and Asgari considered the frames in Hilbert pro-C^*-modules, and extend some of the known results about bases to frames in [18].

During the years, various generalizations of the frame theory in Hilbert pro-C^*-modules have been investigated and proposed, such as the standard frames [16], g-frames [11], fusion frames [2], $*$-frames of multipliers [14], woven frame of multipliers [15] etc. In this paper, we investigate K-$*$-frames of multipliers in Hilbert pro-C^*-modules, and some results for these frames are surveyed. Also, we study an example of K-$*$-frames in super Hilbert pro-C^*-modules.

The paper is organized as follows: In Section 2, we give some the fundamental definitions and basic properties of Hilbert pro-C^*-modules. Also, we recall the $*$-frame of multipliers in Hilbert pro-C^*-modules. In Section 3, we introduce K-$*$-frames of multipliers in Hilbert pro-C^*-modules. In Section 4, some relations between K-$*$-frames and $*$-frames are established. Finally, in the last section, we recall the direct sum of two Hilbert spaces is called super Hilbert space and investigate an example of K-$*$-frames of multipliers in super Hilbert modules over pro-C^*-algebras.

Throughout this paper, let $\mathcal{A}$ be a unital pro-C^*-algebra concerning the family of continuous C^*-seminorms $\rho = \{\rho_\lambda\}_{\lambda\in\Lambda}$ and let E, F be finitely or countably generated Hilbert $\mathcal{A}$-modules. We use I, J to denote finite or countably infinite index sets.

2. Preliminaries

In this section, we remind some basic definitions and properties of Hilbert pro-C^*-modules. Also, the $*$-frame of multipliers in Hilbert pro-C^*-modules are recalled. For more information about Hilbert pro-C^*-modules, we refer to [17, 14].

An operator $T : E \to F$ is adjointable if there exists a map $T^* : F \to E$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for every $x \in E$ and $y \in F$. The set of all adjointable operators from E to F is denoted by $L(E, F)$. If $E = F$, $L(E, F)$ is denoted by $L(E)$ [16].

An $\mathcal{A}$-module map $T : E \to F$ is called bounded if for all $\rho \in S(\mathcal{A})$, there exists $C_\rho > 0$ such that $\bar{\rho}_F(Tx) \leq C_\rho \bar{\rho}_E(x)$ for all $x \in E$ [16].

Definition 2.1. [17] *A bounded operator $T \in L(E, F)$ is strongly bounded if*

$$\sup\{\overline{\rho}_{L(E,F)}(T);\ \rho \in S(\mathcal{A})\} < \infty.$$

The set of all strongly bounded elements in $L(E, F)$, is denoted by $b(L(E, F))$.

Definition 2.2. [11] *Let E and F be two Hilbert pro-C^*-modules over $\mathcal{A}$. Then the operator $T : E \to F$ is called uniformly bounded (below), if there exists $C > 0$ such that for each $\rho \in S(\mathcal{A})$,*

$$\bar{\rho}_F(Tx) \leq C\bar{\rho}_E(x), \text{ for all } x \in E, \tag{1}$$
$$(\bar{\rho}_F(Tx) \geq C\bar{\rho}_E(x), \text{ for all } x \in E). \tag{2}$$

The number C in (1) is called an upper bound for T and we set:

$$\|T\|_\infty = \inf\{C : \ C \textit{ is an upper bound for } T\}.$$

Clearly, in this case $\hat{\rho}(T) \leq \|T\|_\infty$, for all $\rho \in S(\mathcal{A})$ where,

$$\hat{\rho}(T) = \sup\{\bar{\rho}_F(Tx) : \ x \in E, \ \bar{\rho}_E(x) \leq 1\}.$$

In [16] a multiplier of the Hilbert $\mathcal{A}$-module E is an adjointable module morphism from $\mathcal{A}$ to E. The Hilbert $M(\mathcal{A})$-module $L(\mathcal{A}, E)$ is called the multiplier module of E and it is denoted by $M(E)$. For all $h \in M(E)$ and $x \in E$, we set

$$\langle h, x\rangle_{M(E)} = h^*(x).$$

Moreover, if $a \in \mathcal{A}$ and $h \in M(E)$, then $h.a$ can be identified with $h(a)$.

Let the set $H_{\mathcal{A}}$ of all sequences $(a_n)_n$ with $a_n \in \mathcal{A}$, such that $\sum a_n^* a_n$ converges in $\mathcal{A}$ be a Hilbert $\mathcal{A}$-module with inner product

$$\langle (a_n)_n, (b_n)_n\rangle_{H_{\mathcal{A}}} = \sum\nolimits_n a_n^* b_n.$$

Recently, M.N. Irani and A. Nazari introduced frames with algebraic bounds in Hilbert pro-C^*-modules [14]. New frames are called the standard $*$-frames of multipliers in Hilbert pro-C^*-modules.

Definition 2.3. [14] *Let E be a Hilbert pro-C^*-module. The sequence $\{h_i\}_{i\in I}$ in $M(E)$ is called a standard $*$-frame of multipliers for E if for each $x \in E$, the series $\sum_{i\in I} \langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)}$ converges in $\mathcal{A}$ and there exist two strictly nonzero elements C and D in $\mathcal{A}$ such that*

$$C\langle x, x\rangle_E C^* \leq \sum_{i\in I} \langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)} \leq D\langle x, x\rangle_E D^*,$$

for all $x \in E$.

If $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for E with $*$-frame bounds C, D, then the pre-$*$-frame operator $T : E \to H_{\mathcal{A}}$ defined by $T(x) = \{\langle h_i, x\rangle_{M(E)}\}_{i\in I}$ has a unique $*$-frame operator $S : E \to E$ defined by $Sx = \sum_{i\in I} h_i \langle h_i, x\rangle_{M(E)}$. Moreover, S is positive, self-adjoint and invertible. We refer the reader to [14].

3. K-Frames with Pro-C^*-Valued Bounds

In this section, by using the sequence of multipliers, we introduce K-frames with pro-C^*-valued bounds in Hilbert pro-C^*-modules. New frames are called K-$*$-frames of multipliers. Also, some properties of K-$*$-frames in Hilbert pro-C^*-modules are studied.

Definition 3.1. *Let $K \in L(E)$. The sequence $\{h_i\}_{i\in I}$ in $M(E)$ is called a standard K-$*$-frame of multipliers for E if for each x in E, $\sum_{i\in I} \langle x, h_i\rangle_{M(E)} \langle x, h_i\rangle_{M(E)}$ converges in $\mathcal{A}$ and there exist two strictly nonzero elements $A, B \in \mathcal{A}$ such that*

$$A\langle K^*x, K^*x\rangle_E A^* \leq \sum_{i\in I} \langle x, h_i\rangle_{M(E)} \langle x, h_i\rangle_{M(E)} \leq B\langle x, x\rangle_E B^*,$$

for all $x \in E$.

We call A, B lower and upper $*$-frame bounds for K-$*$-frame $\{h_i\}_{i\in I}$. The sequence $\{h_i\}_{i\in I}$ is called a tight K-$*$-frame of multipliers if there exists a nonzero element $C \in \mathcal{A}$ such that $C\langle K^*(x), K^*(x)\rangle_E C^* = \sum_{i\in I} \langle x, h_i\rangle_{M(E)} \langle h_i, x\rangle_{M(E)}$, for all $x \in E$.

Remark 3.2. If $K = I_E$, then every standard K-$*$-frame of multipliers for E is a standard $*$-frame of multipliers for E.

Suppose that $\{h_i\}_{i\in I}$ is a standard K-$*$-frame of multipliers for E. Obviously, it is a $*$-Bessel sequence. So we can define the following operator

$$T : E \to H_{\mathcal{A}}, \quad T(x) = \{\langle h_i, x\rangle_{M(E)}\}_{i\in I}.$$

Then we have

$$T^* : H_{\mathcal{A}} \to E, \quad T^*(\{a_i\}_{i\in I}) = \sum_{i\in I} h_i a_i.$$

Let $S = T^*T$. We obtain $S(x) = \sum_{i\in I} h_i\langle h_i, x\rangle_{M(E)}$.

We call T, T^* and S the K-$*$-pre-frame operator, the synthesis operator and the K-$*$-frame operator, respectively.

Proposition 3.3. *Let the sequence $\{h_i\}_{i\in I}$ in $M(E)$ be a standard K-$*$-frame of multipliers for E, and K be an invertible element in $L(E)$. Then the following statements hold:*

(i) *The K-$*$-frame operator S is invertible and self-adjoint.*

(ii) *Every $x \in E$ can be represented as*

$$x = \sum_{i\in I} h_i\langle S^{-1}h_i, x\rangle_{M(E)} = \sum_{i\in I} S^{-1}h_i\langle h_i, x\rangle_{M(E)}. \tag{3}$$

Proof. (i) Suppose that x is an arbitrary element of E, and $Sx = 0$. Since

$$\langle K^*x, K^*x\rangle_E \leq A^{-1} \sum_{i\in I} \langle x, h_i\rangle_{M(E)} \langle h_i, x\rangle_{M(E)} A^{*^{-1}},$$

therefore

$$\langle K^*x, K^*x\rangle_E \leq A^{-1} \langle Sx, x\rangle_E A^{*^{-1}}.$$

Since K is invertible and $K^*x = 0$, K^* is invertible and $x = 0$. This shows that S is invertible. Also $S = T^*T$, clearly S is self-adjoint.

(ii) Since S is invertible, each $x \in E$ has the representation

$$x = SS^{-1}x = \sum_{i\in I} h_i \big\langle h_i, S^{-1}x\big\rangle_{M(E)}.$$

Since S is self-adjoint, we have $x = \sum_{i\in I} h_i \big\langle S^{-1}h_i, x\big\rangle_{M(E)}$. The second representation in (3) is obtained in the same way, by $x = S^{-1}Sx$. ∎

In Proposition 3.3, the sequence $\{S^{-1}h_i\}_{i\in I}$ is called the canonical dual K-$*$-frame of $\{h_i\}_{i\in I}$.

4. Some Results

In this section, the effect of operators on the standard K-$*$-frames of multipliers for E is examined, and the relation between K-$*$-frames and $*$-frames are obtained.

Proposition 4.1. *Let $\{h_i\}_{i\in I}$ in $M(E)$ be a K-$*$-frame of multipliers and $T \in b(L(E))$. Then $\{Th_i\}_{i\in I}$ is a TK-$*$-frame of multipliers in E.*

Proof. Let x be an arbitrary element in E and C, D be $*$-bounds for $\{h_i\}_{i\in I}$. Since

$$\begin{aligned}\sum_{i\in I} \langle x, Th_i\rangle_{M(E)} \langle Th_i, x\rangle_{M(E)} &= \sum_{i\in I} \langle T^*x, h_i\rangle_{M(E)} \langle h_i, T^*x\rangle_{M(E)} \\ &\leq D\langle T^*x, T^*x\rangle_E D^* \\ &\leq (D\|T\|_\infty)\langle x, x\rangle_E (D\|T\|_\infty)^*,\end{aligned}$$

we have

$$\sum_{i\in I} \langle x, Th_i\rangle_{M(E)} \langle Th_i, x\rangle_{M(E)} \geq C\langle K^*T^*x, K^*T^*x\rangle_E C^*.$$

So $\{Th_i\}_{i\in I}$ is a TK-$*$-frame of multipliers in E. ∎

Theorem 4.2. *Let E be a Hilbert $\mathcal{A}$-module and $\{h_i\}_{i\in I}$ be a standard K-$*$-frame of multipliers in E. If there exists a surjective element U in $b(L(E))$ where $UK = KU$, then $\{Uh_i\}_{i\in I}$ is a standard K-$*$-frame of multipliers in E.*

Proof. Since $\{h_i\}_{i\in I}$ is a standard K-$*$-frame of multipliers for all $x \in E$, we have

$$\sum_{i\in I} \langle x, Uh_i\rangle_{M(E)} \langle Uh_i, x\rangle_{M(E)} = \sum_{i\in I} \langle U^*x, h_i\rangle_{M(E)} \langle h_i, U^*x\rangle_{M(E)}.$$

So the series $\sum_{i\in I} \langle x, Uh_i\rangle_{M(E)} \langle Uh_i, x\rangle_{M(E)}$ converges in $\mathcal{A}$.

And U is a surjective element in $b(L(E))$. Then by Proposition 2.2 in [1], U^* is bounded below in $b(L(E))$ and there exists $m > 0$, such that $m\langle x, x\rangle_E \leq \langle U^*x, U^*x\rangle_E$ for all $x \in E$.

There exist strictly nonzero elements C and D in $\mathcal{A}$. For all $x \in E$, we have:

$$\begin{aligned}\sum_{i\in I} \langle x, Uh_i\rangle_{M(E)} \langle Uh_i, x\rangle_{M(E)} &\geq C\,\langle K^*U^*x, K^*U^*x\rangle\, C^* \\ &= C\,\langle U^*K^*x, U^*K^*x\rangle\, C^* \\ &\geq \sqrt{m}C\,\langle K^*x, K^*x\rangle\,(\sqrt{m}C)^*.\end{aligned}$$

Since $U^* \in b(L(E))$, for all $x \in E$ we have:

$$\begin{aligned}\sum_{i\in I} \langle x, Uh_i\rangle_{M(E)} \langle Uh_i, x\rangle_{M(E)} &\leq D\,\langle U^*x, U^*x\rangle\, D^* \\ &\leq \|U^*\|_\infty D\,\langle x, x\rangle\, \|U^*\|_\infty D^*.\end{aligned}$$

Consequently, $\{Uh_i\}_{i\in I}$ is a standard K-$*$-frame of multipliers in E. ∎

Corollary 4.3. *If the sequence $\{h_i\}_{i\in I}$ in $M(E)$ is a standard K-$*$-frame of multipliers for E and K is a surjective element in $b(L(E))$, then $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for E.*

Remark 4.4. If $\{h_i\}_{i\in I}$ in $M(E)$ is a standard $*$-frame of multipliers for E and $K \in b(L(E))$, then $\{Kh_i\}_{i\in I}$ is a standard K-$*$-frame of multipliers for E.

Theorem 4.5. *Let K be an invertible element in $b(L(E))$. Then the following statements are equivalent:*

(i) *The sequence $\{h_i\}_{i\in I}$ in $M(E)$ is a standard K-$*$-frame of multipliers for E.*
(ii) *The sequence $\{h_i\}_{i\in I}$ in $M(E)$ is a standard $*$-frame of multipliers for E.*

Proof. (i)⇒(ii) For every $x \in E$, we have

$$A\langle K^*x, K^*x\rangle_E A^* \leq \sum_{i\in I} \langle x, h_i\rangle_{M(E)} \langle h_i, x\rangle_{M(E)} \leq B\langle x, x\rangle_E B^*.$$

Since K^* is an invertible element in $b(L(E))$, for every $x \in E$

$$\left\|{K^*}^{-1}\right\|_\infty^{-2} \langle x, x\rangle_E \leq \langle K^*x, K^*x\rangle_E \leq \|K^*\|_\infty^2 \langle x, x\rangle_E.$$

So

$$\begin{aligned}(\left\|{K^*}^{-1}\right\|_\infty^{-1} A)\langle x, x\rangle_E(\left\|{K^*}^{-1}\right\|_\infty^{-1} A)^* &\leq \sum_{i\in I} \langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)} \\ &\leq m\langle x, x\rangle_E m^*,\end{aligned}$$

where $m = \max\{B, \|K\|_\infty 1_{\mathcal{A}}\}$.

This shows that $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for E.

(ii)⇒(i) There exist strictly nonzero elements C, D in $\mathcal{A}$ such that

$$C\langle x, x\rangle_E C^* \leq \sum_{i\in I} \langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)} \leq D\langle x, x\rangle_E D^*.$$

Since K^* is an invertible element in $b(L(E))$, for every $x \in E$

$$\left\|{K^*}^{-1}\right\|_\infty^{-2} \langle x, x\rangle_E \leq \langle K^*x, K^*x\rangle_E \leq \|K^*\|_\infty^2 \langle x, x\rangle_E,$$

so we have

$$\|K^*\|_\infty^{-1} \langle K^*x, K^*x\rangle_E \|K^*\|_\infty^{-1} \leq \langle x, x\rangle_E,$$

and

$$\begin{aligned}(\|K^*\|_\infty^{-1} C)\langle K^*x, K^*x\rangle_E(\|K^*\|_\infty^{-1} C)^* &\leq \sum_{i\in I} \langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)} \\ &\leq B\langle x, x\rangle_E B^*.\end{aligned}$$

This shows that $\{h_i\}_{i\in I}$ is a standard K-$*$-frame of multipliers for E. ■

Proposition 4.6. *Let the sequence $\{h_i\}_{i\in I}$ in $M(E)$ be a standard $*$-frame of multipliers for E. Suppose that $T : E \to F$ is co-isometry and K is an invertible element in $b(L(F))$. Then $\{Th_i\}_{i\in I}$ is a standard K-$*$-frame of multipliers for F.*

Proof. Let C and D be $*$-frame bounds for the standard $*$-frame $\{h_i\}_{i\in I}$. Then for every $x \in E,\ y \in F$, we have:

$$\sum_{i\in I} \langle T^*y, h_i\rangle_{M(E)}\langle h_i, T^*y\rangle_{M(E)} = \sum_{i\in I} \langle y, Th_i\rangle_{M(F)}\langle Th_i, y\rangle_{M(F)}. \tag{4}$$

Since T is a co-isometry operator, we have

$$\begin{aligned}C\langle y, y\rangle_F C^* &= C\langle T^*y, T^*y\rangle_E C^* \\ &\leq \sum_{i\in I} \langle T^*y, h_i\rangle_{M(E)}\langle h_i, T^*y\rangle_{M(E)} \\ &\leq D\langle T^*y, T^*y\rangle_E D^* \\ &= D\langle y, y\rangle_F D^*. \end{aligned} \tag{5}$$

Hence (4) and (5) imply that $\{Th_i\}_{i\in I}$ is a standard $*$-frame of multipliers for F. Now by Theorem 4.5, we obtain the result. ■

The combination of K-$*$-frame and $*$-frame with special conditions is a K-$*$-frame of multipliers, which is shown in the following proposition.

Proposition 4.7. *Let the sequence $\{h_j\}_{j\in J}$ in $M(E)$ be a standard K-$*$-frame of multipliers for E, and the sequence $\{t_i\}_{i\in I}$ in $M(\mathcal{A})$ be a standard $*$-frame of multipliers for $\mathcal{A}$. If for any $j \in J$, h_j is a surjective element in $b(L(\mathcal{A}, E))$, then $\{h_j t_i\}_{i\in I}$ is a standard K-$*$-frame of multipliers in E, for all $j \in J$.*

Proof. Suppose that C, D are lower and upper $*$-frame bounds for $\{h_j\}_{j\in J}$ respectively. Also, assume that A, B be lower and upper $*$-frame bounds for $\{t_i\}_{i\in I}$ respectively. Since for each $j \in J$, h_j is adjointable and $\{t_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $\mathcal{A}$, for any finite subset L of I and for each $j \in J$ and $x \in E$, we have

$$\sum_{i\in L} \langle x, h_j t_i\rangle_{M(E)} \langle h_j t_i, x\rangle_{M(E)} = \sum_{i\in L} \big\langle h_j^* x, t_i\big\rangle_{M(\mathcal{A})} \big\langle t_i, h_j^* x\big\rangle_{M(\mathcal{A})}.$$

So $\sum_{i\in I} \langle x, h_j t_i\rangle_{M(E)} \langle h_j t_i, x\rangle_{M(E)}$ is convergent in $\mathcal{A}$.

For each $j \in J$ and $x \in E$, we have

$$\begin{aligned}
\sum_{i\in I} \langle x, h_j t_i\rangle_{M(E)} \langle h_j t_i, x\rangle_{M(E)} &= \sum_{i\in I} \big\langle h_j^* x, t_i\big\rangle_{M(\mathcal{A})} \big\langle t_i, h_j^* x\big\rangle_{M(\mathcal{A})} \\
&\le B\big\langle h_j^* x, h_j^* x\big\rangle_{\mathcal{A}} B^* \\
&= B\langle x, h_j\rangle_{M(E)} \langle h_j, x\rangle_{M(E)} B^* \\
&\le B \sum_{j\in J} \langle x, h_j\rangle_{M(E)} \langle h_j, x\rangle_{M(E)} B^* \\
&\le BD\langle x, x\rangle_E D^* B^*. \qquad (6)
\end{aligned}$$

This shows that for each $j \in J$, $\{h_j t_i\}_{i\in I}$ is a $*$-Bessel sequence of multipliers in E.

By Proposition 2.2 in [1], there exists $m > 0$ such that for each $j \in J$, $m\langle x, x\rangle_E \le \big\langle h_j^* x, h_j^* x\big\rangle_{\mathcal{A}}$. We have

$$\begin{aligned}
m\langle x, x\rangle_E \le \big\langle h_j^* x, h_j^* x\big\rangle_{\mathcal{A}} &\le A^{-1} \sum_{i\in I} \big\langle h_j^* x, t_i\big\rangle_{M(\mathcal{A})} \big\langle t_i, h_j^* x\big\rangle_{M(\mathcal{A})} A^{*^{-1}} \\
&= A^{-1} \sum_{i\in I} \langle x, h_j t_i\rangle_{M(E)} \langle h_j t_i, x\rangle_{M(E)} A^{*^{-1}}, \qquad (7)
\end{aligned}$$

also

$$\begin{aligned}
m\langle x, x\rangle_E &\ge mD^{-1} \sum_{j\in J} \langle x, h_j\rangle_{M(E)} \langle h_j, x\rangle_{M(E)} D^{*^{-1}} \\
&\ge mD^{-1} C\langle K^* x, K^* x\rangle_E C^* D^{*^{-1}}. \qquad (8)
\end{aligned}$$

Hence (6), (7) and (8) imply that

$$\begin{aligned}(\sqrt{m}D^{-1}CA)\langle K^*x, K^*x\rangle_E(\sqrt{m}D^{-1}CA)^* &\leq \sum_{i\in I}\langle x, h_jt_i\rangle_{M(E)}\langle h_jt_i, x\rangle_{M(E)}\\ &\leq BD\langle x,x\rangle_E D^*B^*.\end{aligned}$$

This shows that for any $j \in J$, $\{h_jt_i\}_{i\in I}$ is a standard K-$*$-frame of multipliers for E. ■

5. K-∗-Frame in Super Hilbert

In this section, we define the super Hilbert modules as a generalization of Hilbert spaces. We investigate K-∗-frames in super Hilbert modules and study an example of K-∗-frames in these spaces.

Definition 5.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two pro-C^*-algebras. Let E be a Hilbert $\mathcal{A}$-module and F be a Hilbert $\mathcal{B}$-module with inner products $\langle .,.\rangle_E$ and $\langle .,.\rangle_F$ respectively. The super Hilbert $\mathcal{A}\oplus\mathcal{B}$-module space H is a direct sum $E\oplus F$ equipped with the inner product $\langle\langle .,.\rangle\rangle_H = \langle .,.\rangle_E + \langle .,.\rangle_F$.*

Let $h^0 : \mathcal{A} \to E$ and $h^1 : \mathcal{B} \to F$ be multipliers in $M(E)$ and $M(F)$ respectively. Then we define adjointable operator h from $\mathcal{A}\oplus\mathcal{B}$ into $E\oplus F$ by

$$\langle\langle h, x\rangle\rangle_{M(E\oplus F)} = \left\langle h^0, x_0\right\rangle_{M(E)} + \left\langle h^1, x_1\right\rangle_{M(F)} = h^{0^*}(x_0) + h^{1^*}(x_1),$$

for all $x = (x_0, x_1) \in E \oplus F$.

Definition 5.2. *Let $K^0 : E \to E$ and $K^1 : F \to F$ be adjointable operators. We define adjointable operator K from $E\oplus F$ to $E\oplus F$ by $K(x) = K^0(x_0)+K^1(x_1)$ for all $x = (x_0, x_1) \in E \oplus F$. If $\{h_i\}_{i\in I}$ is a K-$*$-frame of multipliers for Hilbert $\mathcal{A}\oplus\mathcal{B}$-module $E\oplus F$, then we call it a K-$*$-frame associated with $\{(h_i^0, h_i^1)\}_{i\in I}$.*

Example 5.3. Let $H_{\mathcal{A}\oplus\mathcal{B}}$ be a Hilbert $\mathcal{A}\oplus\mathcal{B}$-module. Then $L(\mathcal{A}\oplus\mathcal{B}, H_{\mathcal{A}\oplus\mathcal{B}})$ is $L(\mathcal{A}\oplus\mathcal{B})$-module with the following operations:

$$\begin{aligned}&UV := \{u'_iv'_i \oplus u_iv_i\}_{i\in\mathbb{N}}, \quad U^* := \{\overline{u'_i} \oplus \overline{u_i}\}_{i\in\mathbb{N}},\\ &\left\langle \{u'_i \oplus u_i\}_{i\in\mathbb{N}}, \{v'_i \oplus v_i\}_{i\in\mathbb{N}}\right\rangle := \sum_{i\in\mathbb{N}} u'_iv'_i \oplus (u_iv_i)^*,\\ &\overline{\rho\oplus q}_{H_{\mathcal{A}\oplus\mathcal{B}}}(U) = (\rho\oplus q(\langle U, U\rangle_{H_{\mathcal{A}\oplus\mathcal{B}}}))^{\frac{1}{2}},\end{aligned}$$

for $U = \{u'_i \oplus u_i\}_{i\in\mathbb{N}},\ V = \{v'_i \oplus v_i\}_{i\in\mathbb{N}}$.

Let $J = \mathbb{N}$ and $h_j = \{h_i^j\}_{i\in\mathbb{N}}$ such that

$$h_i^j(a\oplus b) = \begin{cases} a\oplus b & \text{if} \quad i = j,\\ 0 & \text{if} \quad i \neq j.\end{cases}$$

Then for any $j \in J$, we have:

$$\begin{aligned}
\left(\overline{\rho \oplus q}_{H_{\mathcal{A}\oplus\mathcal{B}}}(h_j(a \oplus b))\right)^2 &= \rho \oplus q\left(\langle h_j(a \oplus b), h_j(a \oplus b)\rangle_{H_{\mathcal{A}\oplus\mathcal{B}}}\right) \\
&= \rho \oplus q\left(\left\langle \{h_i^j(a \oplus b)\}_{i\in\mathbb{N}}, \{h_i^j(a \oplus b)\}_{i\in\mathbb{N}}\right\rangle_{H_{\mathcal{A}\oplus\mathcal{B}}}\right) \\
&= \rho \oplus q\left(\langle a \oplus b, a \oplus b\rangle\right) = \rho \oplus q(aa^* \oplus bb^*) \\
&= \rho(a)^2 q(b)^2 < \infty.
\end{aligned}$$

This shows that h_j is well-defined. It is easy to check that h_j is adjointable and satisfies $h_j^* : H_{\mathcal{A}\oplus\mathcal{B}} \to \mathcal{A} \oplus \mathcal{B}$ defined by $h_j^*(\{x'_i \oplus x_i\}_i) = h_i^{j^*}(\{x'_i \oplus x_i\}_i) = \overline{x'_j} \oplus \overline{x_j}$, and $h_j \in L(\mathcal{A} \oplus \mathcal{B}, H_{\mathcal{A}\oplus\mathcal{B}})$. Then we have:

$$\begin{aligned}
&\langle \{x'_i \oplus x_i\}_i, h_j\rangle_{M(H_{\mathcal{A}\oplus\mathcal{B}})}\langle h_j, \{x'_i \oplus x_i\}_i\rangle_{M(H_{\mathcal{A}\oplus\mathcal{B}})} \\
= \ &\overline{h_j^*(\{x'_i \oplus x_i\}_i)} h_j^*(\{x'_i \oplus x_i\}_i) = x'_j x'_j \oplus \overline{x_j x_j}.
\end{aligned}$$

Also

$$\begin{aligned}
&\sum_{j\in J} \langle x, h_j\rangle_{M(H_{\mathcal{A}\oplus\mathcal{B}})}\langle h_j, x\rangle_{M(H_{\mathcal{A}\oplus\mathcal{B}})} \\
= &\sum_{j\in J} \langle \{x'_i \oplus x_i\}_i, h_j\rangle_{M(H_{\mathcal{A}\oplus\mathcal{B}})}\langle h_j, \{x'_i \oplus x_i\}_i\rangle_{M(H_{\mathcal{A}\oplus\mathcal{B}})} \\
= &\sum_{j\in J} x'_j x'_j \oplus \overline{x_j x_j} = \langle x, x\rangle_{H_{\mathcal{A}\oplus\mathcal{B}}}.
\end{aligned}$$

Hence h_j is a standard normalized $*$-frame in super Hilbert $H_{\mathcal{A}\oplus\mathcal{B}}$. Suppose that $L \in \mathbb{N}$ and invertible element $C \oplus D$ in $\mathcal{Z}(\mathcal{A} \oplus \mathcal{B})$. We define the operator

$$K : H_{\mathcal{A}\oplus\mathcal{B}} \to H_{\mathcal{A}\oplus\mathcal{B}}, \qquad K(\{x'_j \oplus x_j\}_j) = \begin{cases} C \oplus D\{x'_j \oplus x_j\}_j & \text{if } \ j \le L, \\ 0 & \text{if } \ \ j > L. \end{cases}$$

It is easy to check that K is adjointable and satisfies

$$K^*(\{x'_j \oplus x_j\}_j) = \begin{cases} \overline{C \oplus D}\{x'_j \oplus x_j\}_j & \text{if } \ j \le L, \\ 0 & \text{if } \ j > L. \end{cases}$$

For any $\{x_j{}' \oplus x_j\}_j \in H_{\mathcal{A}\oplus\mathcal{B}}$ we have:

$$\begin{aligned}
&\left\langle K^*(\{x'_j \oplus x_j\}_j), K^*(\{x'_j \oplus x_j\}_j)\right\rangle_{H_{\mathcal{A}\oplus\mathcal{B}}} \\
= \ &\langle \overline{C \oplus D}\{x'_j \oplus x_j\}_{j=1}^L, \overline{C \oplus D}\{x'_j \oplus x_j\}_{j=1}^L\rangle_{H_{\mathcal{A}\oplus\mathcal{B}}} \\
= \ &C \oplus D(\sum_{j=1}^L x'_j x'_j \oplus \overline{x_j x_j})\overline{C \oplus D} \\
\le \ &C \oplus D(\sum_{j\in J} x'_j x'_j \oplus \overline{x_j x_j})\overline{C \oplus D} \\
= \ &C \oplus D\langle x, x\rangle_{H_{\mathcal{A}\oplus\mathcal{B}}}\overline{C \oplus D}.
\end{aligned}$$

This shows that for every $L, C \oplus D$, $\{h_i\}_{i\in I}$ is a family of standard K-$*$-frames of multipliers in super Hilbert $H_{\mathcal{A}\oplus\mathcal{B}}$ with bounds $C^{-1} \oplus D^{-1}$ and $1_{\mathcal{A}\oplus\mathcal{B}}$.

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